Exceptional Researcher & Exceptional Points

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Guttmann 2025 - 80 and (still) counting, Melb, 30 June - 1 July 2025

Outline of this talk

1) Exceptional researcher

2) What are exceptional points?

3) Exceptional points in the Z(N) free parafermion model

4) Exceptional points in the XY model

1) Exceptional researcher

A J Guttmann: Most prolific author in J Phys A!

- Journal of Physics A: General Physics (1968-1972)
- Journal of Physics A: Mathematical, Nuclear and General (1973-1974)
- Journal of Physics A: Mathematical and General (1975-2006)
- Journal of Physics A: Mathematical and Theoretical (2007–)

- Anthony J. Guttmann **(130 papers)**
- Giorgio Parisi **(91 papers)**
- Stuart G. Whittington **(86 papers)**
- Michael V Berry **(83 papers)**
- Pavel Winternitz **(71 papers)**

Flashback to Tony Guttmann's 60th, Dunk Island, 10-15 July, 2005

Like most of the audience, I was inspired to work on self-avoiding walks by Tony Guttmann. Much of the progress made from the perspective of exactly solved $O(n)$ models was driven by input from the wealth of knowledge that Tony has amassed over the years with his students, post-docs and colleagues. Fortunately, the exact results are in 2D, where the self-avoiding constraint is most severe.

2) What are exceptional points?

Exceptional points are spectral singularities in the parameter space of a system in which two or more eigenvalues, and their corresponding eigenvectors, simultaneously coalesce.

EPs are level degeneracies induced by non-Hermiticity.

They are central to exotic topological phenomena associated with the winding of eigenvalues and eigenvectors.

A vast and highly active topic!

Of experimental relevance.

3) Exceptional points in the $Z(N)$ free parafermion model

Recall the exact solution:

F C Alcaraz, MTB and Z-Z Liu, J Phys A 50, 16LT03 (2017)

$$-H = \sum_{j=1}^{L} \tau_j + \lambda \sum_{j=1}^{L-1} \sigma_j \sigma_{j+1}^\dagger$$

$$-E = \sum_{j=1}^{L} \omega^{p_j} \epsilon_{k_j}, \quad p_j = 0, 1, \ldots, N-1, \quad \omega = \mathrm{e}^{2\pi i/N}$$

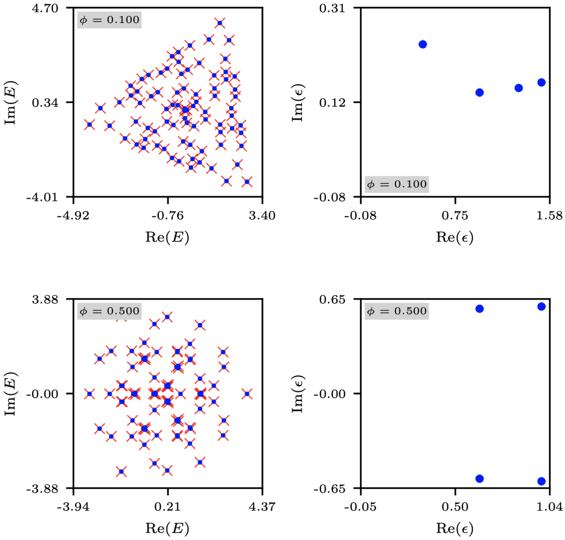
$$\boxed{\epsilon_k = \left(1 + \lambda^N + 2\lambda^{N/2} \cos k\right)^{1/N}}$$

k_j satisfy

$$\boxed{\sin(L+1)k = -\lambda^{N/2} \sin Lk}$$

for $\lambda = 1$, $k_j = \frac{2j\pi}{2L+1}$, $j = 1, \ldots, L$ and $\epsilon_k = \left(2 \cos \frac{k}{2}\right)^{2/N}$.

Free parafermion eigenspectrum ($N = 3,\ L = 4,\ \lambda = e^{2\pi i \phi / N}$)



For real positive λ, the quasi-energies ϵ_j are always positive and distinct.

For **complex** λ, a pair of them may become equal at certain values of λ, which depend on N and L.

We call these *quasi-energy exceptional points*.

We call EPs in the energy spectrum *Hamiltonian exceptional points*.

$\Longrightarrow$ quasi-energy EPs give rise to Hamiltonian EPs.

Moreover, we can calculate them.

A quasi-energy EP will occur when

$$\sin(L+1)k = -\lambda^{N/2} \sin Lk$$

has a repeated root, meaning that both this equation and its derivative are satisfied.

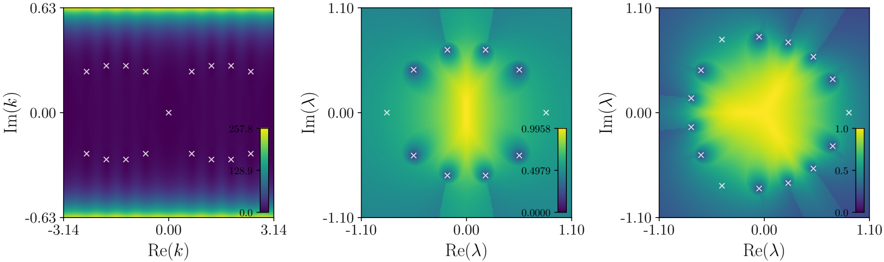
The EPs are pairs of values k_{EP} and λ_{EP} which satisfy these equations simultaneously.

In this way we obtain k_{EP} as the solution to

$$\sin(2L+1)k - (2L+1)\sin k = 0\,,$$

with the corresponding value λ_{EP} given by

$$\lambda^N = \left[\frac{-\sin(L+1)k_{EP}}{\sin Lk_{EP}}\right]^2 \,.$$

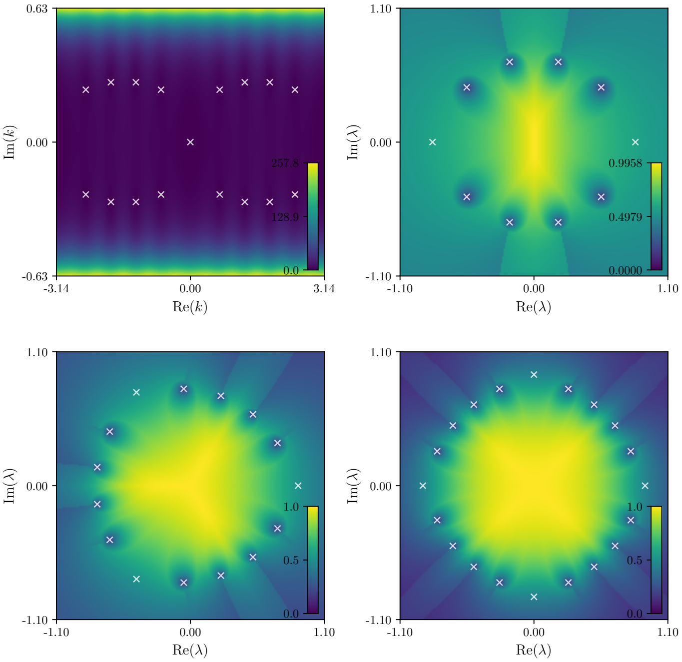


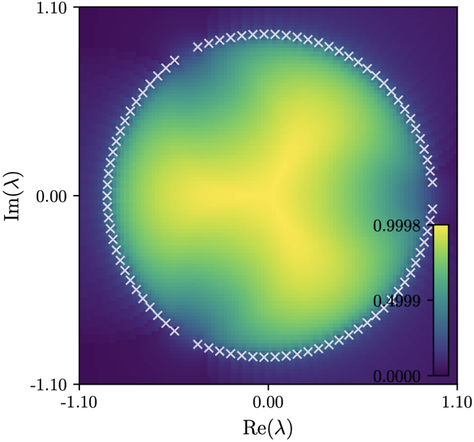
(left) k solutions for $L = 4$

(middle) difference between smallest and second-smallest quasi-energies for $N = 2$

(right) difference between smallest and second-smallest quasi-energies for $N = 3$

The corresponding values of λ_{EP} are also shown as crosses.

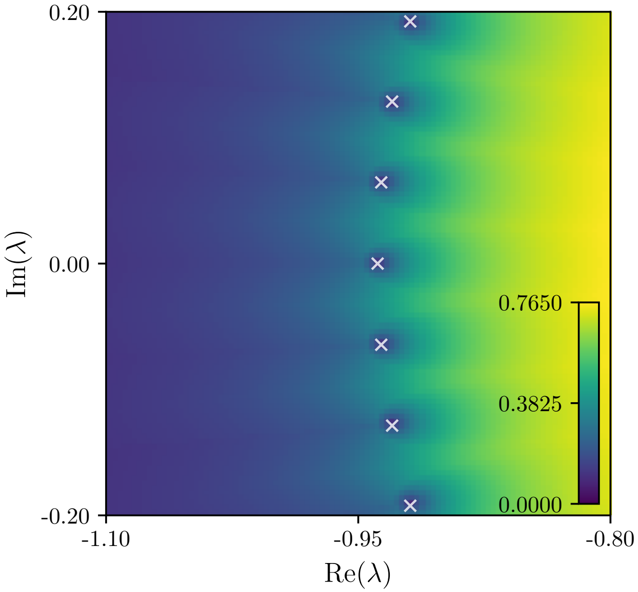

0.63
0.00
-0.63
Im(k)
-3.14
0.00
3.14
Re(k)
257.8
128.9
0.0
1.10
0.00
-1.10
Im(λ)
-1.10
0.00
1.10
Re(λ)
0.9958
0.4979
0.0000
1.10
0.00
-1.10
Im(λ)
-1.10
0.00
1.10
Re(λ)
1.0
0.5
0.0
1.10
0.00
-1.10
Im(λ)
-1.10
0.00
1.10
Re(λ)
1.0
0.5
0.0



$N = 3 \quad L = 50$

Can apply large L expansion results for k to show that λ_{EP} satisfies

$$\lambda^N = \cos\left(\frac{2\pi j}{L}\right) \pm \mathrm{i}\sin\left(\frac{2\pi j}{L}\right) .$$


0.20
0.00
-0.20
Im(λ)
-1.10
-0.95
-0.80
Re(λ)
0.7650
0.3825
0.0000

Summary

- We have located the quasi-energy EPs in the complex plane.
- Numerical tests confirm they correspond to Hamiltonian EPs.
- And also confirm that the corresponding eigenvectors coalesce.
- For large L they are on the unit circle in the complex λ plane.
- There are other degeneracies in the energy eigenspectrum, but they are not EPs.
- The EPs are also present in the $N = 2$ Ising case.

details in R.A. Henry and MTB, SciPost Physics **15**, 016 (2023)

4) Exceptional points in the XY model

The spin-$\frac{1}{2}$ anisotropic XY model is defined by the Hamiltonian

$$H = -\sum_{n=1}^{L-1}\left(\frac{1+\gamma}{2}\sigma_n^x\sigma_{n+1}^x + \frac{1-\gamma}{2}\sigma_n^y\sigma_{n+1}^y\right)$$

Free fermion eigenspectrum

$$E = \pm\epsilon_1 \pm \epsilon_2 \pm \cdots \pm \epsilon_L$$

The quasienergies follow directly from the eigenvalues of the $L \times L$ matrix

$$\begin{pmatrix} y^2 & 0 & x\,y & 0 & \cdots & 0 \\ 0 & x^2+y^2 & 0 & x\,y & \cdots & 0 \\ x\,y & 0 & x^2+y^2 & 0 & \cdots & 0 \\ 0 & x\,y & 0 & x^2+y^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & x^2 \end{pmatrix}$$

where $x = \frac{1}{2}(1+\gamma)$ and $y = \frac{1}{2}(1-\gamma)$.

Solved in 1961 by Lieb, Schultz and Mattis.

For this model

$$\epsilon_j = [1 - (1 - \gamma^2)\sin^2 k_j]^{\frac{1}{2}} .$$

The k_j satisfy

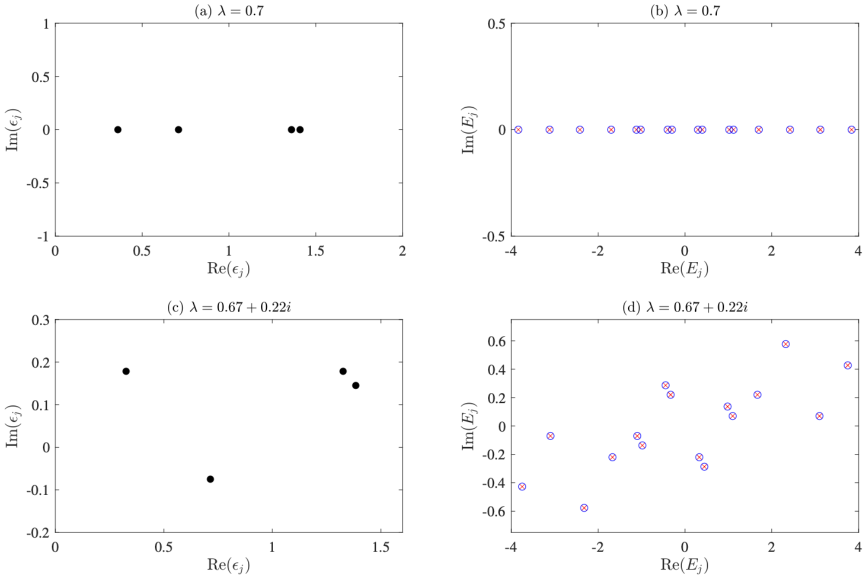
$$\frac{\sin(L+2)k}{\sin Lk} = \lambda^{\pm 1},$$

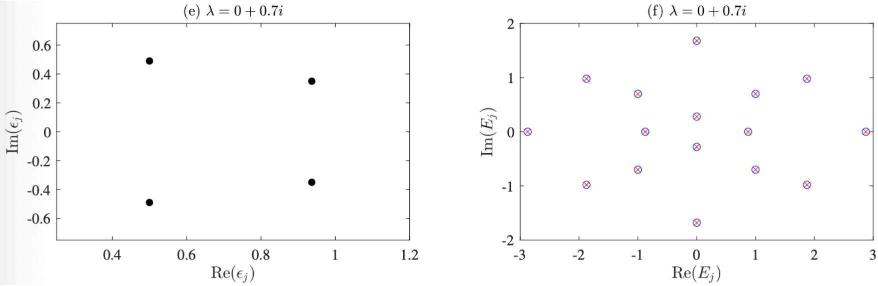
where

$$\lambda = \frac{1-\gamma}{1+\gamma} .$$

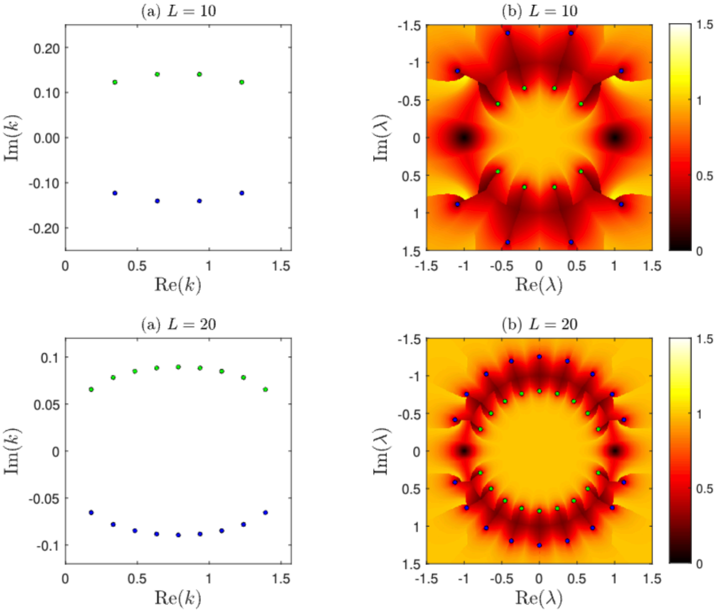
Work through in the same way to obtain the location of the exceptional points!

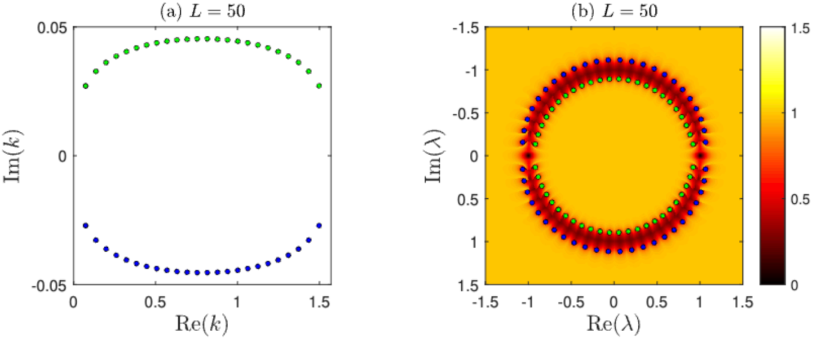
Examples of building the eigenspectrum ($L = 4$)





Can show (broken) PT-Symmetry if λ is pure imaginary.


(a) $L = 10$
(b) $L = 10$
(a) $L = 20$
(b) $L = 20$
Im(k)
Re(k)
Im(λ)
Re(λ)
0.20
0.10
0.00
-0.10
-0.20
0
0.5
1
1.5
-1.5
-1
-0.5
0.1
0.05
-0.05
-0.1



See two concentric rings of EPs which converge to the unit circle in the infinite size limit.

R.A. Henry, D.C. Liu and MTB (to appear soon)