L-convex polyominoes and 201-avoiding ascent sequences

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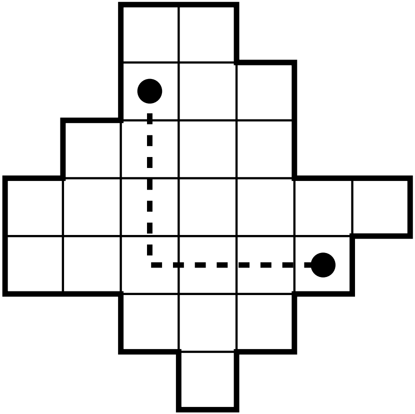
INTRODUCTION.

- L-convex polyominoes are defined by the property that any two cells may be joined by an L-shaped path.



GENERATING FUNCTIONS.

- The perimeter generating function is straightforward:

$$[x^n]P(x) = \frac{(2+\sqrt{2})^{n+1} - (2-\sqrt{2})^{n+1}}{4\sqrt{2}} \sim \frac{1+\sqrt{2}}{4}(2+\sqrt{2})^n.$$

- But the area g.f. is only known as a functional equation (Castiglione et al. 2007):

$$A(q) = 1 + \sum_{k\geq 0} \frac{q^{k+1} f_k(q)}{(1-q)^2(1-q^2)^2 \cdots (1-q^k)^2(1-q^{k+1})} = 1+q+\cdots \tag{1}$$

where

$$f_k(q) = 2f_{k-1}(q) - (1-q^k)^2 f_{k-2},$$

with initial conditions $f_0(q) = 1$, and $f_1(q) = 1 + 2q - q^2$.

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ASYMPTOTICS.

- The asymptotics were completely unknown.
- Together with Vaclav Kotesovec we used this functional equation to, eventually, generate 2000 series coefficients.
- We used these coefficients to obtain the asymptotics:

$$[q^n]A(q) \sim \frac{13\sqrt{2}}{768 \cdot n^{3/2}} \exp(\pi\sqrt{13n/6}).$$

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- A typical L-convex polyomino can be considered as a stack polyomino placed atop an upside-down stack polyomino.
- Stack polyominoes counted by area have generating function

$$S(q) = \sum s_n q^n = \sum_{n \geq 1} \frac{q^n}{(q)_{n-1}(q)_n},$$

where $(q)_n \equiv \prod_{k=1}^{n}(1-q^k)$, and

$$s_n \sim \frac{\exp(2\pi\sqrt{n/3})}{8 \cdot 3^{3/4} \cdot n^{5/4}}, \quad \text{(Auluck 1951).}$$

- Putting two objects together, one expects similar asymptotics:

$$q_n \sim \frac{\exp(a\pi n^{\beta})}{cn^{\delta}}. \tag{2}$$

- We expect exponents β and δ to be simple rationals, and a and c to be products of integers and small fractional powers.

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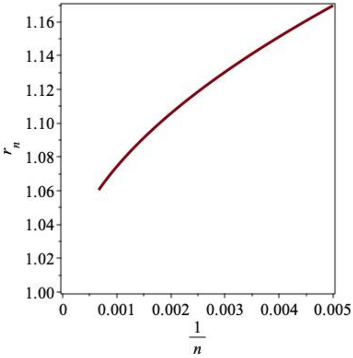
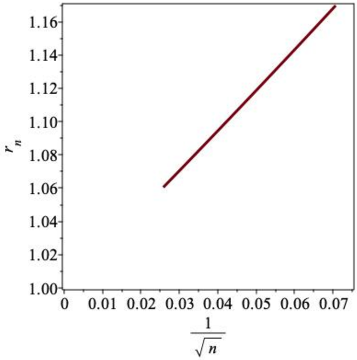
Let's look first at the ratio of successive coefficients, $r_n = q_n/q_{n-1}$. For a power-law singularity, the ratios will approach linearity when plotted against $1/n$, but for stretched exponentials like (2), $r_n = 1 + \frac{a\beta\pi}{n^{1-\beta}} + O(1/n)$.

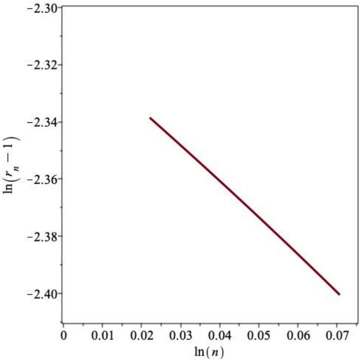


L-convex ratios plotted against $1/n$.

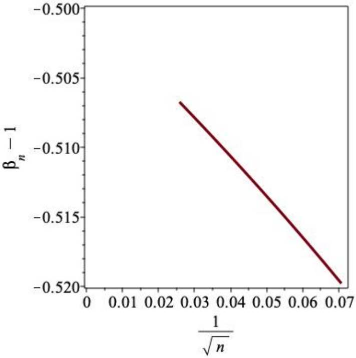


L-convex ratios plotted against $1/\sqrt{n}$.

This suggests $\beta = 1/2$, just as for stack polyominoes. We can refine this. We see that $r_n - 1 = a\beta\pi \cdot n^{\beta-1} + O(1/n)$, so a log-log plot of $r_n - 1$ against n should be linear with gradient $\beta - 1$.



Log-log plot of $r_n - 1$ against n.



Gradient of log-log plot.

- So we have our first conjecture, $\beta = 1/2$.
- From (2), define

$$\lambda_n \equiv \frac{\log(q_n)}{\pi\sqrt{n}} \sim a - \frac{\delta \log(n)}{\pi\sqrt{n}} - \frac{\log c}{\pi\sqrt{n}},$$

- we fit successive triples of coefficients $\lambda_{k-1}, \lambda_k, \lambda_{k+1}$, to the linear equation $\lambda_n = e_1 + e_2\frac{\log n}{\pi\sqrt{n}} + e_3\frac{1}{\pi\sqrt{n}}$, with k increasing until one runs out of known coefficients. Then e_1 should give an estimator of a, e_2 should give an estimator of $-\delta$ and e_3 should give an estimator of $-\log(c)$.

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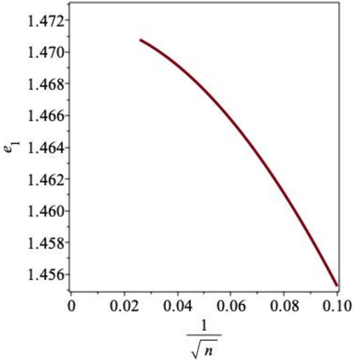
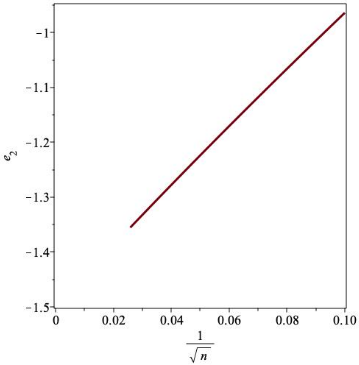
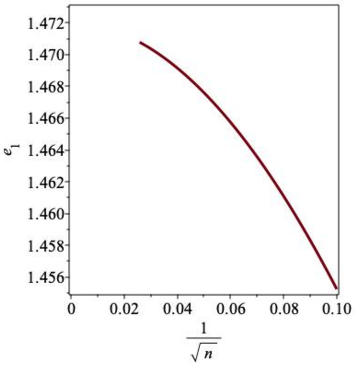


Plot of e_1 against $1/\sqrt{n}$.



Plot of e_2 against $1/\sqrt{n}$.

From these, we estimate $e_1 \approx 1.472$, and $e_2 \approx -1.5$. Comparing to the result for stack polyominoes, we expect e_1 to be the square-root of a rational number, and $e_1^2 = 2.1668$, so we conjecture that $e_1 = \sqrt{13/6}$, and $e_2 = -1.5$.



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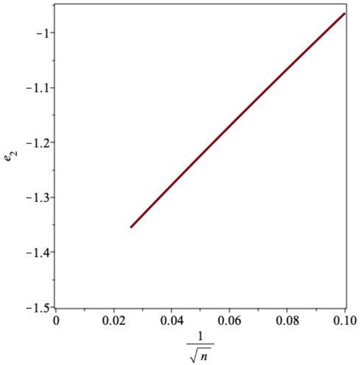

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So now we can conjecture that

$$q_n \sim \frac{\exp(\pi\sqrt{13n/6})}{c \cdot n^{3/2}}.$$

We reached this stage based on only 100 terms. In order to both gain more confidence in the conjectured form, and to calculate the constant, we needed more terms, and eventually generated 2000 terms. To calculate the constant, define

$$c_n \equiv \frac{\exp(\pi\sqrt{13n/6})}{q_n \cdot n^{3/2}},$$

and extrapolate the sequence c_n using any standard method. We used the Bulirsch-Stoer method, with parameter $1/2$.

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Because we had so many terms available, we considered only the terms $Q_n \equiv q_{n^2}$. Now we have for the conjectured asymptotic form

$$Q_n = q_{n^2} =\sim \frac{\exp(n\pi\sqrt{13/6})}{c \cdot n^3}.$$

We have 44 coefficients of the series Q_n available. Extrapolating gave $c \approx 0.023938510821419$. In analogy with the stack polyomino result, we expect this number to include a square root, cube root or fourth-root of a small integer. Here we got lucky. Dividing c by $\sqrt{2}$, gave a result that the Maple command *identify* reported as $13/768$. This implies $c = 13\sqrt{2}/768 = 0.023938510821419577\cdots$, agreeing to all quoted digits. The occurrence of 13 in this fraction, as well as in the exponent square-root, is reassuring, as is the factorisation of 768 as $3 \cdot 2^8$. Thus we conclude with the conjecture that the asymptotic form of the coefficients of L-convex polyominoes is

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Ascent sequences

An ascent sequence of length n is a sequence of non-negative integers $a_1, a_2, \ldots, a_n$, s.t.

- $a_1 = 0$,
- for each $i > 2$,

$$0 \leq a_i \leq 1 + asc(a_1, a_2, \ldots, a_{i-1}),$$

  where *asc* counts the number of ascents in the sequence to a_{i-1}. An ascent at position j occurs if $a_j < a_{j+1}$.
- There are 5 ascent sequences of length 3: $(0, 0, 0)$, $(0, 0, 1)$, $(0, 1, 0)$, $(0, 1, 1)$, and $(0, 1, 2)$.
- They are enumerated by the Fishburn numbers, and encode various combinatorial structures (like certain posets or permutations avoiding patterns).

Pattern-avoiding ascent sequences.

- Asking how many ascent sequences avoid a given pattern is related to set partition problems and stack-sorting problems.
- Example: The ascent sequence $(0, 1, 0, 2, 3, 1)$ has three occurrences of the pattern 001, namely 002, 003, and 001. It avoids the sequence 201.
- For patterns 001, 010, 011, and 012 the number of ascent sequences avoiding these patterns is just 2^{n-1}.
- For patterns 101 and 021the number is given by the n^{th} Catalan number.

AVOIDING THE PATTERN 201.

- The number of pattern-avoiding ascent sequences in this case was known for a.s. of size ≤ 27. (OEIS A202062).
- Using the *gfun* package of Maple, we immediately found the recurrence

$$\begin{aligned}
&(2n^2+n)u_n + (6n^2+45n+60)u_{n+1} - (34n^2+263n+480)u_{n+2} \\
&+ (44n^2+421n+984)u_{n+3} + (-20n^2-235n-684)u_{n+4} \\
&+ (2n^2+31n+120)u_{n+5} = 0, \\
&\text{with } u_0 = 1, u_1 = 1, u_2 = 2, u_3 = 5, u_4 = 15. \qquad (3)
\end{aligned}$$

We converted this to a third-order homogeneous ODE, using the gfun command *diffeqtohomdiffeq*, giving

$$P_3(x)f'''(x) + P_2(x)f''(x) + P_1(x)f'(x) + P_0(x)f(x) = 0,$$

where

$$P_3(x) = -2x^2(x^3 + 5x^2 - 8x + 1)(4x^4 - 30x^3 + 48x^2 - 36x + 15)(x-1)^2,$$

$$P_2(x) = -3x(x-1)(12x^8 - 30x^7 - 652x^6 + 2734x^5 - 4767x^4 + 4758x^3$$

$$- 2843x^2 + 870x - 85),$$

$$P_1(x) = -24x^9 + 30x^8 + 2754x^7 - 13278x^6 + 28884x^5 - 38106x^4$$

$$+ 32436x^3 - 16620x^2 + 4350x - 420,$$

$$P_0(x) = 30(3x-2)(3x^5 - 10x^4 + 19x^3 - 28x^2 + 24x - 7).$$

The smallest root of the cubic factor in $P_3(x)$ is $x = 0.1370633395\ldots$ and is the radius of convergence of the solution, and of course the reciprocal of the growth constant $\mu = 7.295896946\ldots$. Explicitly,

$$\mu = \frac{14}{3}\cos\left(\frac{\arccos(\frac{13}{14})}{3}\right) + \frac{8}{3}.$$

Use the Maple package *DEtools*. Convert the ODE to differential operator form using *de2diffop*, then factor into the direct sum of two differential operators using *DFactorLCLM*. One of these operators is first order and one is second order. The solution of the first order ODE is given by *dsolve*, and is the rational function

$$y_1(x) = \frac{x^4 + 26x^3 - 45x^2 + 18x + 1}{12(x-1)x^3}. \tag{4}$$

To solve the second-order ODE, we obtain a series solution, the first term of which is $O(x^{-3})$. So multiply by x^3 to obtain a regular power series, then use the *gfun* command *seriestoalgeq* to discover the cubic,

$$\begin{aligned}
&4(x-1)^3 y_2(x)^3 - \\
&3(x-1)(x^2-x+1)(x^6 - 235x^5 + 1430x^4 - 1695x^3 + 270x^2 + 229x + 1)y_2(x) \\
&+ x^{12} + 510x^{11} - 14631x^{10} + 80090x^9 - 218058x^8 + 316290x^7 - 253239x^6 \\
&+ 131562x^5 - 70998x^4 + 37950x^3 - 8955x^2 - 522x + 1 = 0.
\end{aligned} \tag{5}$$

Use Maple's *solve* command. Choose the appropriate solution.

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$$+ 131562x^5 - 70998x^4 + 37950x^3 - 8955x^2 - 522x + 1 - 24\sqrt{3x(x-1)(x^3+5x^2-8x+1)^7},$$

$$P_2 = (x^2 - x + 1)(x-1)^4(x^6 - 235x^5 + 1430x^4 - 1695x^3 + 270x^2 + 229x + 1), \text{ and}$$

$$P_3 = (3^{5/6}i + 3^{1/3}), \; P_4 = (-3^{5/6}i + 3^{1/3}),$$

then

$$y_2(x) = \frac{-3^{2/3}\left[P_4\left(-P_1\cdot(x-1)^6\right)^{2/3} + P_2\cdot P_3\right]}{12(-P_1\cdot(x-1)^6)^{1/3}(x-1)^3}. \tag{6}$$

The solution to the original ODE is then

$$y(x) = \frac{y_2(x)}{12x^3} - y_1(x) = 1 + x + 2x^2 + 5x^3 + 15x^4 + \cdots.$$

THE AMPLITUDE

We next obtained the first 5000 terms in only a few minutes by expanding this solution. We know that the coefficients behave asymptotically as

$$u(n) \sim C\frac{\mu^n}{n^{9/2}}. \tag{7}$$

Equivalently, the generating function behaves as

$$U(x) = \sum u(n)x^n = A(1-\mu\cdot x)^{7/2},$$

where $C = A/\Gamma(-7/2) = 105A/(16\sqrt{\pi})$. We estimate C by assuming a pure power law, so that

$$\frac{u(n)\cdot n^{9/2}}{\mu^n} = C(1+\sum_{k\geq 1} a_k/n^k).$$

We calculated the first twenty coefficients of this expansion, which gave $C = 13.4299960869\cdots$ with 74-digit accuracy.

- This time, *identify* didn't help.
- In favourable cases such constants are a product of rational numbers and square roots of small integers, sometimes with integer or half-integer powers of π.
- These powers of π usually arise from the conversion factor in going from the g.f. amplitude A to the coefficient amplitude C. We might expect the amplitude A to be simpler than C.
- And, to eliminate square-roots, we will try and identify A^2.
- We seek the minimal polynomial with root A^2, using the command *MinimalPolynomial*.
- In fact, one only requires 20 digit accuracy in the estimate of A^2 to find the minimal polynomial, $A^6 - 1369A^4 + 17839A^2 + 1$, which gives

$$C = \frac{35}{16}\left(\frac{4107}{\pi} - \frac{84}{\pi}\sqrt{9289}\cos\left(\frac{\pi}{3} + \frac{1}{3}\arccos\left[\frac{255709\sqrt{9289}}{24653006}\right]\right)\right)$$

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CONCLUSION

- We have shown how experimental mathematics can be used to conjecture exact asymptotics, in the case of L-convex polyominoes, and to conjecture an exact solution, in the case of 201-avoiding ascent sequences.
- We hope that the methods will be more widely applied, as there are many outstanding combinatorial problems that lend themselves to such an approach.