On braided Hopf structures on exterior algebras

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Outline

Introduction

Consider a braided $\mathbb{F}$-vector space (V, τ), $\tau \in \mathrm{Aut}(V \otimes V)$ satisfies the Yang–Baxter equation

$$\tau'\tau''\tau' = \tau''\tau'\tau'', \quad \tau' := \tau \otimes \mathrm{id}_V, \; \tau'' := \mathrm{id}_V \otimes \tau.$$

The tensor algebra $T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ is canonically a braided Hopf algebra with an invertible antipode with all the elements of V being primitive

$$\Delta u = u \otimes 1 + 1 \otimes u, \quad \forall u \in V$$

The Nichols algebra associated with (V, τ) is the quotient braided Hopf algebra

$$\mathfrak{B}(V) = T(V)/\mathfrak{J}_V$$

where $\mathfrak{J}_V$ is the maximal Hopf ideal generated by primitive elements of degree ≥ 2, where elements of degree n are those of $T^n(V) := V^{\otimes n}$.

We are interested in the cases when the Nichols algebra is finite-dimensional.

A braided Hopf algebra H is endowed with five structural morphisms:

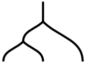
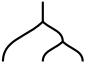
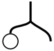
$$\nabla\colon H\otimes H\to H,\quad \eta\colon I\to H,\quad \Delta\colon H\to H\otimes H,\quad \epsilon\colon H\to I,\quad S\colon H\to H$$

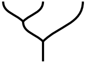
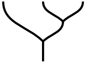
called, respectively, the product, unit, coproduct, counit and antipode.

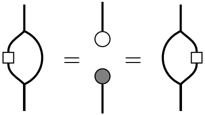
∇ =  (product), Δ =  (coproduct), $\beta_{H,H}$ =  (braiding)

η =  (unit), ϵ =  (counit), S =  (antipode).

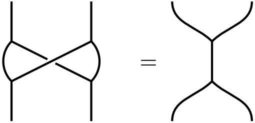
Axioms of a braided Hopf algebra:

 =  (associativity),  = =  (unitality),

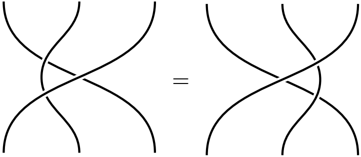
 =  (coassociativity),  = =  (counitality),

 (invertibility),

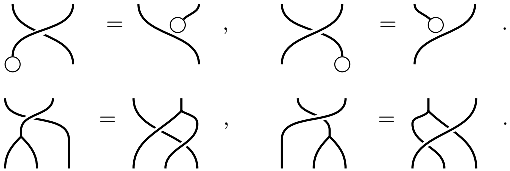
and

 (compatibility).

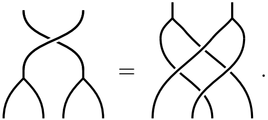
The Yang–Baxter equation for the braiding

 .

All morphisms are braided linear maps and the following relations hold (and many others):



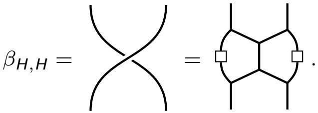
‘Fusion relation’



The braiding $\beta_{H,H}$ can be expressed entirely in terms of the structural maps, via the formula

$$\beta_{H,H} = (\nabla \otimes \nabla)(S \otimes (\Delta\nabla) \otimes S)(\Delta \otimes \Delta),$$

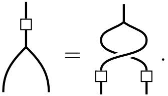
or in graphical form:



The following relation between the braiding and the antipode holds:

$$S\nabla = \nabla\beta_{H,H}(S \otimes S),$$

or in graphical form



Let (V, τ) be a braided vector space. The tensor algebra $T(V)$ is a braided Hopf algebra in which all the elements of V are primitive. The braiding

$$\hat{\tau} := \beta_{T(V),T(V)} \colon T(V) \otimes T(V) \to T(V) \otimes T(V)$$

is induced from τ in the sense that $\hat{\tau}|_{V\otimes V} = \tau$, and it satisfies the compatibility conditions with the unit $\eta \colon \mathbb{F} \to T(V)$

$$\hat{\tau}(\eta \otimes \mathrm{id}) = \mathrm{id} \otimes \eta, \quad \hat{\tau}(\mathrm{id} \otimes \eta) = \eta \otimes \mathrm{id}\,.$$

An immediate consequence of the fusion formula is the *preservation of degree* along the strands of the induced braiding $\hat{\tau}$, in the sense that

$$\forall m, n \in \mathbb{Z}_{\geq 0} \colon \hat{\tau}(T^m(V) \otimes T^n(V)) = T^n(V) \otimes T^m(V).$$

Let V be a $\mathbb{F}$-vector space, and let $\mathbb{B}$ be a linearly ordered basis of V. Define the Heaviside theta symbol

$$\theta_{a,b} \in \{0, 1\}, \quad a, b \in \mathbb{B},$$

by setting $\theta_{a,b} = 1$ if $a > b$ and $\theta_{a,b} = 0$ otherwise.

For any scalar $p \neq 0$, we define a linear map $\tau\colon V \otimes V \to V \otimes V$ by

$$\tau(a \otimes b) = \begin{cases} -a \otimes a & \text{if } a = b; \\ -(1-p)a \otimes b - b \otimes a & \text{if } a > b; \\ -pb \otimes a & \text{if } a < b, \end{cases}$$

for all $a, b \in \mathbb{B}$. Since τ is invertible and satisfies the quantum Yang–Baxter equation over V, it qualifies as an R-matrix. Accordingly, (V, τ) forms a braided $\mathbb{F}$-vector space.

When $\dim(V) = N$, the braiding τ corresponds to the R-matrix of the quantum group $U_q(\mathfrak{sl}_N)$ evaluated at the N-dimensional fundamental representation.

The R-matrix τ defined above, has the eigenvalue -1 with the corresponding eigenspaces spanned by

$$\{a \otimes b + b \otimes a \mid a < b\} \cup \{a \otimes a \mid a \in \mathbb{B}\}$$

Any primitive element of degree two of the tensor algebra $T(V)$ is an eigenvector of τ corresponding to the eigenvalue -1:

$$\Delta w = w \otimes 1 + 1 \otimes w \quad \Rightarrow \quad \tau w = -w, \quad w \in T^2(V) \simeq V \otimes V.$$

This follows from the formula for the coproduct of $T(V)$ in degree two:

$$\forall u, v \in V \colon \Delta(uv) = uv \otimes 1 + 1 \otimes uv + u \otimes v + \tau(u \otimes v).$$

The exterior algebra $\bigwedge V$ is a braided Hopf algebra where all the elements of V are primitive and the braiding is induced by τ

$$\bigwedge V \simeq T(V)/\mathfrak{J}_2 = \mathbb{F}\langle \mathbb{B} \mid \{ba = -ab \mid a, b \in \mathbb{B}\} \cup \{a^2 \mid a \in \mathbb{B}\}\rangle.$$

We denote by $\Lambda_p(V)$ the exterior algebra endowed with this braided Hopf algebra structure, and by $\hat{\tau}$ its associated braiding. For any $k \in \mathbb{Z}_{\geq 0}$, we write $\Lambda_p^k(V)$ for the subspace of $\Lambda_p(V)$ consisting of all the elements of degree k:

Set-theoretic bases in exterior algebras

We consider a linearly ordered basis $\mathbb{B} \subset V$ and identify with $\mathbb{B}$ the set of integers $\{1, \ldots, |\mathbb{B}|\}$, which is equipped with the natural order.

Example: $N = 3$
Basis of V is given by $\{f_{\{1\}}, f_{\{2\}}, f_{\{3\}}\}$ and

$$\Lambda_p(V) \equiv \{f_\emptyset, f_{\{1\}}, f_{\{2\}}, f_{\{3\}}, f_{\{1,2\}}, f_{\{1,3\}}, f_{\{2,3\}}, f_{\{1,2,3\}}\}.$$

This is an 8-dim space and, for any N, $\dim(V) = N$, $\dim(\Lambda_p(V)) = 2^N$.

For a set E, we denote the set of all *subsets* of E by

$$\mathcal{P}_{\mathrm{fin}}(E) := \{A \mid A \subseteq E,\ |A| < \infty\}.$$

The set of all *k-element subsets* of E is denoted by

$$\binom{E}{k} := \{A \subseteq E \mid |A| = k\}.$$

For any two finite subsets A, B of a linearly ordered set, define the Heaviside theta-symbol

$$\theta_{A,B} = \sum_{a \in A} \sum_{b \in B} \theta_{a,b}.$$

We will also write $f_{E,F,\dots,G}$ instead of $f_E \otimes f_F \otimes \cdots \otimes f_G$.

Theorem

Let $\mathbb{B}$ be a linearly ordered basis of a vector space V, where $p \in \mathbb{F}_{\neq 0}$ a nonzero scalar, and let $\{f_E \mid E \in \mathcal{P}_{\mathrm{fin}}(\mathbb{B})\}$ be the (canonical) basis of $\bigwedge V$ given by words in the alphabet $\mathbb{B}$ with strictly increasing order. Then, the braided Hopf algebra $\Lambda_p(V)$ has the following structure maps:
the product

$$\nabla(f_E \otimes f_F) =: f_E f_F = \delta_{|E\cap F|,0}(-1)^{\theta_{E,F}} f_{E\cup F}, \quad \forall E, F \in \mathcal{P}_{\mathrm{fin}}(\mathbb{B});$$

the coproduct

$$\Delta f_E = \sum_{A\subseteq E} (-p)^{\theta_{A,E\setminus A}} f_A \otimes f_{E\setminus A}, \quad \forall E \in \mathcal{P}_{\mathrm{fin}}(\mathbb{B});$$

the antipode

$$S f_E = \gamma_{|E|} f_E, \quad \forall E \in \mathcal{P}_{\mathrm{fin}}(\mathbb{B}),$$

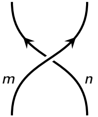
where the integers $\theta_{E,F} \in \mathbb{Z}_{\geq 0}$ are defined above, and the signed Gaussian exponential is

$$\gamma_k := (-1)^k p^{k(k-1)/2}.$$

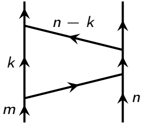
Theorem

The action of the braiding $\hat{\tau}$ of $\Lambda_p(V)$ correspond to the MOY diagrammatic identity

$$\hat{\tau}(\pi_m \otimes \pi_n) =$$



$$= \sum_{k=0}^{\min(m,n)} \gamma_k$$



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A similar formula has been derived by Murakami, Ohtsuki, and Yamada (MOY) (1998) and Cautis, Kamnitzer, and Morrison (2014). Their proofs are much more involved because they did not use a Hopf algebra structure with antipode.

Matrix elements are given by complicated combinatorial sums. Explicit calculations for $N = 2, 3, 4$ show that there are many cancelations. The goal is to produce a much nicer formula with factorised matrix elements.

Matrix coefficients of the braiding

For any finite subsets E and F of $\mathbb{B}$, and any G and H such that

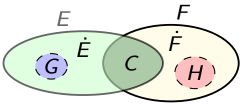
$$G \subseteq E \setminus F, \quad H \subseteq F \setminus E,$$

we define

$$C := E \cap F, \quad \dot{E} := (E \setminus F) \setminus G, \quad \dot{F} := (F \setminus E) \setminus H$$

and

$$E' := (E \setminus G) \cup H, \quad F' := (F \setminus H) \cup G.$$



The braiding of $\Lambda_p(V)$ can be decomposed into a sum

$$\hat{\tau} = \sum_{k \in \mathbb{Z}_{\geq 0}} \tau_k, \quad \tau_k f_{E,F} = (-1)^{|E||F|} \sum s_{E,G;F,H} f_{F',E'}, \quad f_{E,F} \in \Lambda_p(V)^{\otimes 2}.$$

where the summation runs over the subsets $G \in \binom{E \setminus F}{k}$ and $H \in \binom{F \setminus E}{k}$. The conditions on G and H imply the equalities

$$|E'| = |E|, \quad |F'| = |F|$$

Matrix coefficients of the braiding

Theorem

For $G \in \binom{E \setminus F}{k}$, $H \in \binom{F \setminus E}{k}$, we have the following formula for the coefficient $s_{E,G;F,H}$

$$s_{E,G;F,H} = \beta_{E,G;F,H},$$

which does not vanish for generic p if and only if

$$\theta_{A,H} > \theta_{A,G}, \quad \forall A \in \binom{G}{1}.$$

Coefficients $\beta_{E,G;F,H}$ are given by

$$\beta_{E,G;F,H} = (-1)^{\theta_{F,E}+\theta_{F',E'}} p^{\theta_{G \sqcup C,E}+\theta_{\dot{F},E'}} \alpha_{G,H},$$

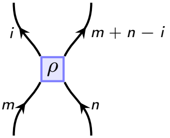
$$\alpha_{G,H} := \prod_{A \in \binom{G}{1}} (p^{\theta_{A,H}-\theta_{A,G}} - 1),$$

where the product is over the subsets A of degree 1.

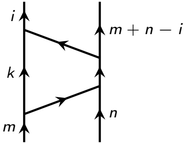
A two-parametric R-matrix from the Nichols algebra

Theorem

There is a two-parametric solution of the YBE $\rho(p,t)$, $t \in \mathbb{F}$ which corresponds the MOY diagrammatic equation



$$= t^i (tp^m; p)_{n-i} \sum_{k=0}^{\min(i,m)} \gamma_k$$



.

We define the matrix coefficients of ρ as follows:

$$\rho f_{E,F} = (-1)^{|E||F|} \sum r_{E,G;F,H} f_{F',E'}$$

where the summation runs over $G \subseteq E \setminus F$ and $H \subseteq F \setminus E$ such that $|H| \geq |G|$ and

$$E' := (E \setminus G) \sqcup H, \quad F' := (F \setminus H) \sqcup G.$$

We have the following formula for the coefficients of the matrix ρ :

$$r_{E,G;F,H} = t^{|F'|} (tp^{|E|}; p)_{|H|-|G|} \beta_{E,G;F,H},$$

where $\beta_{E,G;F,H}$ is given on the previous slide.

Discussion

Conjecture

Let $\dim(V) = N$*. Then the invariant of long knots* J_ρ *associated with the R-matrix* $\rho(p,t)$ *is of the form*

$$J_\rho = \mathrm{LG}^{(N)}(p,t)\,\mathrm{id}_{\Lambda_p(V)}$$

where $\mathrm{LG}^{(N)}(p,t)$ *is the Links–Gould invariant (1993) associated with a* 2^N*-dimensional representation of the super quantum group* $U_q(\mathfrak{gl}(N|1))$*.*

The calculations for a few examples of knots for the values $N = 2, 3, 4$ and a comparison with the results of De Wit (2001) are consistent with the Conjecture.

Future directions include:

- extend our computation of the R-matrix to derive the associated knot invariants in specific examples, without fixing the dimension N, and study their behaviour as functions of N;
- motivated by the V_n invariants, exploring deeper ties with the representation theory of $U_q(\mathfrak{gl}(N|1))$ and constructing coloured versions of the Links–Gould invariants.

These developments promise to deepen the algebraic and topological applications of Nichols algebras.