The finite lattice method.

- here applied to enumeration of $d=2$

SAWs SAPs θ's

- also applicable to problems involving connected graph expansions e.g. Ising model
- mini-review - original work by Ian Enting, Tom de Neef, Tony Guttmann and Iwan Jensen
- goal: to extend count of c_n using

$$c_n = 2d\,c_{n-1} + \sum_{m=2}^{n} \pi_m c_{n-m}$$

λ^{n-1}

- For SAW $c_n \sim \mu^n n^{\gamma-1}$
- For SAP $p_{2n} \sim \mu^{2n} n^{\alpha-3}$
- $d=2$ $\mu = 2.638....$
- i.e. exponential growth $\longrightarrow$ hard problem

Best approach:

- Find exact solution
  (or at least a polynomial time algorithm)

Simplest approach:

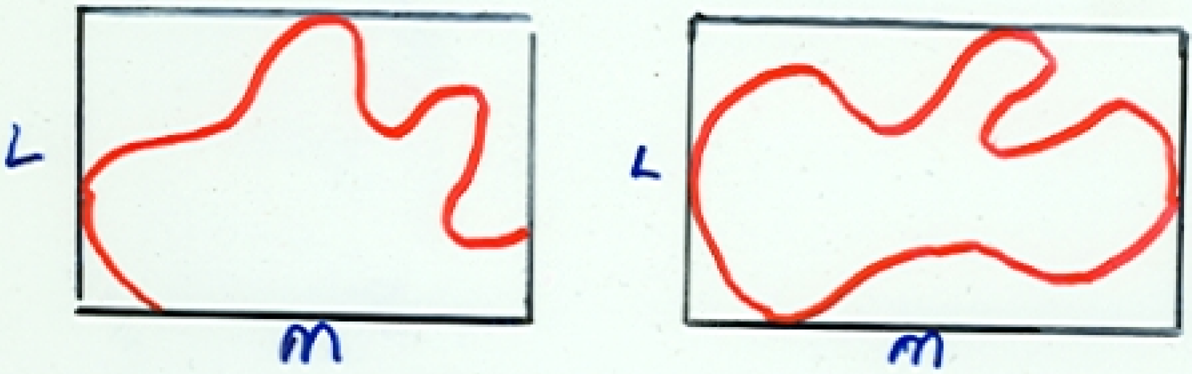
- Direct enumeration using backtracking
- time $O(c_n) = O(\mu^n)$
- Best known method for $d \geq 3$

Finite Lattice Method:

- still exponential $\left(O(\alpha^n)\right)$ for $d=2$
- BUT $\alpha < \mu$
- SAW $\alpha \simeq 1.3$, enumerated c_n to $n=71$
- SAP $\alpha \simeq 1.20$, enumerated p_n to $n=110$

Method – step 1

- Any SAW or SAP has an enclosing rectangle of smallest size



- c_n = # of SAWs of length n ←

  $= \sum_{\substack{L \times M \\ \text{rectangles}}}$ (# of SAWs of length n which touch all boundaries of an $L \times M$ rectangle)

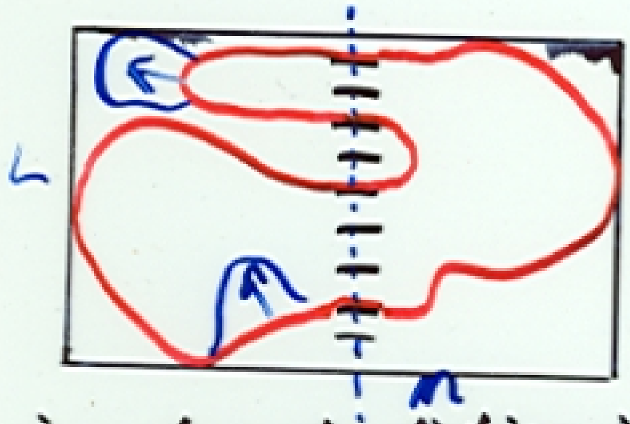
- SAW of length n fits in a rectangle of perimeter $\leq 2n$. i.e $\underline{L+M \leq n}$
- SAP of length $2n$ fits in a rectangle of perimeter $\leq 2n$ i.e $L+M \leq n$
- In terms of generating funtitions

$$X_n(z) = \sum_{j=0}^{n} c_j z^j$$

$$= \sum_{\substack{L \times m \\ L+M \leq n}} X_n^{(L \times m)}(z)$$

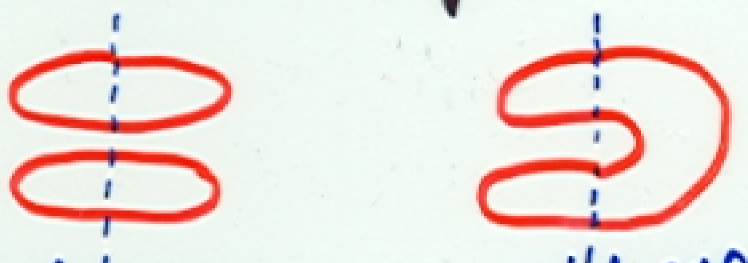
Method - step 2

- conditional independence of partial generating functions across a boundary
- for SAPs: (slightly harder for SAWs)
- draw a boundary through the edges of the finite lattice



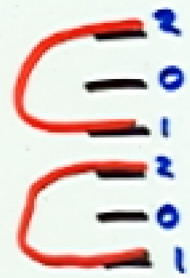
- can change shape, but not topology, on LHS and RHS independently

- i.e. have to distinguish between:



not an SAP   valid SAP

- fix topology by specifying whether an edge is empty (0), or forms the upper part of a loop (2), or the lower part of a loop.

∴ 

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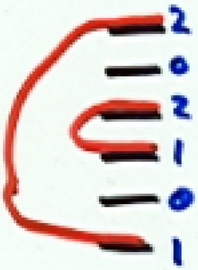
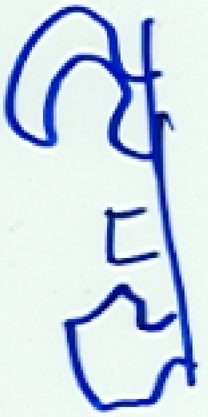


[202101]

- loops can never intertwine

- independence ⇒ for a fixed boundary the full generating function is the convolution of the partial generating functions (PGF) on LHS and RHS

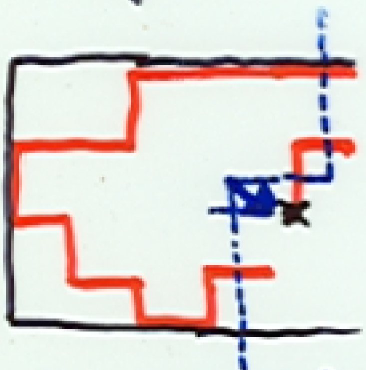
i.e. $f_B(z) = (f_{L_B} * f_{R_B})(z)$ with states on boundary fixed.



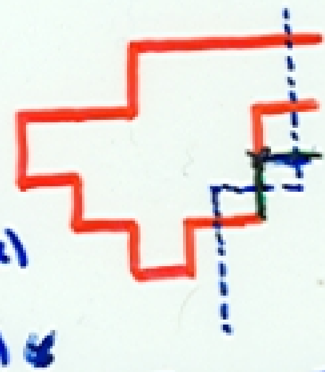
$$\chi^{(L\times m)} = \sum_B f_B(z)$$

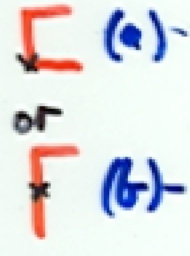
Method - step 3

- need an efficient algorithm to calculate PGF
- Transfer Matrix algorithm - build up PGF one site at a time by moving the boundary
- each legal boundary (balanced parentheses) has its own generating function. L edges on boundary $\to \sim 3^L$ PGFs. 0, 1, 2



$\longrightarrow$





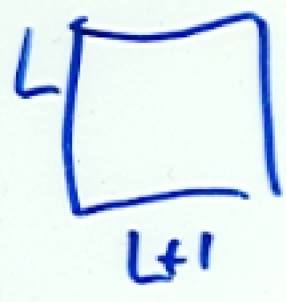
[221010] f

$f_{(a)} = zf + f_{(a)}$

$f_{(b)} = zf + f_{(a)}$

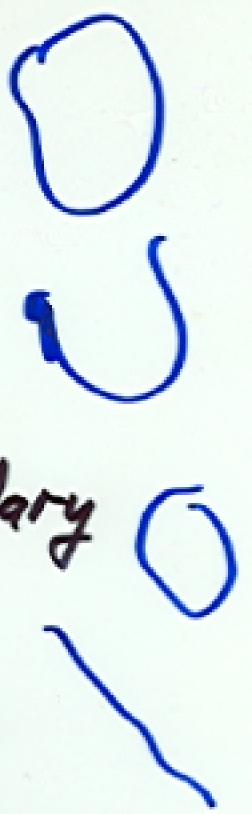
[221010] or [220120]
(a) (b)

- iterate until whole rectangle is to the left of the boundary
- takes time $O(3^L)$, memory $O(3^L)$
  For polygons $n_{max} = 4L+2$
  $\Rightarrow O(3^{n/4}) \simeq O(1{\cdot}31^n)$



$\mu = 2{\cdot}638\ldots$

- SAWs more complicated due to free ends
- need an extra state to make 4
- at most 2 of these states in boundary

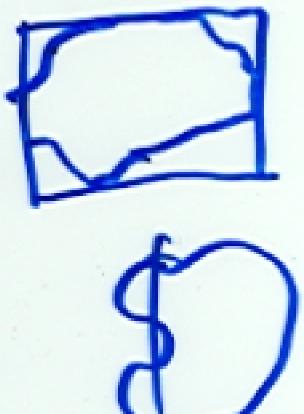


$\Rightarrow$ complexity still $O(3^{n/4})$

(not the whole story: if $n_{max} = 2L+1$ $\Rightarrow O(3^{n/2})$, but ingenious method by Conway et. al. overcame this)

- **Pruning** (due to Jensen) improves algorithm still further

n



- throw away boundary states which won't contribute to final G.F.
- SAPs: $O(1.20^n)$ c.f. $O(1.31^n)$
- SAWs: $O(1.33^n)$ c.f $O(1.31^n)$

but for 'small' n up to 70, pruned algorithm much better

$?$

$n_{config} \geq n$

50 $\quad$ 50× more memory

Can we apply the FLM to the lace expansion for SAWs?

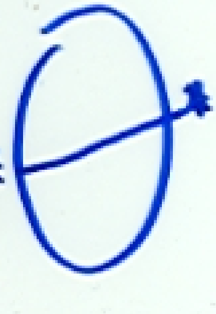
- lace expansion graphs are less numerous and more compact, than SAPs
- would like to calculate $\Pi_m = \sum_N (-1)^N \Pi_m^{(N)}$ to the same order as $\Pi_m^{(0)}$ (SAPs)
- then, using

$$c_n = 2d\,c_{n-1} + \sum_{m=2}^{n} \Pi_m c_{n-m} \leftarrow$$

we have c_n to same order as Π_m

$\Rightarrow$ 71 term SAW series $\longrightarrow$ 110 term SAW series

- θ graphs easy – similar technically to SAWs as we have 2 special vertices



- , , ... problematic

[12347][345]

  - walks which don't avoid $\Rightarrow$ multiply occupied bonds
  - need extra states, $O(q^{n/4})$

110 55[illegible] $\to \sim 53$

- more promising: direct enumeration for $d \geq 3$ to extend c_n and $1/d$ expansion for μ (with Richard Liang and Gordon Slade)