Convergence rate of critical mean-field $O(N)$ magnetization distribution

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Joint work with:

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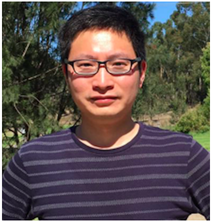

Eric Zhou

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- But what if $X_1, X_2, \ldots, X_n$ are **dependent**?

Curie-Weiss model - limit theorems

- Random spin configuration $X \in \{-1, 1\}^n$ with distribution:

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If $0 \le \beta < 1$ *then*

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$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\sqrt{1-\beta}\, n^{-1/2}\, \mathcal{M}_n \leq x\right) - \mathbb{P}(Z \leq x) \right| \leq C\, n^{-1/2}$$

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- What about higher-dimensional spin models?

Mean-field $O(N)$ model

- Let $\mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$
- Random spin configuration $X \in (\mathbb{S}^{N-1})^n$ with distribution:

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Let $0 \le \beta < N$. *If* $\mathcal{W}_n := \sqrt{N - \beta}\, n^{-1/2} \mathcal{M}_n$ *and* $Z \sim \mathcal{N}(0, I_N)$ *then*

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- Proof via multivariate normal extension of Stein's method due to Chatterjee-Meckes (2008)

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- Proof via an extension of multivariate nonnormal Stein's method

Stein's method (Barbour's generator version)

- Markov process in $\mathbb{R}^N$ satisfying

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- Bounding derivatives of $P_t h$ gives bounds on $\mathcal{A}f_h$

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$$d(\mathcal{L}(W), \mu) \leq C\left(\alpha\mathbb{E}|\delta|^3 \max(|\log\delta|, 1) + \mathbb{E}|R_1| + \mathbb{E}|R_2|\right)$$

- For critical $O(N)$ model, $V(x) = |x|^4$ is not strictly convex
- We weaken the assumption on V, so it is convex outside a ball
- Covers critical mean-field $O(N)$ model

Theorem (G.-Zhou-Perez (2025+))

Let $\mu \propto e^{-V(x)}\mathrm{d}x$ *with* $V \in \mathcal{C}^4(\mathbb{R}^N;\mathbb{R})$ ***eventually*** *convex (+ tech.)*
Let $(W, W') \stackrel{d}{=} (W', W)$*, let* $\delta := W' - W$*, fix* α *and set*

$$R_1 := \alpha\mathbb{E}[\delta|W] + \nabla V(W)$$
$$R_2 := \alpha\mathbb{E}[\delta\delta^T|W] - 2I$$

Then there exists $C > 0$ *such that*

$$d(\mathcal{L}(W), \mu) \leq C\left(\alpha\mathbb{E}|\delta|^3 \max(|\log\delta|, 1) + \mathbb{E}|R_1| + \mathbb{E}|R_2|\right)$$

- For critical $O(N)$ model, $V(x) = |x|^4$ is not strictly convex
- We weaken the assumption on V, so it is convex outside a ball
- Covers critical mean-field $O(N)$ model
  - Let W' be one-step update of W via single-spin Glauber process

Some details

- Fang-Shao-Xu bound derivatives of $P_t h(x)$ via Malliavin calculus

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- We use the Bismut-Elworthy-Li formula to differentiate $P_t h(x)$
  - This then requires bounding spatial derivatives of X_t^x

Happy Birthday Tony!