Vintage Tony

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2 September 1992

Dear Mr Warnaar,

Congratulations on winning ANZIAM's T.M. Cherry Prize. I have been tasked by the Australian Mathematical Society with transferring the prize money into your account. Please send me the following at your earliest convenience: bank account name, account number and, most importantly, your password.

Yours sincerely,
Prof. A.J. Guttmann


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A modern rendition of Tony's #1 paper on MathSciNet

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition and $\alpha = (\alpha_1, \ldots, \alpha_s)$ a composition such that $|\lambda| := \lambda_1 + \cdots + \lambda_r = \alpha_1 + \cdots + \alpha_s =: |\alpha|$. Then a semistandard Young tableau of shape λ and content α is a filling of the Young diagram of λ such that rows are weakly increasing from left to right, columns are strictly increasing from top to bottom, and exactly α_i boxes contain the number i. For example,

$$\begin{array}{|c|c|c|c|}\hline 2 & 3 & 4 & 6 \\ \hline 4 & 4 & 6 \\ \cline{1-3} 5 & 6 \\ \cline{1-2}\end{array}$$

is a semistandard Young tableau of shape $(4, 3, 2)$ and content $(0, 1, 1, 3, 1, 3)$.

The weight x^T of a semistandard Young tableau T is the monomial

$$x^T = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_s^{\alpha_s},$$

so that the weight of the tableau in the example is $x_2 x_3 x_4^3 x_5 x_6^3$.

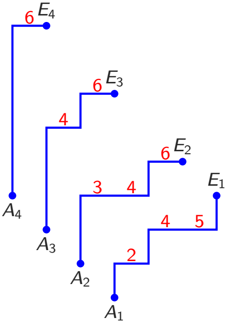
There is a correspondence between semistandard Young tableaux and configurations of vicious walkers.

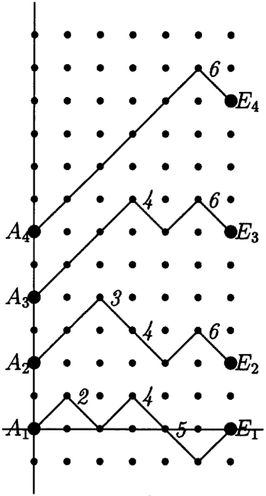
Fix a positive integer n (possibly $n = \infty$) and let T be a semistandard Young tableaux of shape λ such that $n \geqslant \lambda'_1$.

If T consists of k columns (i.e., $\lambda_1 = k$), draw k paths in the xy-plane traversed by vicious walkers, such that walker i starts at $A_i = (1-i, i-1)$ and finishes at $E_i = (\lambda'_i - i + 1, n - \lambda'_i + i - 1)$. Here walker i takes λ'_i unit steps to the right and $n - \lambda'_i$ unit steps up, where the numbers in the ith column of T encode at which steps the walker goes right.
For example, if $n = 6$,

2	3	4	6
4	4	6	
5	6		

$\longleftrightarrow$





a. A star

2	3	4	6
4	4	6	
5	6		

b. A tableau

Figure 3. (*a*) A typical star configuration and (*b*) associated tableau.

The weight of a path $A = (a, b) \to E = (a + r, b + s)$ of length $r + s$ taken by a vicious walker is a monomial $x_{i_1} x_{i_2} \cdots x_{i_r}$, where $i_1, \ldots, i_r$ are the labels of the horizontal steps. The generating function of configurations of single-walker paths from $A = (a, b) \to E = (a + r, b + s)$ is

$$e_r(x_1, \ldots, x_n) = \prod_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n} x_{i_1} \ldots x_{i_r},$$

where e_r is the rth elementary symmetric function.

The weight of a configuration of vicious walkers is the product of the individual weights. This makes the map between Young tableaux and configurations of vicious walkers weight-preserving. By the Lindström–Gessel–Viennot lemma, the generating function of k walkers such that $A_i = (1 - i, i - 1) \to E_i = (\lambda'_i - i + 1, n - \lambda'_i + i - 1)$ for $1 \leqslant i \leqslant k$ is given by

$$\det_{1 \leqslant i,j \leqslant k} \left(e_{\lambda'_i - i + j}(x_1, \ldots, x_n) \right).$$

This implies the (dual) Jacobi–Trudy identity

$$s_\lambda(x_1, \ldots, x_n) := \sum_{T \in \mathrm{SSYT}(\lambda, \cdot)} x^T = \det_{1 \leqslant i,j \leqslant k} \big(e_{\lambda'_i - i + j}(x_1, \ldots, x_n)\big),$$

where s_λ is the Schur function.

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$$s_\lambda(x_1,\dots,x_n) := \sum_{T\in\mathrm{SSYT}(\lambda,\cdot)} x^T = \det_{1\leqslant i,j\leqslant k}\big(e_{\lambda'_i-i+j}(x_1,\dots,x_n)\big),$$

where s_λ is the Schur function.

An important generalisation of the Schur function is the Hall–Littlewood polynomial:

$$P_\lambda(x_1,\dots,x_n;t) = \sum_{T\in\mathrm{SSYT}(\lambda,\cdot)} \Psi_T(t)\,x^T$$

where $\Psi_T(t)$ is a simple, combinatorially defined, polynomial of the form

$$(1-t)^{n_1}(1-t^2)^{n_2}\cdots(1-t^{\lambda'_1})^{n_{\lambda'_1}}$$

such that, for $T\in\mathrm{SSYT}(\lambda,\alpha)$,

$$\Psi_T(1) = \begin{cases} 1 & \text{if } \mathrm{sort}(\alpha)=\lambda, \\ 0 & \text{otherwise.}\end{cases}$$

Hence $P_\lambda(0)=s_\lambda$ and $P_\lambda(1)=m_\lambda$, the monomial symmetric function.

Can we use vicious walkers to find a Jacobi–Trudi-like formula for the $P_\lambda(t)$?

Each configuration of vicious walkers now has a weight $\Psi_T(t)x^T$, where we know from before how to distribute x^T among the individual walkers. But what about $\Psi_T(t)$?

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Conjecture: Affine Jacobi–Trudi identity

We have

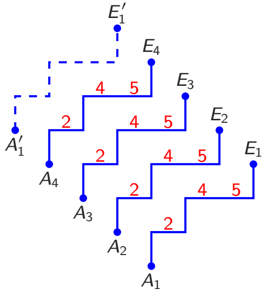
$$P_{(k^r)}(x_1,\dots,x_n;t) = \sum_{\substack{y_1,\dots,y_k\in\mathbb{Z}\\ y_1+\cdots+y_k=0}} \det_{1\leqslant i,j\leqslant k}\Big(t^{k\binom{y_i}{2}+iy_i}e_{r-i+j-ky_i}(x_1,\dots,x_n)\Big).$$

Note that for the rectangular partition (k^r), all k vicious walkers take exactly r steps to the right and $n-r$ steps up.

According to Gessel and Krattenthaler (*Cylindric partitions*, 1997),

$$\sum_{\substack{y_1,\dots,y_k\in\mathbb{Z}\\ y_1+\cdots+y_k=0}} \det_{1\leqslant i,j\leqslant k}\Big(e_{r-i+j-ky_i}(x_1,\dots,x_n)\Big).$$

is the generating function of configurations of k vicious walkers $(P_1,\dots,P_k)$ where P_i starts at $A_i=(1-i,i-1)$ and ends at $E_i=(r-i+1,n-r+i-1)$, such that $(P_1,\dots,P_k,P_1')$ is also a configuration of vicious walkers. Here P_1' is P_1 translated by $(-k,k)$:



Hence

$$\sum_{\substack{y_1,\ldots,y_k\in\mathbb{Z}\\ y_1+\cdots+y_k=0}} \det_{1\leqslant i,j\leqslant k}\Big(e_{r-i+j-ky_i}(x_1,\ldots,x_n)\Big) = e_r\left(x_1^k,\ldots,x_n^k\right)$$

$$= m_{(k^r)}(x_1,\ldots,x_n)$$

as it should.

In his work on vicious walkers, Tony also considered walkers in the presence of one or two walls.

Let n be fixed, and let T be a symplectic tableau with entries at most $2n+1$. The weight $\boldsymbol{x}^T$ of the symplectic tableau T is defined by

$$\boldsymbol{x}^T = \prod_{l=1}^{n} x_l^{|\{T_{i,j}=2l-1\}|-|\{T_{i,j}=2l\}|} \tag{4.10}$$

where T_{ij} denotes the entry in cell (i, j) of T. Note that entries $2n+1$ do not contribute to the weight. Given this terminology, the (even) symplectic character $sp_\lambda(x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1})$ is also given by (see [42, theorem 4.2], [48, theorem 2.3]),

$$sp_\lambda(x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}) = \sum_T \boldsymbol{x}^T \tag{4.11}$$

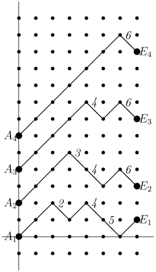
where the sum is over all symplectic tableaux T of shape λ with entries $\leqslant 2n$, whereas the odd symplectic character $sp_\lambda(x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}, 1)$ is also given by (see [42, theorem 4.2]),

$$sp_\lambda(x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}, 1) = \sum_T \boldsymbol{x}^T \tag{4.12}$$

where the sum is over all symplectic tableaux T of shape λ with entries $\leqslant 2n+1$.

The formula for symplectic characters needed here is (see [10, (3.27)], [39, proposition 3.2], [48, theorem 4.5(1)])

$$\begin{aligned} sp_\lambda(\underbrace{1, 1, \ldots, 1}_{m}) &= \prod_{1\leqslant i<j\leqslant m} \frac{\lambda_i - i - \lambda_j + j}{j-i} \prod_{1\leqslant i\leqslant j\leqslant m} \frac{\lambda_i + \lambda_j + m - i - j + 2}{m+2-i-j} \\ &= \prod_{\rho\in\lambda} \frac{m - f_\rho}{h_\rho} \end{aligned} \tag{4.13}$$



$1 \leq$	2	3	4	6
$3 \leq$	4	4	6	
$5 \leq$	5	6		

Fix an integer n and let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition such that $r \leqslant n$, and $\alpha = (\alpha_1, \ldots, \alpha_{2n})$ a composition such that $|\lambda| = |\alpha|$. Then a semistandard Young tableau of shape λ and content α is said to be symplectic if all the entries in row i are at least $2i - 1$.

2	3	4	6
4	4	6	
5	6		

is a symplectic tableau of shape $(4, 3, 2)$ and content $(0, 1, 1, 3, 1, 3)$ for $n = 3$.

The weight x^T of a symplectic tableau T is the monomial

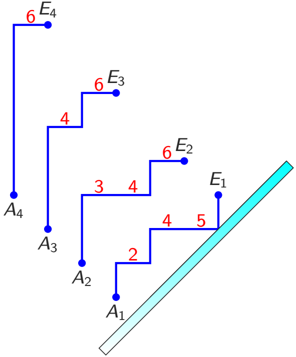
$$x^T = x_1^{\alpha_1 - \alpha_2} x_2^{\alpha_3 - \alpha_4} \cdots x_n^{\alpha_{2n-1} - \alpha_{2n}},$$

so that the weight of the tableau in the example is $x_1^{-1} x_2^{-2} x_3^{-2}$.

As you can read in Tony's beautiful paper, there is a bijection between the set of symplectic Young tableaux of shape λ such that $\lambda'_1 \leqslant k$ and configurations of k vicious walkers with starting points $A_i = (1-i, i-1)$ and end points $E_i = (\lambda'_i - i + 1, 2n - \lambda'_i + i - 1)$ such that no path crosses the wall $y = x - 1$. Here walker i takes λ'_i unit steps to the right and $2n - \lambda'_i$ unit steps up, where the numbers in the ith column of T encode at which steps the walker goes right.
For example, if $n = 3$,

2	3	4	6
4	4	6	
5	6		

$\longleftrightarrow$



This implies a Jacobi–Trudy identity for symplectic Schur functions

$$\begin{aligned}
\mathrm{sp}_{2n,\lambda}(x_1,\ldots,x_n) &:= \sum_{T \in \mathrm{SympYT}(\lambda,\cdot)} x^T \\
&= \det_{1\leqslant i,j\leqslant k}\Big(e_{\lambda'_i-i+j}\big(x_1^{\pm},\ldots,x_n^{\pm}\big) - e_{\lambda'_i-i-j}\big(x_1^{\pm},\ldots,x_n^{\pm}\big)\Big).
\end{aligned}$$

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The symplectic Schur function also has a Hall–Littlewood generalisation, denoted as $P_\lambda^{\mathrm{C}_n}(x_1,\dots,x_n;t)$. Can we use vicious walkers in the presence of a wall to find a Jacobi–Trudi-like formula for these?

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Conjecture: Symplectic affine Jacobi–Trudi identity

For k a nonnegative integer, let $K := 2k+2$. Then

$$\begin{aligned}P_{(k^n)}^{\mathrm{C}_n}(x_1,\dots,x_n;t) = \sum_{y_1,\dots,y_k\in\mathbb{Z}} \det_{1\leqslant i,j\leqslant k}\Big(& t^{\frac{1}{2}Ky_i^2-jy_i}\, e_{n-i+j-Ky_i}\big(x_1^{\pm},\dots,x_n^{\pm}\big) \\ &- t^{\frac{1}{2}Ky_i^2+jy_i}\, e_{n-i-j-Ky_i}\big(x_1^{\pm},\dots,x_n^{\pm}\big)\Big).\end{aligned}$$

From Tony et al.:

2. THE NUMBER OF VICIOUS WALKER CONFIGURATIONS WITH ARBITRARY FIXED STARTING AND END POINTS

The Lindström–Gessel–Viennot determinant[17, 18] in the case of the presence of two walls yields the following result. It appears already in ref. 14, Eq. (13), in an equivalent form.

Theorem 1. Let $0 \leqslant a_1 < a_2 < \cdots < a_p \leqslant h$, all a_i's of the same parity, and $0 \leqslant e_1 < e_2 < \cdots < e_p \leqslant h$, all e_i's of the same parity, such that $a_i + e_i \equiv m \pmod 2$, $i = 1, 2, \ldots, p$. The number of vicious walkers with p branches of length m, the ith branch running from $A_i = (0, a_i)$ to $E_i = (m, e_i)$, $i = 1, 2, \ldots, p$, which do not go below the x-axis nor above the line $y = h$, is given by

$$\det_{1 \leqslant s, t \leqslant p} \left(\sum_{k=-\infty}^{\infty} \left(\binom{m}{\frac{m+e_t-a_s}{2} + k(h+2)} - \binom{m}{\frac{m+e_t+a_s}{2} + k(h+2)+1} \right) \right). \tag{2.1}$$

What is it all good for?

What is it all good for?

Theorem: Littlewood identity (Rains & W, 2021)

For k a nonnegative integer,

$$P^{\mathrm{C}_n}_{(k^n)}(x_1,\dots,x_n;t)=(x_1\cdots x_n)^{-k}\sum_{\lambda\subseteq(k^n)}P_{2\lambda}(x_1,\dots,x_n;t).$$

Combined with the conjecture this implies

$$\begin{aligned}
&(x_1\cdots x_n)^{-k}\sum_{\lambda\subseteq(k^n)}P_{2\lambda}(x_1,\dots,x_n;t)\\
&\quad=\sum_{y_1,\dots,y_k\in\mathbb{Z}}\det_{1\leqslant i,j\leqslant k}\Big(t^{\frac{1}{2}Ky_i^2-jy_i}\,e_{n-Ky_i-i+j}\big(x_1^{\pm},\dots,x_n^{\pm}\big)\\
&\qquad\qquad-t^{\frac{1}{2}Ky_i^2+jy_i}\,e_{n-Ky_i-i-j}\big(x_1^{\pm},\dots,x_n^{\pm}\big)\Big).
\end{aligned}$$

Specialising $x_i = q^{i-1/2}$ for $1 \leqslant i \leqslant n$ and using

$$e_{n-r}\big(q^{n-1/2}, q^{n-3/2}, \dots, q^{1/2-n}\big) = q^{\frac{1}{2}r^2-\frac{1}{2}n^2} \begin{bmatrix} 2n \\ n-r \end{bmatrix}_q,$$

yields

$$\begin{aligned}\sum_{\lambda \subseteq (k^n)} & q^{|\lambda|} P_{2\lambda}\big(1, q, \dots, q^{n-1}; t\big) \\ &= \sum_{y_1,\dots,y_k \in \mathbb{Z}} \det_{1 \leqslant i,j \leqslant k} \Bigg(t^{\frac{1}{2}Ky_i^2 - jy_i} q^{\frac{1}{2}(Ky_i+i-j)^2} \begin{bmatrix} 2n \\ n-Ky_i-i+j \end{bmatrix}_q \\ &\qquad\qquad - t^{\frac{1}{2}Ky_i^2 + jy_i} q^{\frac{1}{2}(Ky_i+i+j)^2} \begin{bmatrix} 2n \\ n-Ky_i-i-j \end{bmatrix}_q \Bigg).\end{aligned}$$

In the large-n limit, the right-hand side may be written in product form by the $\mathrm{C}_k^{(1)}$ Macdonald identity.

Let $\theta(z;p) = (z;p)_\infty (q/z;p)_\infty$ be a Jacobi theta function.

q,t-Rogers–Ramanujan identity (W, 2025)

For k a nonnegative integer,

$$\sum_{\substack{\lambda \\ \lambda_1 \leqslant k}} q^{|\lambda|} P_{2\lambda}(1,q,q^2,\dots;t)$$
$$= \frac{(tq^{2k+2};tq^{2k+2})_\infty^k}{(q;q)_\infty^k} \prod_{i=1}^{k} \theta(q^{2i};tq^{2k+2}) \prod_{1\leqslant i<j\leqslant k} \theta(q^{j-i},q^{i+j};tq^{2k+2}).$$

For $k=1$ and $t=q$ this is the first Rogers–Ramanujan identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{(q^2,q^3,q^5;q^5)_\infty}{(q;q)_\infty}.$$

More generally, for $t = q^{2n-1}$ we may use the theta function identity

$$\frac{(p;p)_\infty^k}{(q;q)_\infty^k}\prod_{i=1}^{k}\theta\big(q^{2i};p\big)\prod_{1\leqslant i<j\leqslant k}\theta\big(q^{j-i},q^{i+j};p\big)$$

$$=\frac{(p;p)_\infty^n}{(q;q)_\infty^n}\prod_{i=1}^{n}\theta\big(q^{i+k};p\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j-1};p\big)$$

for $p = q^{2k+2n+1}$ to obtain:

$\mathrm{A}_{2n}^{(2)}$ Rogers–Ramanujan identity (Griffin, Ono, W, 2016)

For k, n positive integers, let $p = q^{2k+2n+1}$. Then

$$\sum_{\substack{\lambda \\ \lambda_1\leqslant k}} q^{|\lambda|}P_{2\lambda}\big(1,q,q^2,\dots;q^{2n-1}\big)$$

$$=\frac{(p;p)_\infty^n}{(q;q)_\infty^n}\prod_{i=1}^{n}\theta\big(q^{i+k};p\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j-1};p\big).$$

Conjecturally, the GOW theorem is the generating function of partitions in which parts can take n distinct colours such that the frequencies of parts of a given colour satisfy some simple restrictions.

For simplicity, let $n = 3$ with colours red, blue and green ordered as *red* $>$ *blue* $>$ *green*.

An example of a partition for $n = 3$ is $\lambda = (5, 3, 3, 2, 2, 2, 1)$ or

$$
\begin{array}{ll}
f_5 = 1 & \bullet\ \bullet\ \bullet\ \bullet\ \bullet \\
f_3 = 1 & \bullet\ \bullet\ \bullet \\
f_3 = 1 & \bullet\ \bullet\ \bullet \\
f_2 = 2 & \left\{ \begin{array}{l} \bullet\ \bullet \\ \bullet\ \bullet \end{array} \right. \\
f_2 = 1 & \bullet\ \bullet \\
f_1 = 1 & \bullet
\end{array}
$$

An n-coloured partition is an $\mathrm{A}_{2n}^{(2)}$ partition for the standard module $L(k\Lambda_n)$ if for *any* path on the associated frequency array the sum of the entries along the path does not exceed k.

Two examples of paths on the frequency array for $n = 3$ are shown below:

$$
\begin{array}{cccccccccccc}
k & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & \dots \\
 & 0 & & f_2 & & f_4 & & f_6 & & f_8 & & \\
0 & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & \dots \\
 & 0 & & f_2 & & f_4 & & f_6 & & f_8 & & \\
0 & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & \dots \\
 & 0 & & f_2 & & f_4 & & f_6 & & f_8 & &
\end{array}
$$

I will conclude with some homework for  .

- Check that the partition $\lambda = (5, 3, 3, 2, 2, 2, 1)$ is not an $\mathrm{A}_{2n}^{(2)}$ partition associated with $L(6\Lambda_3)$.

$$\begin{array}{cccccccccccc}
6 & & 0 & & 1 & & 0 & & 0 & & 0 & \dots \\
& 0 & & 1 & & 0 & & 0 & & 0 & & \\
0 & & 0 & & 0 & & 1 & & 0 & & 0 & \dots \\
& 0 & & 2 & & 0 & & 0 & & 0 & & \\
0 & & 1 & & 1 & & 0 & & 0 & & 0 & \dots \\
& 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

- Count all $\mathrm{A}_{2n}^{(2)}$ partitions associated with $L(6\Lambda_3)$ of size n, where $1 \leqslant n \leqslant 1000$.