Convex Hulls of Dyck Paths

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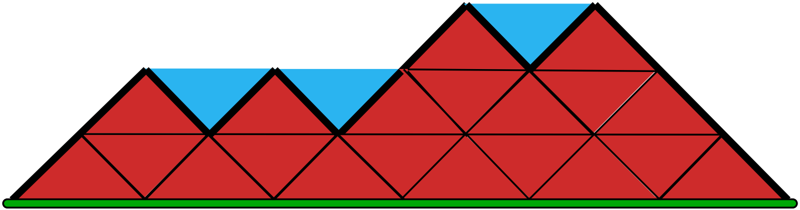
Guttmann 2025: 80 and (still) counting

Melbourne, June 30 - July 1

The Enumeration Problem

Convex Hulls of Dyck Paths

Enumeration of paths with respect to area below path and area between path and its convex hull (“water capacity”)

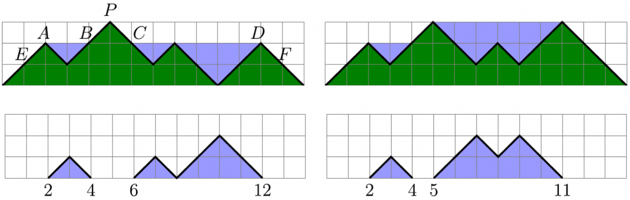


Half-length $n = 6$, height $h = 3$, water capacity $w = 3$, and area $a = 20$

Water Capacity of Dyck Paths

A. Blecher, C. Brennan, A. Knopfmacher, *Adv. Appl. Math.* **112** 101945 (2020)

- Idea: view water cells as Dyck paths



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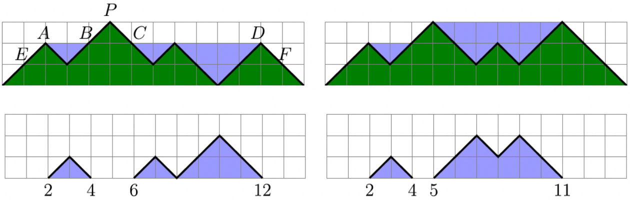

- The generating function for the capacity of Dyck paths of half-length n is given by

$$-\frac{(1+u)^2}{(1-u)u}\left(\frac{u}{1-u}-(1+u)\sum_{r=1}^{\infty}\frac{u^r}{1-u^r}+2(1-u)\sum_{r=1}^{\infty}\frac{ru^{2r}}{(1-u^r)^2}\right)$$

where $z = u/(1+u)^2$ is the half-length generating variable

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- The exact capacity of Dyck paths of half-length n is given by

$$\sum_{r=1}^{n+1} d_0(r)\binom{2n+2}{n-r+1} - 4\sum_{r=1}^{n+1}\left(rd_0(r)-d_1(r)\right)\frac{r}{n+r+1}\binom{2n+1}{n-r+1} - 4^n$$

with the sum-of-divisors function $d_s(r) = \sum_{d|r} d^s$

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- The average water capacity of Dyck paths of half-length n, as $n\to\infty$ is

$$\left(\frac{\pi^{5/2}}{3}-3\sqrt{\pi}\right)n^{3/2} - n + \left(-\frac{73\sqrt{\pi}}{24}+\frac{3\pi^{5/2}}{8}\right)\sqrt{n} - \frac{5}{6} + O(n^{-1/2})$$

Motivation

- There is non-trivial asymptotics $n^{3/2}$, in line with the expected average area growth
- The existing computation leads to the first moment of the water capacity, with unit area weights
- The competition between the two areas in a weighted model will be intriguing and novel
- In principle, the tools to do the two-variable area computation exist

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Let's do it!

The Ingredients

A. L. Owczarek, T. Prellberg, *Australas. J. Combinat.* **54** 13 (2012)

The generating function for height-restricted Dyck paths

$$D_h(x,q) = \sum_{n,d} d_{h,n,d} x^n q^d$$

for the number of paths $d_{h,n,d}$ with height bounded by h, half-length n and number of diamonds d below the path (so that area $a = 2d + n$) is given by

$$D_h(x,q) = \frac{Q_h(qx,q)}{Q_h(x,q)}$$

where

$$Q_k(x,q) = \sum_{m=0}^{\infty} (-x)^m q^{m(m-1)} \begin{bmatrix} l-m \\ m \end{bmatrix}_q$$

The Concatenation Argument

- Take triangle $x^h y^h q^{h^2}$
- At height k, insert inverted Dyck path of height at most k:

$$D_k(q^{2k} xp/q, p^2/q^2)$$

- Two insertions at heights $k < h$ and one insertion at peak height h:

$$G_h(x, p, q) = x^h y^h q^{h^2} \left[\prod_{k=1}^{h-1} D_k(q^{2k} xp/q, p^2/q^2) \right]^2 D_h(q^{2h} xp/q, p^2/q^2)$$

- Sum over all heights:

$$G(x, y, p, q) = 1 + \sum_{h=1}^{\infty} y^h G_h(x, p, q)$$

The Generating Function

The generating function for Dyck paths with half-length n, height h, water capacity w, and area a below the path,

$$G(x,y,p,q) = \sum_{n,h,w,a} c_{n,h,w,a} x^n y^h p^w q^a$$

is given by

$$G(x,y,p,q) = \sum_{h=0}^{\infty} \frac{x^h y^h q^{h^2} \prod_{k=1}^{h} \frac{Q_k(q^{2k-3}p^3x,\, p^2/q^2)^2}{Q_k(q^{2k-3}px,\, p^2/q^2)^2}}{Q_h(q^{2h-3}p^3x,\, p^2/q^2)\, Q_{h+1}(q^{2h-1}px,\, p^2/q^2)}$$

with the polynomials

$$Q_k(x,q) = \sum_{m=0}^{\infty} (-x)^m q^{m(m-1)} \begin{bmatrix} l-m \\ m \end{bmatrix}_q$$

A $p \leftrightarrow q$ near-symmetry

After some gentle massaging and using $\begin{bmatrix} n \\ m \end{bmatrix}_{1/q} = q^{-k(n-k)} \begin{bmatrix} n \\ m \end{bmatrix}_q$ we find

$$G(x,y,p,q) = \sum_{k=0}^{\infty} x^h y^h q^{h^2} \frac{\left[\prod_{k=0}^{h-1} Q_k(p^{2k-3}qp^2x, q^2/p^2)\right]^2 Q_h(p^{2h-3}qp^2x, q^2/p^2)}{\left[\prod_{k=0}^{h} Q_k(p^{2k-3}qx, q^2/p^2)\right]^2 Q_{h+1}(p^{2h-1}qx, q^2/p^2)}$$

but also

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The denominators are polynomials in p, q, x independent of y and symmetric under exchange of p and q

The locus of singularities $x_c(p, q)$ is symmetric under exchange of p and q

Special Cases

Known scenarios:

- $p = 1$, $q = 1$: standard Dyck path enumeration, Calatan generating function
- $p = 1$, general q: Dyck paths enumerated by area, q-Airy
  N. Haug, T. Prellberg, J. Math. Phys. **56** 043301 (2015)

$$x_c(q) \sim \frac{1}{4} + C(1-q)^{2/3} \,, \quad q \to 1$$

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New scenarios:

- $p = q$: Dyck paths enumerated by convex hull area
- $p \neq q$: Dyck paths enumerated by area and water capacity

Convex Hull Enumeration ($p = q$)

The generating function

$$G(x,y,q,q) = \sum_{h=0}^{\infty} \frac{x^h y^h q^{h^2} \prod_{k=1}^{h} \frac{Q_k(q^{2k}x,1)^2}{Q_k(q^{2k-2}x,1)^2}}{Q_h(q^{2h}x,1)Q_{h+1}(q^{2h}x,1)}$$

simplifies to

$$G(x,y,q,q) = \frac{1}{\sqrt{x}} \sum_{h=0}^{\infty} \frac{y^h \prod_{k=0}^{h} \frac{U_k(\frac{q^{-k}}{2\sqrt{x}})^2}{U_k(\frac{q^{1-k}}{2\sqrt{x}})^2}}{U_h(\frac{q^{-h}}{2\sqrt{x}})U_{h+1}(\frac{q^{-h}}{2\sqrt{x}})}$$

where $U_k(t)$ are the Chebyshev polynomials of the second kind

Singularities

- We find simple poles at

$$x_k(q) = \frac{1}{4q^{2k}\cos(\pi/(k+2))^2}$$

- The poles $x_k(q)$ and $x_{k+1}(q)$ coalesce at (q_k, x_k) given by

$$q_k = \frac{\cos(\pi/(k+2))}{\cos(\pi/(k+3))}\,, \quad x_k = \frac{\cos(\pi/(k+3))^{2k}}{4\cos(\pi/(k+2))^{2k+2}}$$

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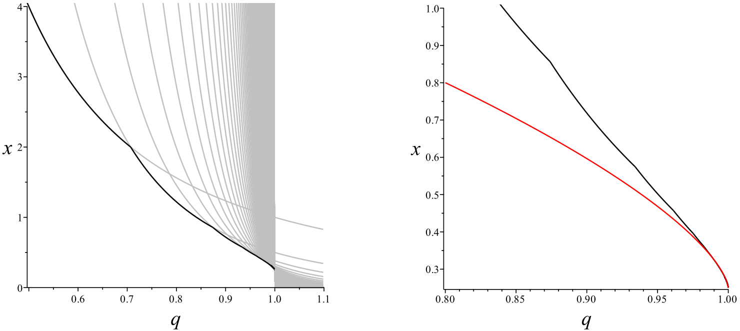
- Asymptotically, for k large,
  $q_k \sim 1 - \pi^2/k^3\ , \quad x_k \sim 1/4 + 3\pi^2/(4k^2)$ so that

$$x_c(q) \sim \frac{1}{4} + \frac{3\pi^{2/3}}{4}(1-q)^{2/3}\ , \quad q \to 1$$

Singularity Structure

We find a sequence of first-order phase transitions accumulating at $(q, x) = (1, 1/4)$

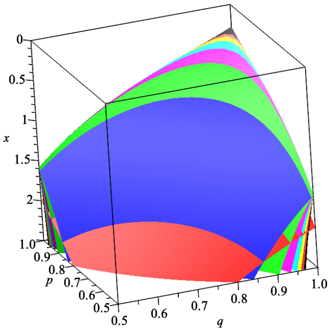


The red line is

$$x = \frac{1}{4} + \frac{3\pi^{2/3}}{4}(1-q)^{2/3}$$

General Case

The scenario of a sequence of phase transitions extends to $p \neq q$



Each surface can be computed and the intersection lines determined, e.g.

$$x_1 = \frac{1}{pq}\,, \quad x_2 = \frac{1}{pq(p^2+q^2)} \quad : \quad p^2+q^2=1$$

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- Singularity $x_k(p, q)$ only appears in G_h for $h \geq k$
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Some further analysis gives

- Phase corresponding to $x_k(p, q)$ is dominated by configurations with height h

Water or Mustard?

It's 8 degrees and raining, but Aleks Owczarek has braved the elements and driven eight kilometres to Leo's supermarket in Kew from his home in Northcote.

He emerges with two jars of mustard. Not just any mustard, but Marcel Recorbet-branded goodies, imported from the French Alps.

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The Age, June 26 2025

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Mustard on Sandwiches

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- Solvable model with two competing areas
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- I started playing with these lattice models in 1991 as Tony's postdoc

Thank you, Tony!