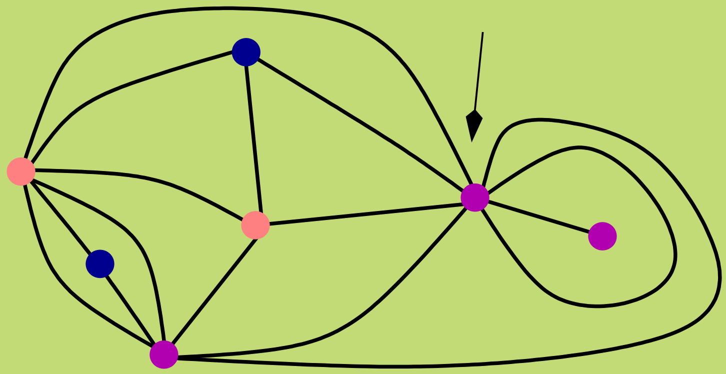
The 3-state Potts model on planar maps



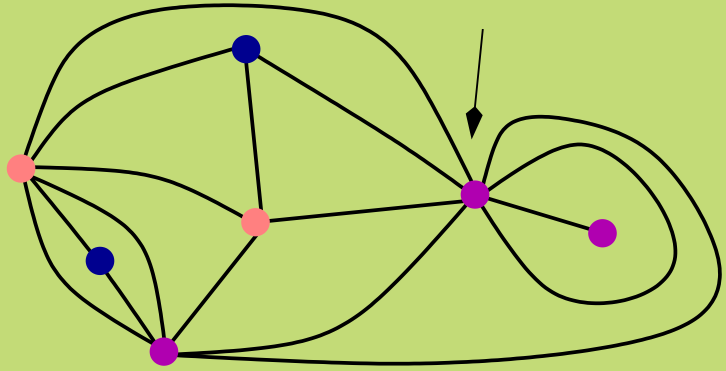
Mireille Bousquet-Mélou
CNRS, Université de Bordeaux, F

Hadrien Notarantonio
Université Paris Cité, F

Dress code at the Puffing Billy race, 1998

Tony
et
le petit Laurent

The 3-state Potts model on planar maps



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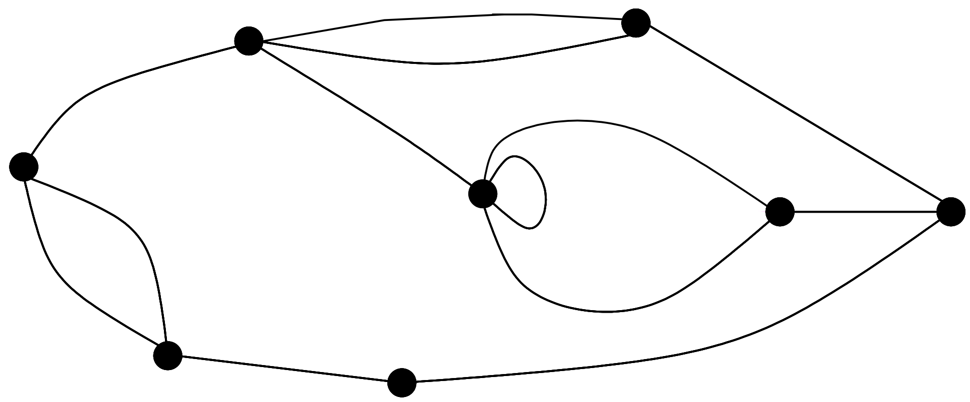
Planar maps

Def. A connected planar (multi)graph, given with a proper embedding in the plane, taken up to continuous deformation.



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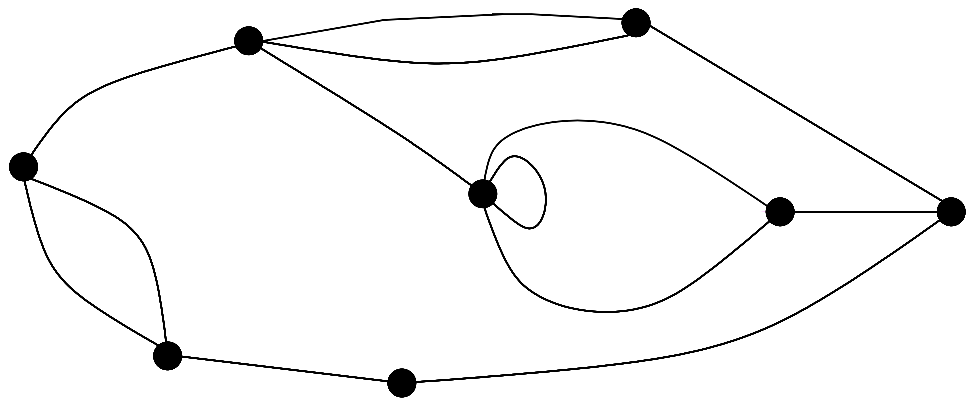

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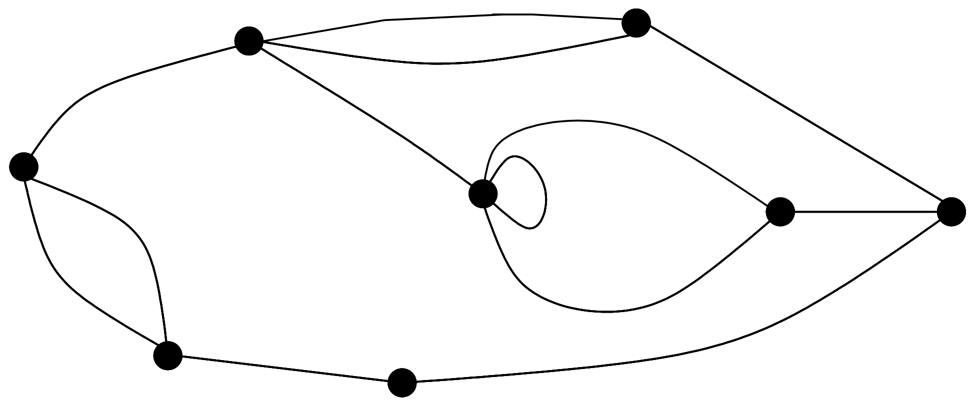

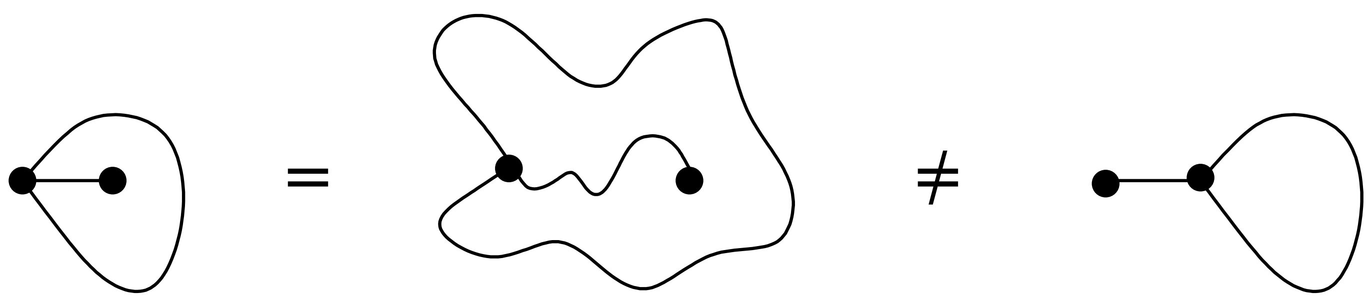

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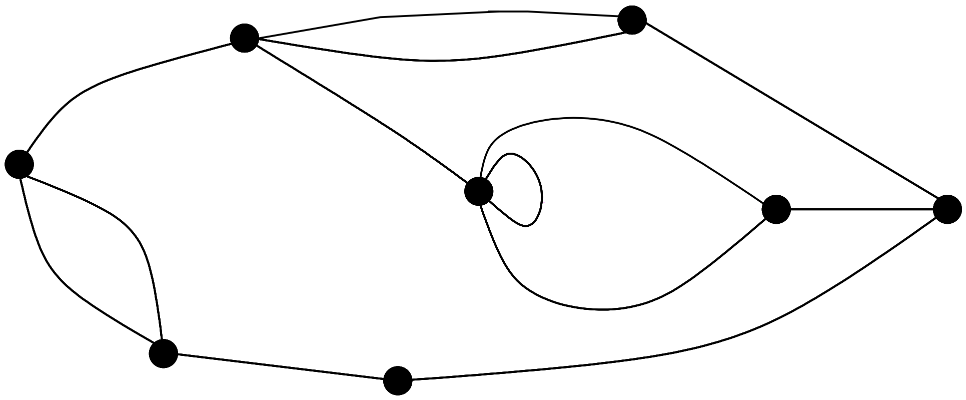

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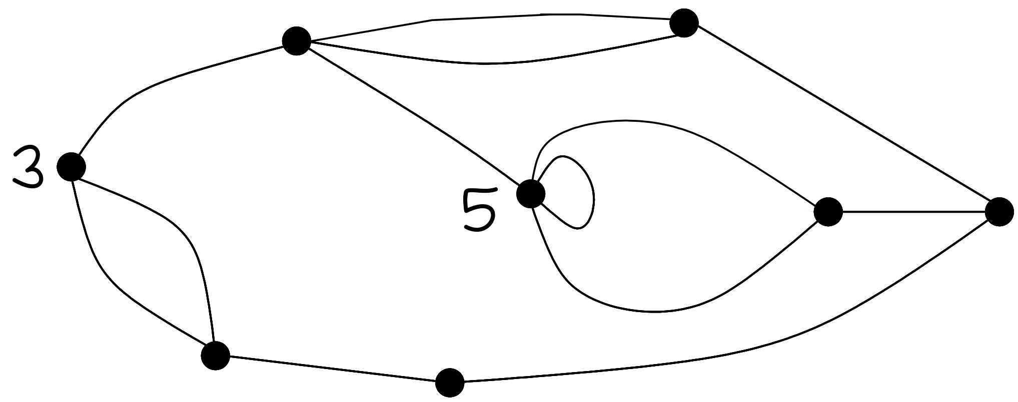

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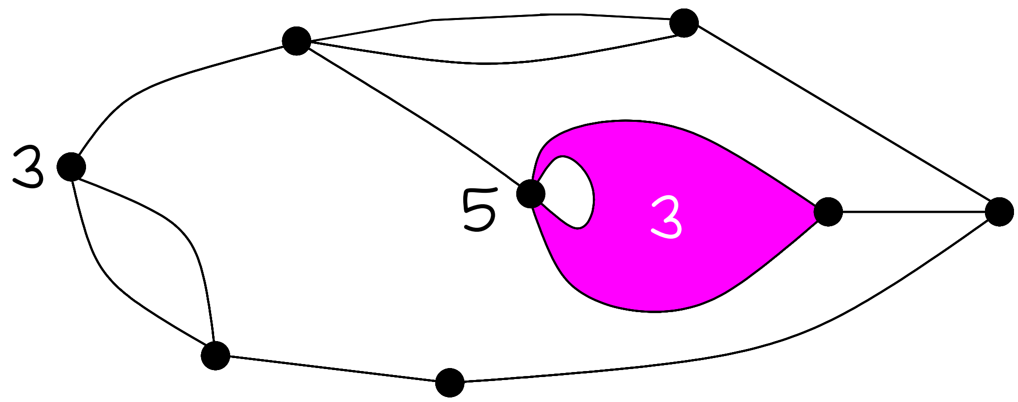

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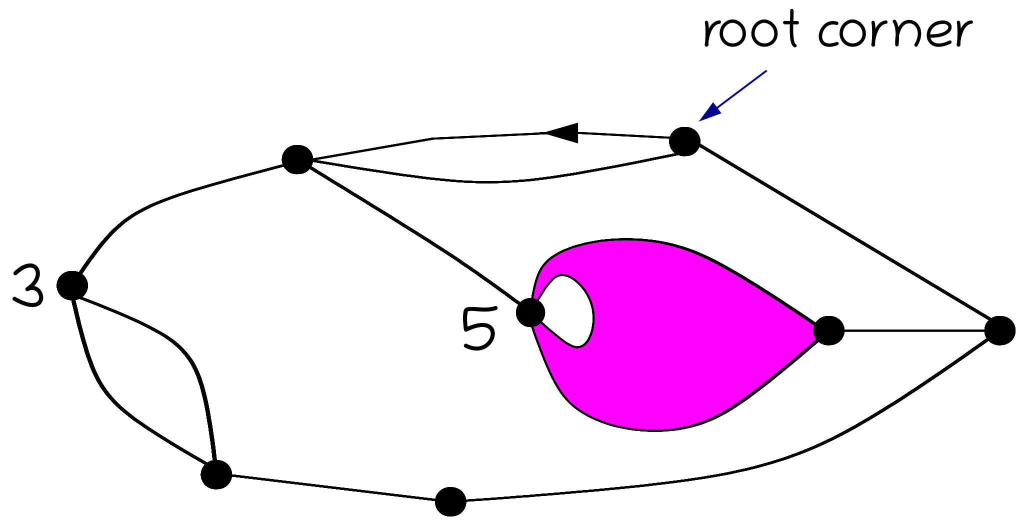

Rooted map: a distinguished corner in the outer face

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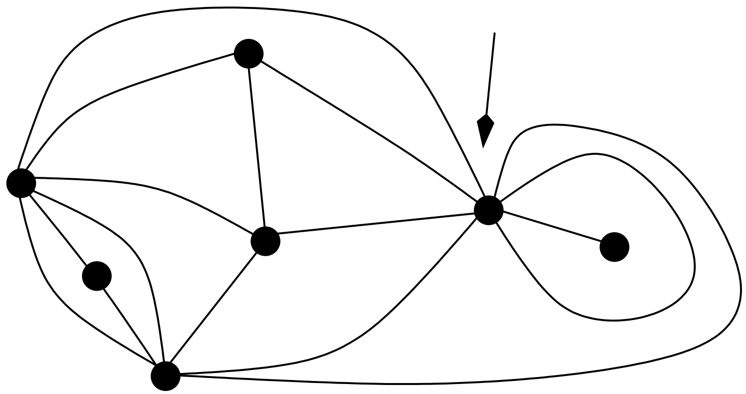

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Triangulation: all faces have degree 3

Near-triangulation: all **finite** faces have degree 3

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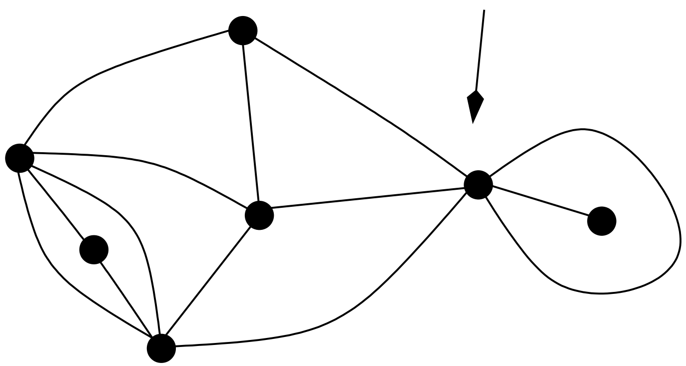

Rooted map: a distinguished corner in the outer face

Triangulation: all faces have degree 3

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Generating functions

- For a class of maps $\mathcal{C}$, equipped with some size (edge number...),

$$C := \sum_{M \in \mathcal{C}} t^{e(M)}.$$

- Multivariate versions, with more variables.

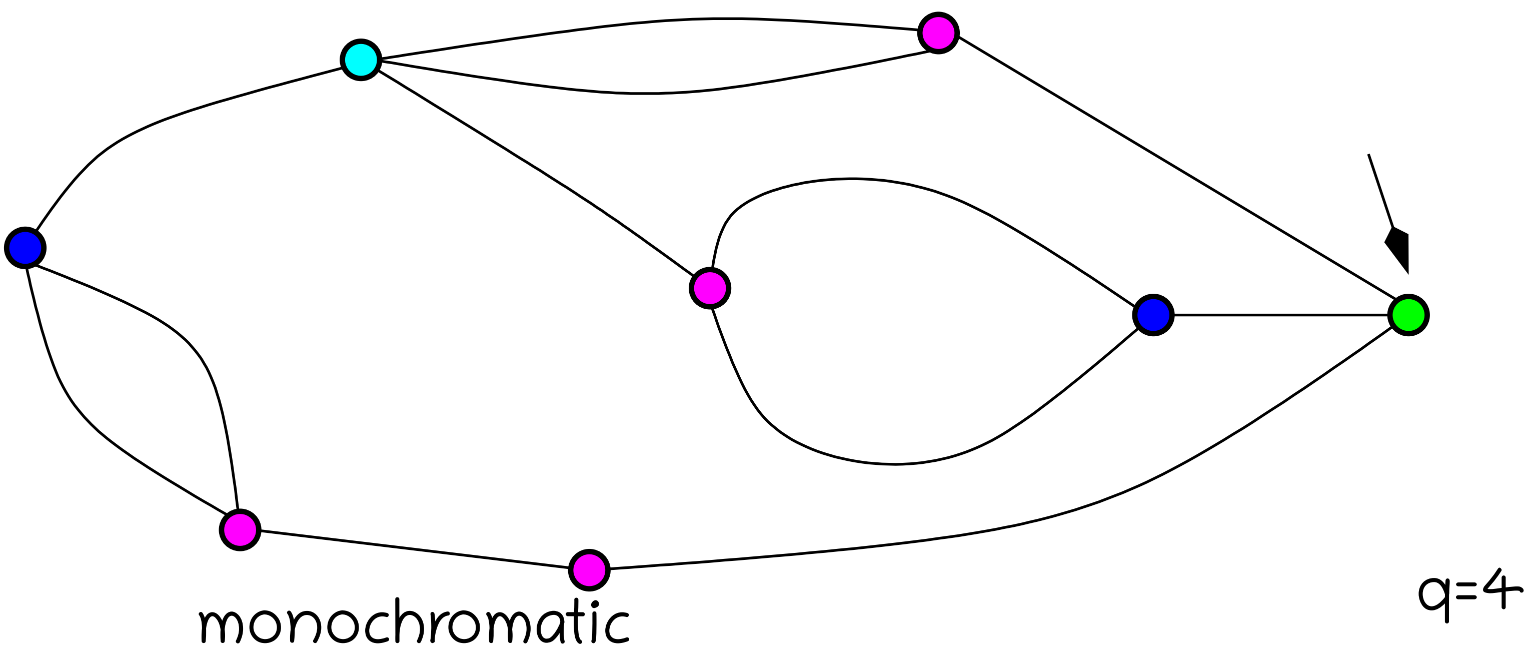
- The series C is **algebraic of degree k** if

$$P(C, t) = 0$$

for some irreducible polynomial P of degree k in its first variable.

Vertex colourings of maps

Definition. Vertices are coloured in q colours



Proper colouring: neighbour vertices get different colours.
Potts model: a generalisation

The q-state Potts model on planar maps

Definition. Let q be positive integer, M a map. The partition function of the (q-state) Potts model on M (or: **Potts polynomial** of M) is

$$P_M(q, \nu) := \sum_{c:V(M)\to\{1,\dots,q\}} \nu^{m(c)},$$

where m(c) is the number of monochromatic edges in the colouring c.

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Example.

$$P_M(q, \nu) := q\nu + q(q-1).$$

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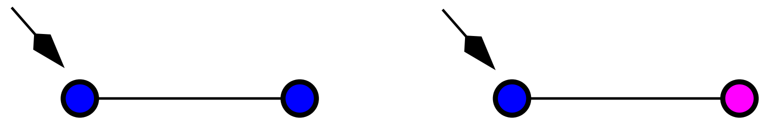

Properties

- polynomial in q and ν
- duality: for $q = (\nu-1)(\nu^*-1)$,

$$(\nu^*-1)^{f(M)-1} P_M(q,\nu) = (\nu-1)^{f(M^*)-1} P_{M^*}(q,\nu^*).$$

The Potts GF of near-triangulations

The Potts GF of (planar) near-triangulations is

$$T(y) \equiv T(q, \nu, t; y) = \sum_M P_M(q, \nu) t^{v(M)} y^{\mathrm{od}(M)},$$

where the sum runs over all near-triangulations M and od(M) is the outer degree.

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$$T(y) = \sum_{M,c} \nu^{m(c)} t^{v(M)} y^{\mathrm{od}(M)},$$

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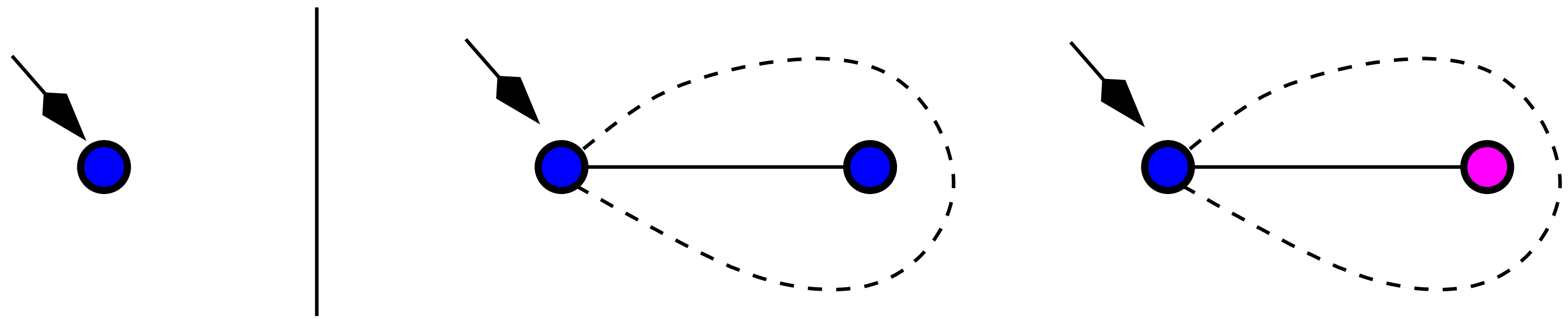
First coefficients:

$$T(y) = qt + yq(\nu + q - 1)(\nu + y)t^2 + \mathcal{O}(t^3).$$



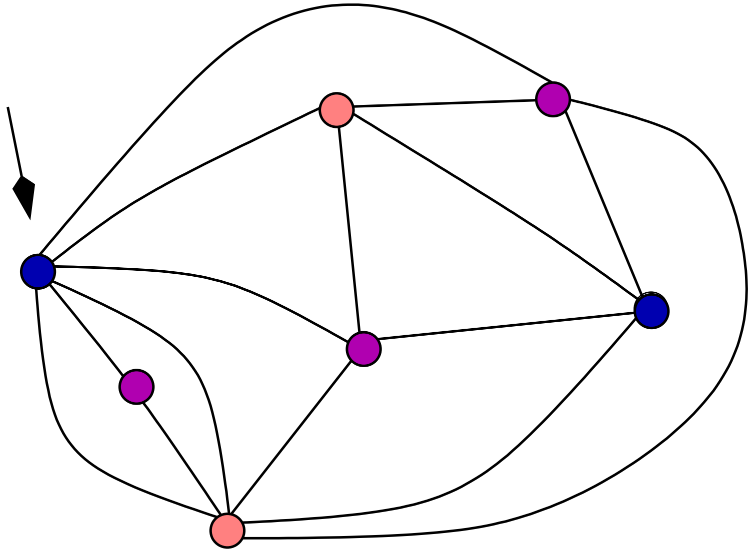
I. An old result, and a new one

Properly 3-coloured triangulations

An old result [Tutte 1963]

The generating function T_3 of properly 3-coloured triangulations (counted by vertices) is algebraic of degree 2:

$$18t^4 - 2t^3 + (24t^2 - 12t + 1)T_3 + 8T_3^2 = 0.$$



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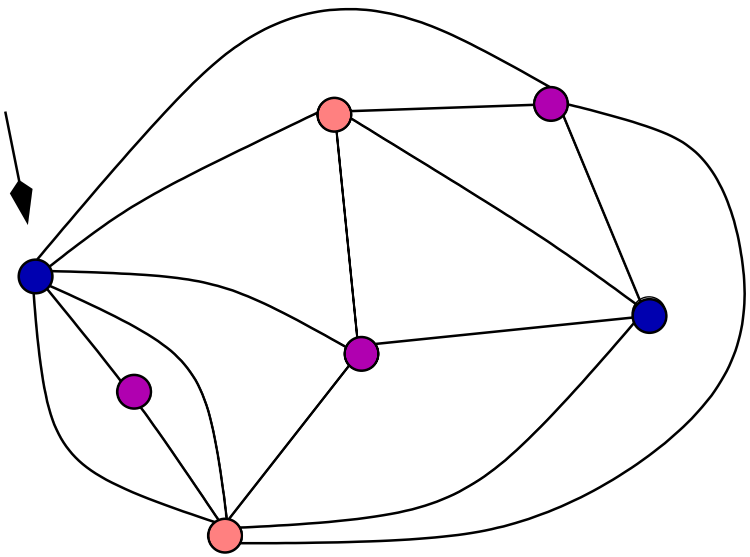

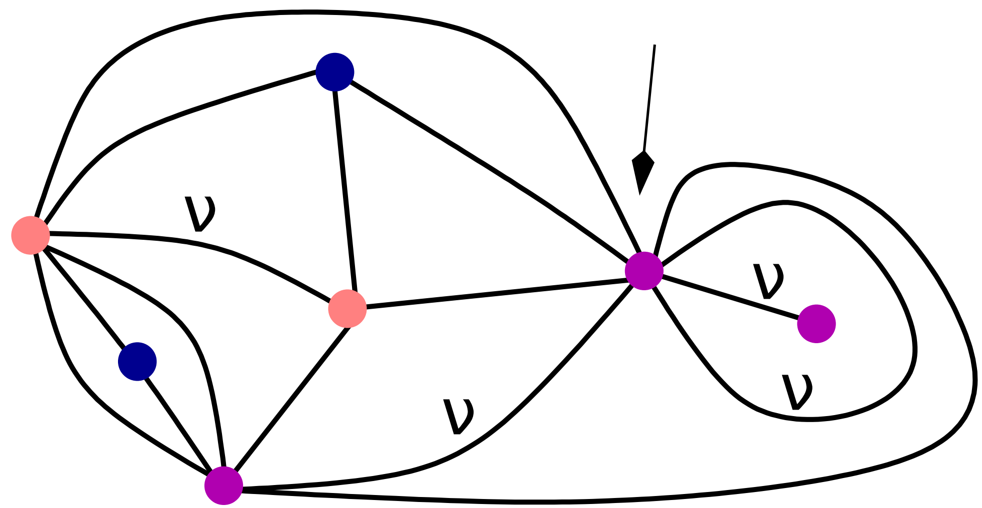

A new result [mbm-Notarantonio 2025]

The 3-Potts generating function of triangulations (counted by vertices) is algebraic of degree 11.

A new result [mbm-Notarantonio 25]

Proposition. For any $i \geq 1$, the 3-Potts generating function T_i of near-triangulations of outer degree i is algebraic of degree 11.

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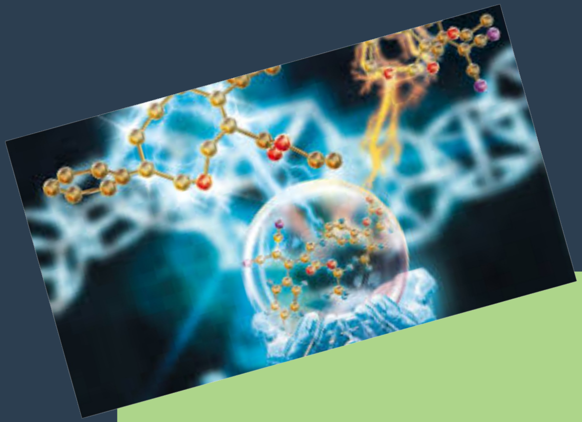
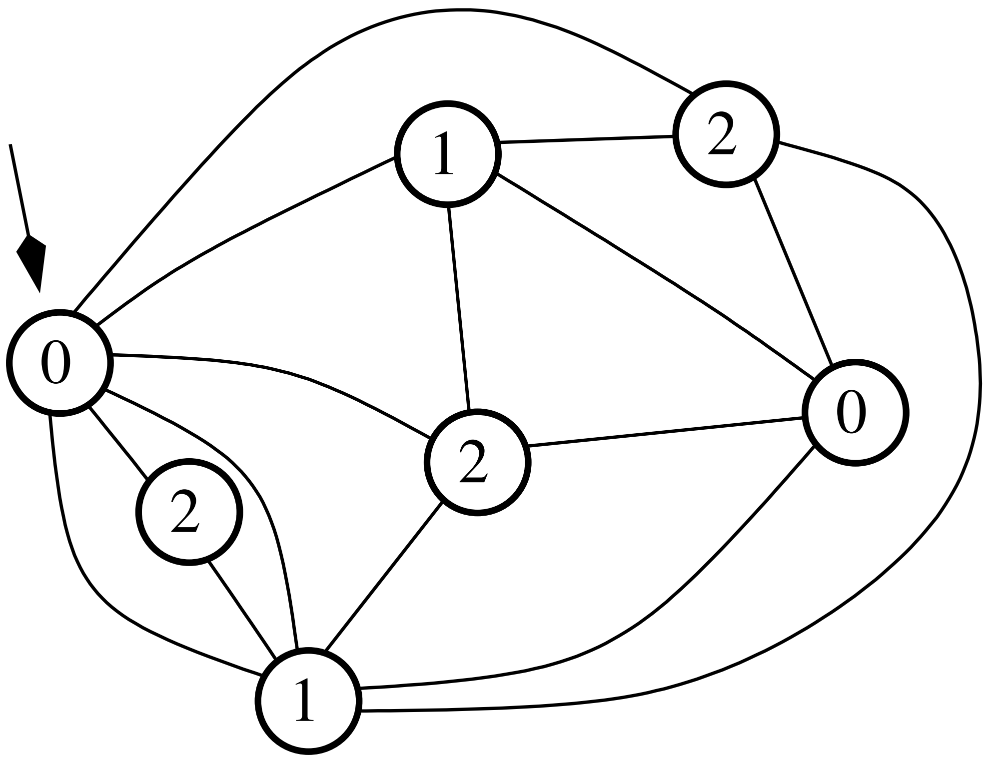
Minimal polynomial of the derivative of T_1 (degree 2 in t):

$$
\begin{aligned}
&276480\dot{T}_1^{11}\nu^7 - 27648\nu^6(31\nu+24)\dot{T}_1^{10} + 1152\nu^5\left(1021\nu^2+1678\nu+541\right)\dot{T}_1^9 \\
&\quad - 18\nu^4\left(46080\nu^3 t + 51935\nu^3 + 138243\nu^2 + 92253\nu + 17089\right)\dot{T}_1^8 \\
&+72\nu^3\left(1920\nu^3(17\nu+7)t + 6545\nu^4 + 25755\nu^3 + 26863\nu^2 + 10253\nu + 1144\right)\dot{T}_1^7 \\
&-4\nu^2\left(1008\nu^3\left(727\nu^2+586\nu+127\right)t + 38596\nu^5 + 219355\nu^4 + 322318\nu^3 + 190022\nu^2 + 43274\nu + 2915\right)\dot{T}_1^6 \\
&+4\nu\left(216\nu^3\left(2433\nu^3+2879\nu^2+1255\nu+153\right)t + 8027\nu^6 + 67626\nu^5 + 134820\nu^4 + 109109\nu^3 + 38007\nu^2\right. \\
&\left.+5103\nu+188\right)\dot{T}_1^5 + \left(41472\nu^6(\nu-1)t^2 - 12\nu^3\left(78871\nu^4+122456\nu^3+80010\nu^2+19688\nu+1375\right)t\right. \\
&\left.-3876\nu^7 - 53138\nu^6 - 145202\nu^5 - 151460\nu^4 - 71656\nu^3 - 14332\nu^2 - 958\nu - 18\right)\dot{T}_1^4 + \left(-13824\nu^5(5\nu+1)(\nu-1)t^2\right. \\
&+8\nu^2(5\nu+1)\left(6823\nu^4+11843\nu^3+9045\nu^2+2429\nu+100\right)t + 208\nu^7 + 6088\nu^6 + 24600\nu^5 + 31836\nu^4 + 19256\nu^3 \\
&\left.+5040\nu^2+440\nu+12\right)\dot{T}_1^3 + \left(1728\nu^4(\nu-1)(5\nu+1)^2t^2 - 12\nu(3\nu+1)\left(1358\nu^5+2771\nu^4+2504\nu^3+868\nu^2\right.\right. \\
&\left.\left.+58\nu+1\right)t - 312\nu^6 - 2401\nu^5 - 3747\nu^4 - 2821\nu^3 - 899\nu^2 - 78\nu - 2\right)\dot{T}_1^2 + \nu\left(-96\nu^2(\nu-1)(5\nu+1)^3t^2\right. \\
&\left.+4(\nu+1)\left(1229\nu^5+2390\nu^4+2114\nu^3+697\nu^2+49\nu+1\right)t + \left(104\nu^4+189\nu^3+177\nu^2+67\nu+3\right)\right)\dot{T}_1 \\
&+2\nu^2(\nu-1)(5\nu+1)^4t^2 - 2\nu^2(\nu+2)\left(104\nu^4+189\nu^3+177\nu^2+67\nu+3\right)t = 0.
\end{aligned}
$$



II. Equations, equations, equations

Properly 3-coloured triangulations



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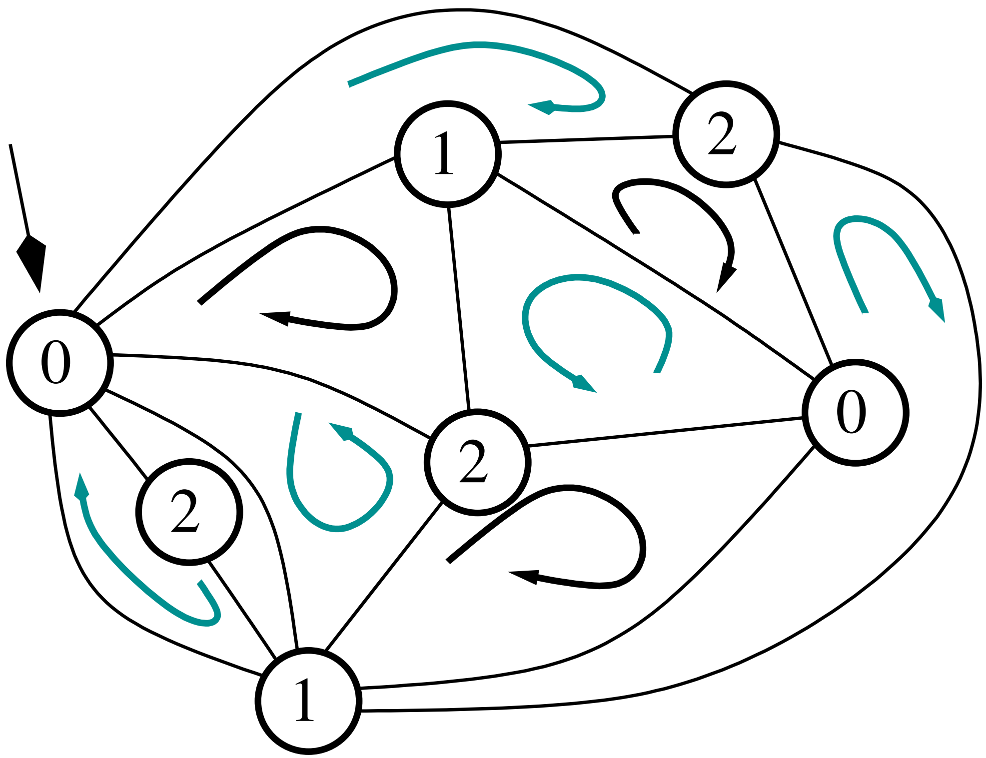

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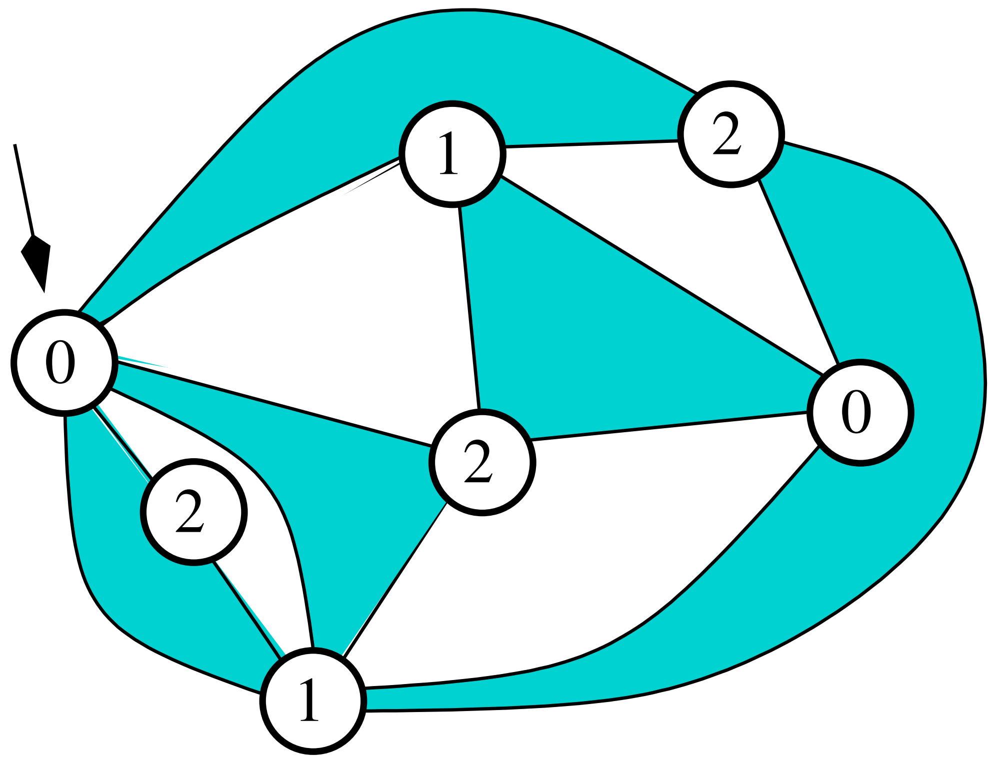
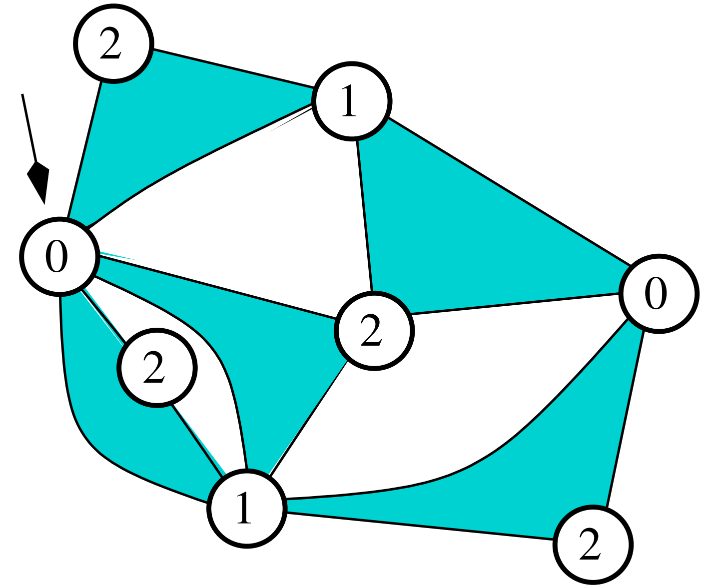


Properly 3-coloured triangulations ⇔ face-bicoloured triangulations

Properly 3-coloured triangulations are 1-catalytic

Face-bicoloured **near-triangulations**,
by vertices and **outer degree:**

$$B(t;y) \equiv B(y) = \sum_M t^{v(M)} y^{od(M)/3}$$



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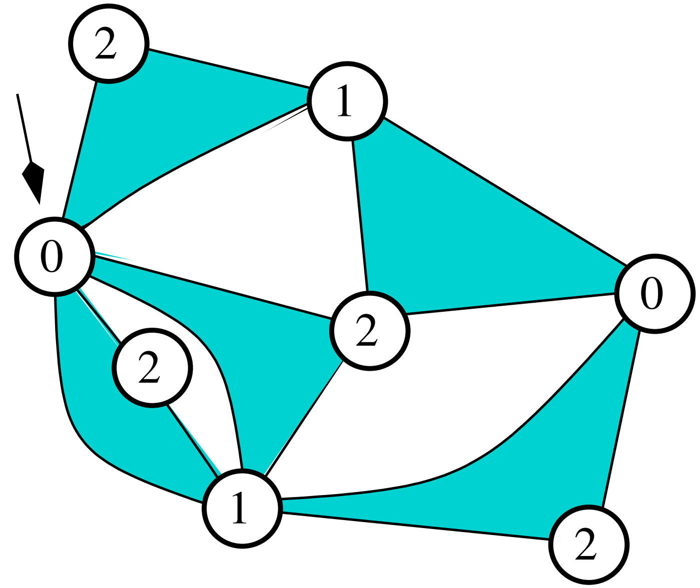

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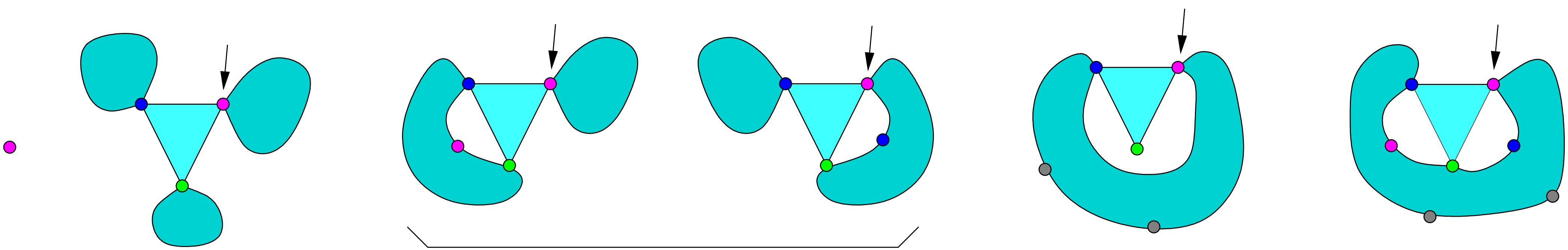

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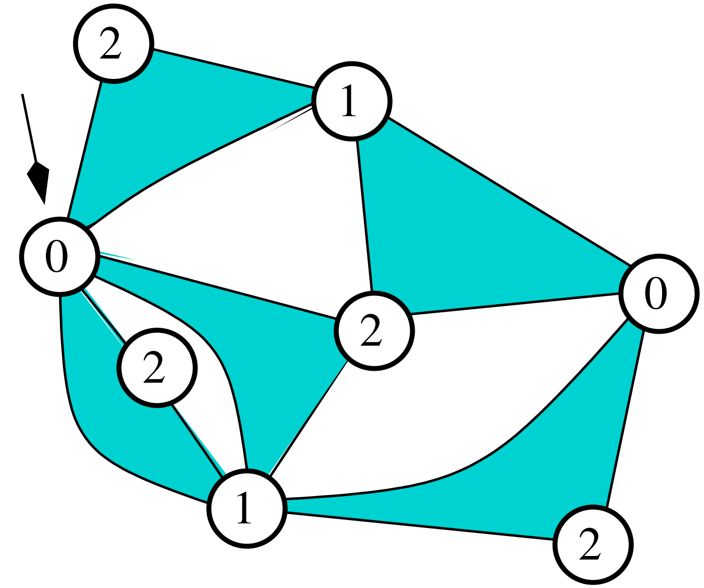

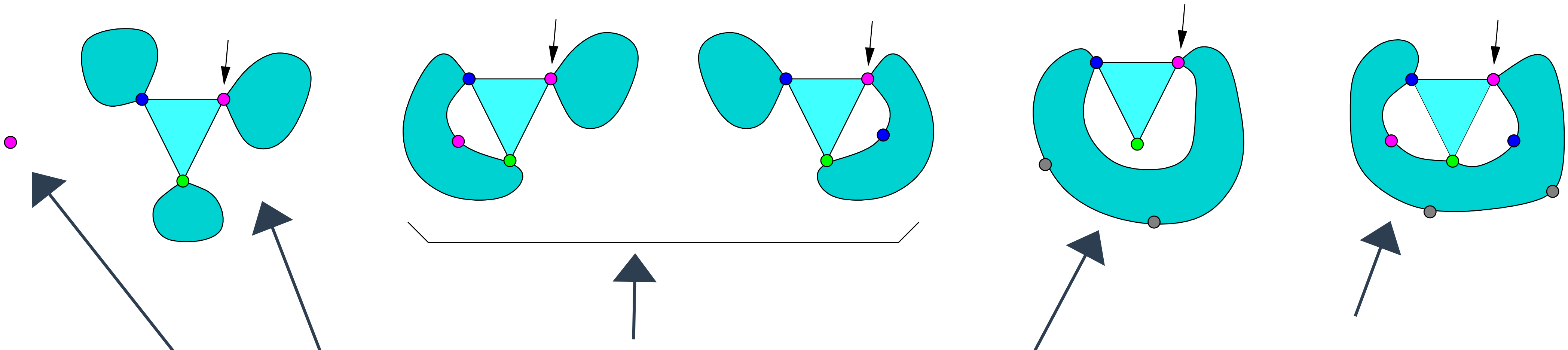

$$B(y) = t + yB(y)^3 \; + \; 2B(y)(B(y)-t) \; + \; t(B(y)-t) \; + \; \frac{B(y) - t - yB_1}{y}$$

$B_1 = [y]B(y)$: near-triangulations of outer degree 3

Properly 3-coloured triangulations are 1-catalytic

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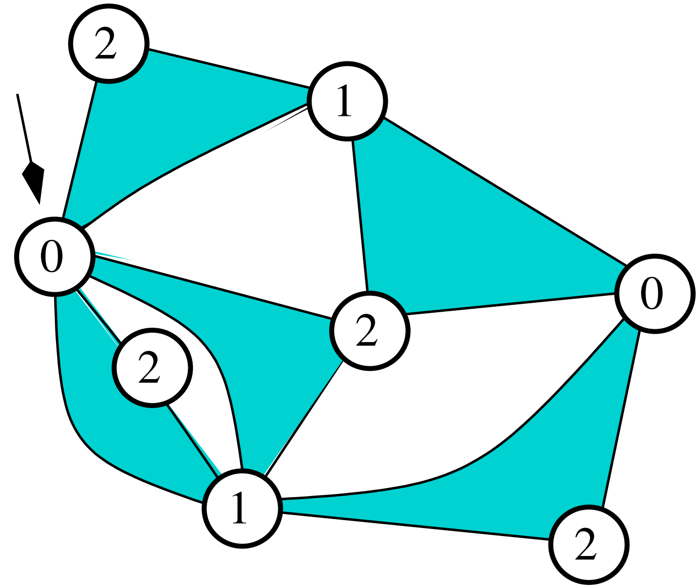

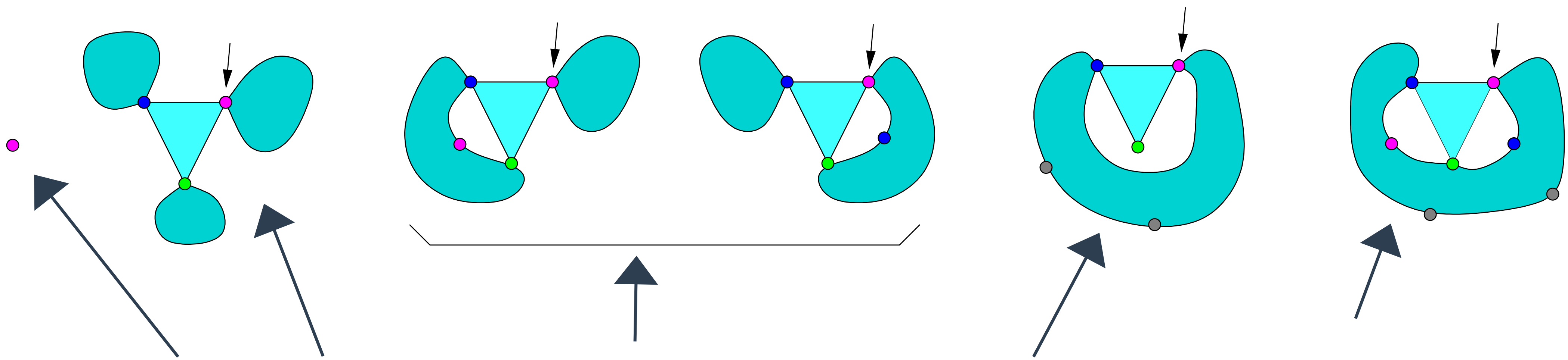

$$\mathrm{Pol}(B(y), B_1, t, y) = 0$$

An equation in one catalytic variable, y

3-Potts on triangulations is 1-catalytic as well

... but this is a more complicated story ! [Bernardi-mbm 11]

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when $q \neq 0,4$ is of the form $4\cos(k\pi/m)^2$. Includes $q=2, 3$.

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Proposition. There exists an explicit polynomial Pol such that the 3-Potts GF of near-triangulations satisfies:

$$\mathrm{Pol}(T(y), T_1, T_3, T_5, T_7, \nu, t, y) = 0$$

where

$$T_i = [y^i]T(y)$$

is the contribution of near-triangulations with outer degree i.

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No combinatorial explanation

One-catalytic implies algebraic !

Theorem [Popescu 86] [mbm-Jehanne 06]

If a polynomial equation

$$\mathrm{Pol}(S(t;y), A_1(t), \ldots, A_k(t), t, y) = 0$$

with coefficients in some field $\mathbb{F}$ has a **unique solution** $S(t;y)$, $A_1(t)$, ..., $A_k(t)$ in **formal power series**, then all these series are **algebraic** over $\mathbb{F}(t,y)$.

[mbm-Jehanne 06] **An effective procedure.**

3-Potts on near-triangulations is algebraic

Proposition. Let q=3. There exists an explicit polynomial such that

$$\mathrm{Pol}(T(y), T_1, T_3, T_5, T_7, \nu, t, y) = 0 \qquad (1)$$

where

$$T_i = [y^i]T(y).$$

Corollary. The 3-Potts GF of near-triangulations T(y) is algebraic.

[Bernardi-mbm 11]

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[Bernardi-mbm 11]

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Better algorithms than [mbm-Jehanne 06]: Bostan, Chyzak, Notarantonio, Safey el Din (2022-) ... but (1) was still too big.

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Now a solution... What happened?

III. Solving 1-catalytic equations

General approach to 1-catalytic equations [mbm-AJ]

Consider the 1-catalytic equation

$$\mathrm{Pol}(S(y), A_1, A_2, A_3, A_4, t, y) = 0.$$

Theorem: Let $\Delta(a_1, a_2, a_3, a_4, t, y)$ be the discriminant of $\mathrm{Pol}(s, a_1, a_2, a_3, a_4, t, y)$ in its first variable.

Then, as a polynomial in y, $\Delta(A_1, A_2, A_3, A_4, t, y)$ has 4 double roots Y_1, Y_2, Y_3, Y_4.

General approach to 1-catalytic equations [mbm-AJ]

Consider the 1-catalytic equation

$$\mathrm{Pol}(S(y), A_1, A_2, A_3, A_4, t, y) = 0.$$

Theorem: Let $\Delta(a_1, a_2, a_3, a_4, t, y)$ be the discriminant of $\mathrm{Pol}(s, a_1, a_2, a_3, a_4, t, y)$ in its first variable.

Then, as a polynomial in y, $\Delta(A_1, A_2, A_3, A_4, t, y)$ has 4 double roots Y_1, Y_2, Y_3, Y_4.

⇒ system of 8 polynomial equations for $A_1, A_2, A_3, A_4, Y_1, Y_2, Y_3, Y_4$

$$\Delta(A_1, A_2, A_3, A_4, t, Y_i) = \partial_y \Delta(A_1, A_2, A_3, A_4, t, Y_i) = 0, \quad i = 1 \ldots 4.$$

General approach to 1-catalytic equations [mbm-AJ]

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Elimination via resultants ⇒ **each T_i has degree 11**

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Solution for 3-Potts on near-triangulations

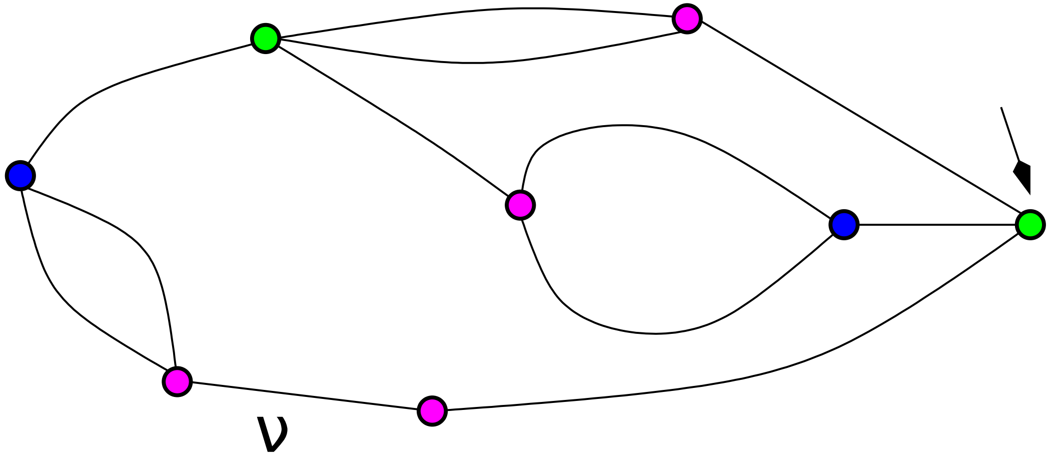
Elimination via resultants ⇒ each T_i has degree 11

The case of general planar maps

Proposition. The 3-Potts generating function $M_1(\nu,t)$ of **general planar maps** is algebraic of **degree 22**, with an explicit minimal polynomial.

[mbm-Notarantonio 25]

- Same starting point with D_+, D_-
- Alternative solution technique



IV. Asymptotics

Asymptotics for 3-Potts on near-triangulations

Proposition. Fix $\nu > 0$. The 3-Potts GF T_i of near-triangulations of outer degree i has radius of convergence ρ_ν where

$$\Delta_1(\nu, \rho_\nu) = 0 \quad \text{for} \quad 0 < \nu \leq \nu_c := 1 + 3/\sqrt{47},$$
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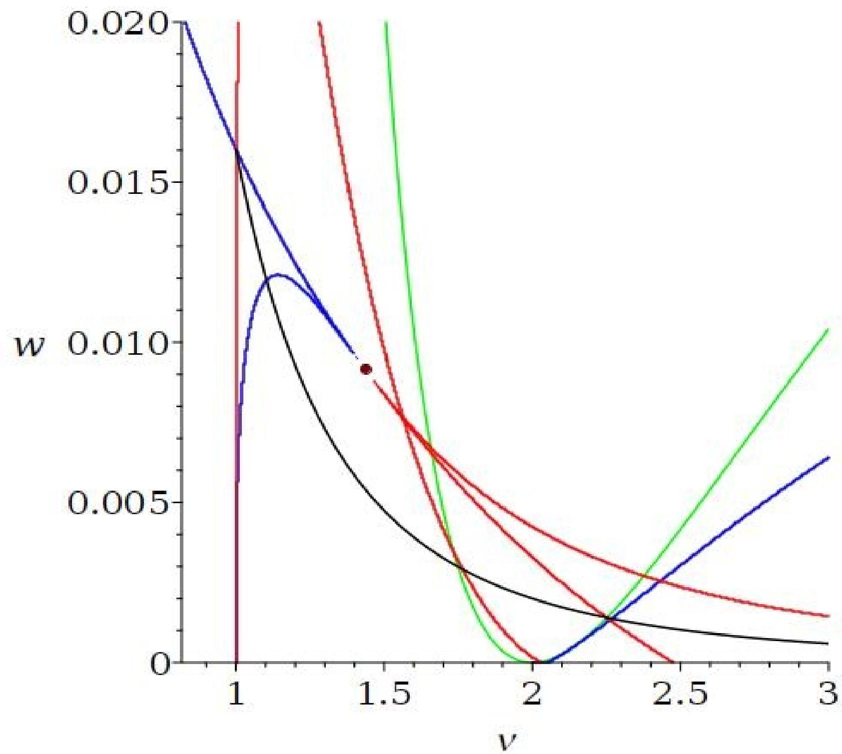

Roots of
the discriminant

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As t approaches ρ,

$$T_i = \alpha_\nu + \beta_\nu(1 - t/\rho_\nu) + \gamma_\nu(1 - t/\rho_\nu)^\alpha (1 + o(1)),$$

with

$$\alpha = 3/2 \quad \text{if} \quad \nu \neq \nu_c, \qquad \alpha = 6/5 \quad \text{if} \quad \nu = \nu_c.$$

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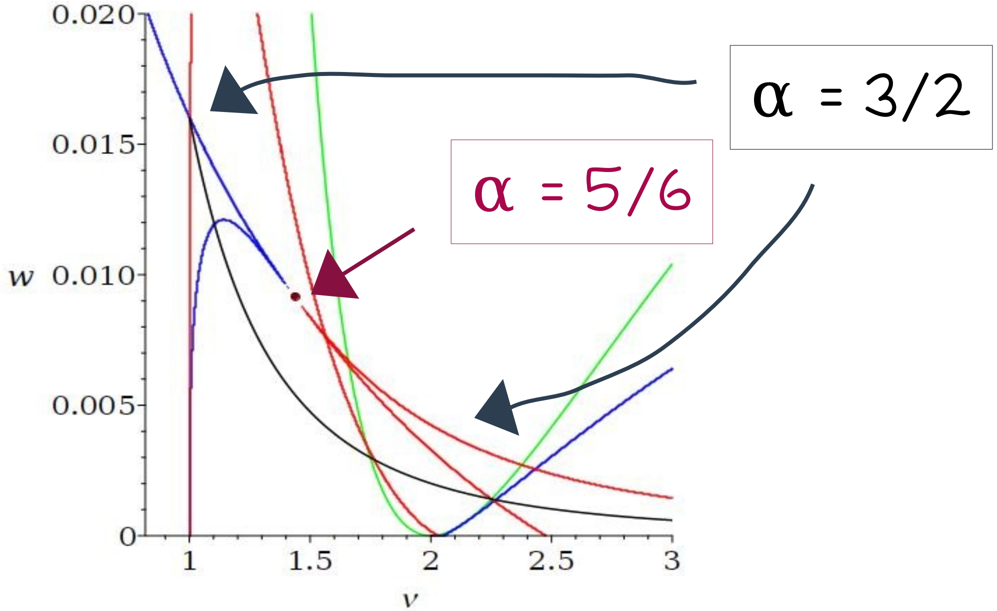

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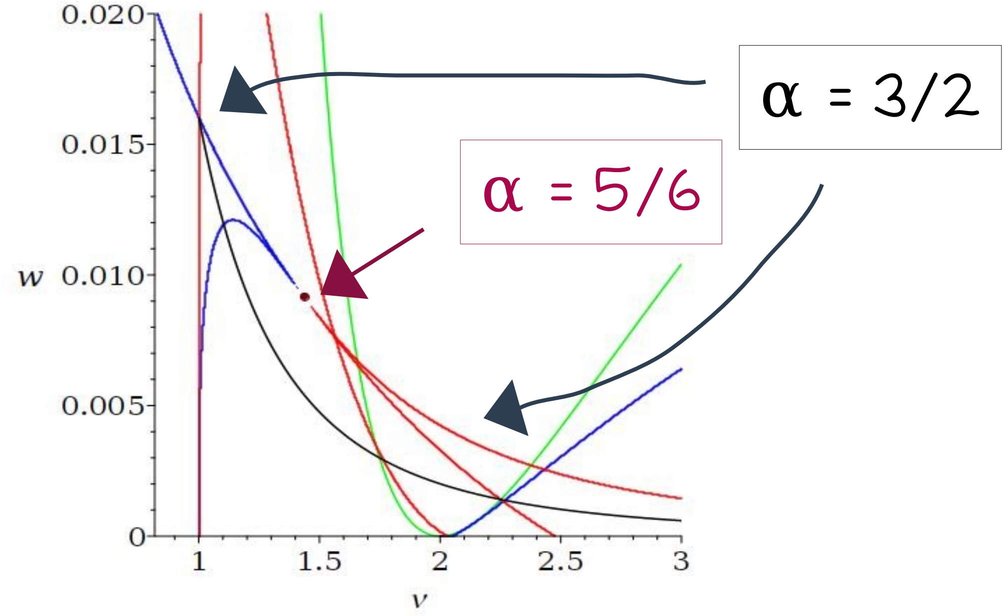

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Merci !