Exact three and four-point correlation functions in the $O(n)$ loop model

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Tale of two loop models

Q-state Potts model

- Q-state spins; interactions have S_Q permutation symmetry.
- Equivalent loop model on medial lattice [Baxter-Kelland-Wu 1976].
- Respects fixed orientation of lattice edges: $PSU(n)$ symmetry.
- Related to integrable 6-vertex model and Temperley-Lieb algebra.
- S_Q commutes with partition algebra $\mathscr{P}_L(Q)$, descending to Potts–Temperley–Lieb algebra $P\mathscr{TL}_{2L}(\sqrt{Q})$ in $d = 2$.
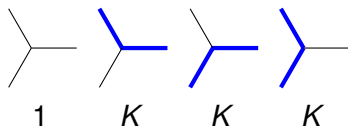
$O(n)$ model

- Vector spins $\in \mathbb{R}^n$; interactions have $O(n)$ symmetry.
- Equivalent loop model in $d = 2$ after modification [Nienhuis 1982].
- Related to integrable 19-vertex model and Motzkin algebra.
- $O(n)$ commutes with Brauer algebra $\mathscr{B}_L(n)$, descending to unoriented Jones–Temperley–Lieb algebra $u\mathscr{JTL}_L(n)$ in $d = 2$.

$O(n)$ model [Nienhuis 1982]

loop weight n
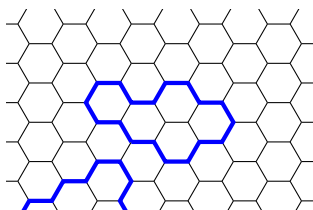
Vertex weights 1 and K

$$1 \qquad K \qquad K \qquad K$$

All configurations can be built by a transfer matrix:

$$\check{R}_k = \rangle\!\langle + K \rangle\!\langle + K \rangle\!\langle + K^2 \smile + K^2 \frown + K^2 \rangle\!\langle + K^2 \rangle\!\langle + K^2 \asymp$$

Define the partition function

$$Z(K, n) = \sum_{\text{loops}} K^{\#\text{monomers}} n^{\#\text{loops}}.$$

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Monomer fugacity at the critical point:

$$K_c = \left(2 \pm \sqrt{2 - n}\right)^{-1/2},$$

where $-2 \leq n \leq 2$. Plus (minus) sign for the dilute (dense) phase.

Special cases:

- $n = 1$ dense: Site percolation
- $n = 1$ dilute: Ising model
- $n = 0$ dilute: Self-avoiding walks
- $n = 2$ either: Gaussian free field, XY model

Most of there are really *logarithmic* CFTs.

Our first objective is to understand the case of 'generic' n.

Conformal Field Theory of the $O(n)$ model

Central charge

$$c = 13 - 6\beta^2 - 6\beta^{-2} \qquad \text{with} \qquad \begin{cases} \Re\beta^2 > 0 , \\ \beta^2 \notin \mathbb{Q} . \end{cases}$$

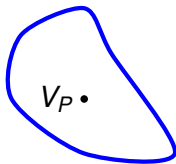Conformal weight Δ and momentum P:

$$\Delta = P^2 - P_{(1,1)}^2 , \quad \Delta_{(r,s)} = P_{(r,s)}^2 - P_{(1,1)}^2 , \quad P_{(r,s)} = \tfrac{1}{2}\left(-\beta r + \beta^{-1}s\right) .$$

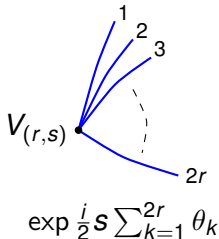Field content, with left- and right-moving conformal weights $(\Delta, \bar{\Delta})$:

Name	Notation	Parameters	$(\Delta, \bar{\Delta})$
Degenerate	$V_{\langle r,s \rangle}^d$	$r = 1; s \in 2\mathbb{N} + 1$	$\left(\Delta_{(r,s)}, \Delta_{(r,s)}\right)$
Diagonal	V_P	$P \in \mathbb{C}$	$\left(P^2 - P_{(1,1)}^2, P^2 - P_{(1,1)}^2\right)$
Non-diagonal	$V_{(r,s)}$	$r \in \tfrac{1}{2}\mathbb{N}^*; s \in \tfrac{1}{r}\mathbb{Z}$	$\left(\Delta_{(r,s)}, \Delta_{(-r,s)}\right)$

Interpretation of fields within the loop model:

Diagonal and non-diagonal fields

$$w(P) \qquad\qquad \exp \tfrac{i}{2} s \sum_{k=1}^{2r} \theta_k$$
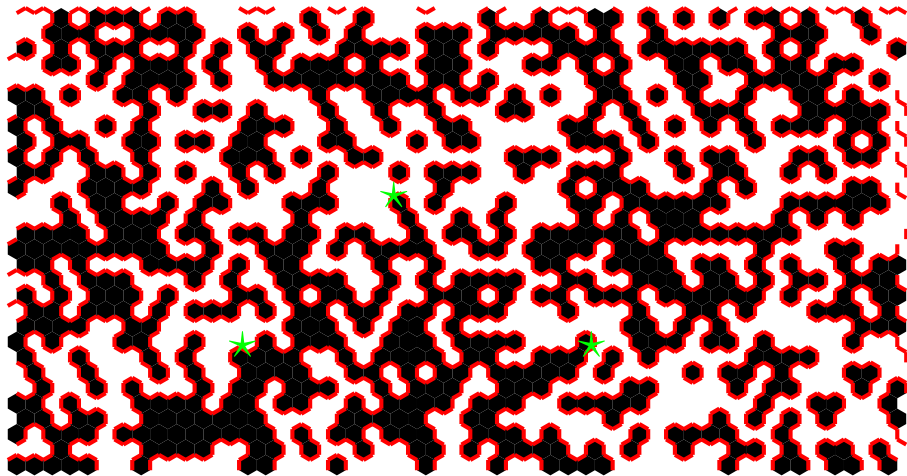
$V^d_{\langle 1,3 \rangle}$ is the energy operator.

The dense $O(n)$ model has a CFT limit iff $V^d_{\langle 1,3 \rangle}$ is irrelevant:

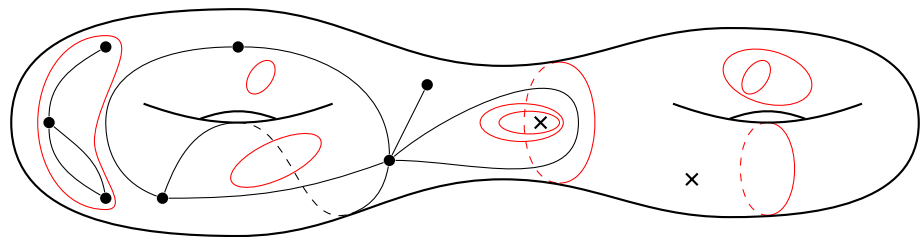$$\Re\Delta_{(1,3)} > 1 \iff \Re\beta^{-2} > 1 .$$

Correlation functions

For example: Site percolation ($n = 1$, dense phase)

$P(z_1, z_2, z_3 \in \text{same hull}) = \frac{C_{2,2,2}}{|z_1 - z_2|^{2\Delta}|z_2 - z_3|^{2\Delta}|z_3 - z_1|^{2\Delta}}$ by global conf. inv.

$\Delta = \frac{1}{8}$ and $C_{2,2,2} = ???$

Here • are $V_{(r,s)}$ insertions, and × are V_P insertions.

Open curves define a *combinatorial map* on a Riemann surface.

Segal's axioms: Three basic building blocks
1) Annulus with one insertion, 2) Disk with two insertions, 3) Pants.
The blocks are glued by integrating over eigenstates.

Progress this far

- Fields related to irreps of affine Temperley-Lieb algebra, $\mathscr{ATL}_L(n)$.
- Bijection between correlation functions and combinatorial maps.
- Conformal symmetry enhanced to *interchiral symmetry* via $V^d_{\langle 1,3 \rangle}$.
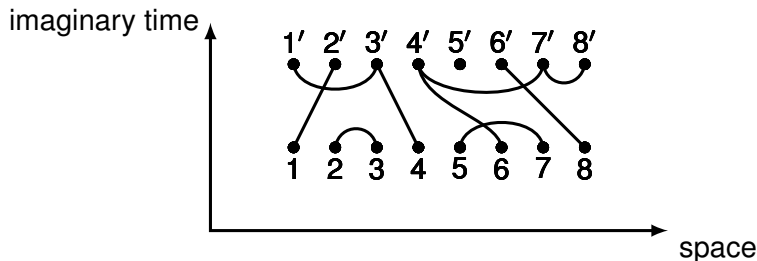- Global $O(n)$ symmetry in interplay with conformal symmetry.

First goal is to understand $N \leq 4$ points on the sphere.

- $N = 2$ understood from critical exponents.
- $N = 3$ conjecturally understood in all cases.
  And two cases are rigorously understood [Ang–Cai–Sun–Wu, 2024 preprint].
- $N = 4$ from conformal bootstrap. Partial analytical control.

The talk summarises this progress.

Diagrammatic algebras

Partition algebra:

Generators:

$$p_i = \Big| \; \cdots \; \Big| \underset{i\;i+1}{\times} \Big| \; \cdots \; \Big|, \quad s_i = \Big| \; \cdots \; \Big| \overset{i}{\vdots} \Big| \Big| \; \cdots \; \Big|, \quad s_{i+\frac{1}{2}} = \Big| \; \cdots \; \Big| \underset{i\;i+1}{\cup\cap} \Big| \; \cdots \; \Big|$$

$$1 \le i \le L-1 \qquad\qquad 1 \le i \le L \qquad\qquad 1 \le i \le L-1$$

From this we can construct the TL generator:

$$e_i = s_{i+\frac{1}{2}} s_i s_{i+1} s_{i+\frac{1}{2}} = $$

To get the periodic algebra $\mathscr{ATL}_L(n)$ we add two more generators:

$$e_L = \qquad , \quad u = $$

Define also the pseudo-translation t of the $2r \in \mathbb{N}^*$ through-lines:

$\mathscr{ATL}_L(n)$ is ∞-dimensional. A finite-dimensional quotient, the unoriented Jones–Temperley–Lieb algebra $u\mathscr{JTL}_L(n)$, is obtained by replacing non-contractible loops by n and imposing

$$t^{2r} \underset{u\mathscr{JTL}_L(n)}{=} 1 .$$

The standard modules $W^{(L)}_{(r,s)}$ are irreps of $u\mathscr{JTL}_L(n)$, spanned by link patterns with $2r$ defects. E.g. for $W^{(10)}_{(1,s)}$:

We have

$$\left(t - e^{\pi i s} \right) W^{(L)}_{(r,s)} = 0 .$$

The labels (r, s) carry over to the CFT.

Conformal partition function on the torus

Obtained by Di Francesco-Saleur-Zuber in 1987.
Let $q = e^{2\pi i \tau}$ with τ the modulus, and $\eta(q)$ is the Dedekind function.

$$Z^{O(n)}(q) = \sum_{s \in 2\mathbb{N}+1} \chi_{\langle 1,s \rangle}(q) + \sum_{r \in \frac{1}{2}\mathbb{N}^*} \sum_{s \in \frac{1}{r}\mathbb{Z}} L_{(r,s)}(n) \chi^N_{(r,s)}(q)$$

with the diagonal degenerate characters

$$\chi_{\langle r,s \rangle}(q) = \left| \frac{q^{P^2_{(r,s)}} - q^{P^2_{(r,-s)}}}{\eta(q)} \right|^2 ,$$

and the non-diagonal characters

$$\chi^N_{(r,s)}(q) = \frac{q^{P^2_{(r,s)}} \bar{q}^{P^2_{(r,-s)}}}{|\eta(q)|^2} .$$

$$Z^{O(n)}(q) = \sum_{s \in 2\mathbb{N}+1} \chi_{\langle 1,s \rangle}(q) + \sum_{r \in \frac{1}{2}\mathbb{N}^*} \sum_{s \in \frac{1}{r}\mathbb{Z}} L_{(r,s)}(n) \chi^N_{(r,s)}(q)$$

We have the Virasoro representations:

$\mathscr{R}_{\langle 1,s \rangle} =$ diagonal level-s degenerate rep. with character $\chi_{\langle 1,s \rangle}(q)$,

$\mathscr{W}_{(r,s)} \underset{r,s \in \mathbb{N}^*}{=}$ indecomposable rep. with character $\chi^N_{(r,s)}(q) + \chi^N_{(r,-s)}(q)$,

$\mathscr{W}_{(r,s)} \underset{r \notin \mathbb{Z}^* \text{ or } s \notin \mathbb{Z}^*}{=}$ Verma module with character $\chi^N_{(r,s)}(q)$.

The multiplicities $L_{(r,s)}(n)$ were obtained [Read-Saleur, 2001] and proved combinatorially [J-Richard, 2007]:

$$L_{(r,s)}(n) = \delta_{r,1}\delta_{s\in2\mathbb{Z}+1} + \frac{1}{2r}\sum_{r'=0}^{2r-1} e^{\pi i r' s} x_{(2r)\wedge r'}(n) ,$$

with polynomials $x_d(n)$ defined by

$$x_0(n) = 2 \quad , \quad x_1(n) = n \quad , \quad n x_d(n) = x_{d-1}(n) + x_{d+1}(n) .$$

Global $O(n)$ symmetry

This looks like Schur-Weyl duality.
Indeed we have both CFT and $O(n)$ symmetry.

$O(n)$ can be defined for $n \in \mathbb{C}$ (Deligne category).
Under the global $O(n)$ symmetry, primary operators transform in irreps:

$$[\,] : \quad \bullet \quad , \quad [2] : \quad \square\square \quad , \quad [11] : \quad \begin{array}{c}\square\\\square\end{array} \quad , \quad [5421] : \quad \text{(Young diagram)} .$$

Known dimensions and tensor products (Newell-Littlewood numbers).

Loop-model interpretation:
Each loop carries [1], the fundamental (defining) representation.
Empty space corresponds to [ ], the trivial representation.

$$Z^{O(n)}(q) = \sum_{s \in 2\mathbb{N}+1} \chi_{\langle 1,s \rangle}(q) + \sum_{r \in \frac{1}{2}\mathbb{N}^*} \sum_{s \in \frac{1}{r}\mathbb{Z}} L_{(r,s)}(n) \chi^N_{(r,s)}(q)$$

The proper way to understand it is that the $O(n)$ CFT has a space of states (spectrum)

$$\mathcal{S}^{O(n)} = \bigoplus_{s \in 2\mathbb{N}+1} [\,] \otimes \mathcal{R}_{\langle 1,s \rangle} \oplus \bigoplus_{r \in \frac{1}{2}\mathbb{N}^*} \bigoplus_{s \in \frac{1}{r}\mathbb{Z}} \Lambda_{(r,s)} \otimes \mathcal{W}_{(r,s)}$$

acted upon by $O(n) \times \mathfrak{C}$, where $\mathfrak{C}$ is conformal symmetry.

So $\dim_{O(n)} \Lambda_{(r,s)} = L_{(r,s)}(n)$. And of course $\dim_{O(n)}[\,] = 1$.

Introduce the formal alternating hook representations

$$\Lambda_t = \delta_{t \equiv 0 \bmod 2}[\,] + \sum_{k=0}^{t-1} (-1)^k [t-k, 1^k] \,.$$

We find then

$$\Lambda_{(r,s)} = \delta_{r,1} \delta_{s \in 2\mathbb{Z}+1}[\,] + \frac{1}{2r} \sum_{r'=0}^{2r-1} e^{\pi i r' s} x_{(2r) \wedge r'} \left(\Lambda_{\frac{2r}{(2r) \wedge r'}} \right) \,.$$

There exists an equivalent formula which makes clear that the expansion coefficients of Young tableaux $\in \mathbb{N}$.

Let us have a closer look:

$$\Lambda_{(\frac{1}{2},0)} = [1] \,,$$
$$\Lambda_{(1,0)} = [2] \,,$$
$$\Lambda_{(1,1)} = [11] \,,$$
$$\Lambda_{(\frac{3}{2},0)} = [3] + [111] \,,$$
$$\Lambda_{(\frac{3}{2},\frac{2}{3})} = [21] \,,$$
$$\Lambda_{(2,0)} = [4] + [22] + [211] + [2] + [\,] \,,$$
$$\Lambda_{(2,\frac{1}{2})} = [31] + [211] + [11] \,,$$
$$\Lambda_{(2,1)} = [31] + [22] + [1111] + [2] \,.$$

We also have e.g. $[1] \otimes [1] = [2] + [11] + [\,]$.
This tells us how to decompose two loop lines on $O(n)$ irreps.

Consequences for correlation functions

Two-point functions are given by the conformal dimensions, up to normalisation of the field.

Three-point functions are also fixed by global conformal invariance, up to structure constants.

Four-point functions could be determined by differential equations, if both $V^d_{\langle 1,s \rangle}$ and $V^d_{\langle r,1 \rangle}$ were present, but we only have the former!

Therefore we need the *conformal bootstrap*.

But we can do better than usual for two reasons:

- $V^d_{\langle 1,3 \rangle}$ generates an *interchiral symmetry*.
- We can exploit the global $O(n)$ symmetry.

Consider a four-point function of non-diagonal primary fields, and its *s*-channel decomposition into conformal blocks:

$$\left\langle \prod_{i=1}^{4} V_{(r_i,s_i)} \right\rangle = \sum_{s\in 2\mathbb{N}+1} D_s \mathscr{G}^D_{\langle 1,s \rangle} + \sum_{r\in\frac{1}{2}\mathbb{N}^*} \sum_{s\in\frac{1}{r}\mathbb{Z}} D_{(r,s)} \mathscr{G}_{(r,s)} \,.$$

The blocks are known from Zamolodchikov's recursion relation.

Degenerate shift equations using $V^d_{\langle 1,3 \rangle}$ determine $\frac{D_{(r,s+1)}}{D_{(r,s-1)}}$ and $\frac{D_{s+1}}{D_{s-1}}$.

So rewrite

$$\left\langle \prod_{i=1}^{4} V_{(r_i,s_i)} \right\rangle = D_{s_0} \mathscr{H}_{s_0} + \sum_{r\in\frac{1}{2}\mathbb{N}^*} \sum_{s\in\frac{1}{r}\mathbb{Z}\cap(-1,1]} D_{(r,s)} \mathscr{H}_{(r,s)} \,,$$

in terms of interchiral blocks

$$\mathscr{H}_{s_0} = \sum_{s\in s_0+2\mathbb{N}} \frac{D_s}{D_{s_0}} \mathscr{G}^D_{\langle 1,s \rangle} \quad , \quad \mathscr{H}_{(r,s)} = \sum_{j\in 2\mathbb{N}} \frac{D_{(r,s+j)}}{D_{(r,s)}} \mathscr{G}_{(r,s+j)} \,.$$

Solve then the crossing equations

$$\sum_{V\in\mathcal{S}^{(s)}} D_V^{(s)} \quad = \quad \sum_{V\in\mathcal{S}^{(t)}} D_V^{(t)} \quad = \quad \sum_{V\in\mathcal{S}^{(u)}} D_V^{(u)}$$

s-channel   t-channel   u-channel

We know the spectrum. And we can constrain the solution space by fixing the $O(n)$ symmetry of the exchanged fields V.

In favourable cases this gives a unique (numerical) solution.

Conjecture: Each solutions to the crossing equations gives a valid correlation function in the $O(n)$ CFT.

We have computed the 30 correlation functions with $\sum_{i=1}^{4} r_i = 2, 3, 4$.

Proofs

We have two ways to prove

$$\Lambda_{(r,s)} = \delta_{r,1}\delta_{s\in 2\mathbb{Z}+1}[\,] + \frac{1}{2r}\sum_{r'=0}^{2r-1} e^{\pi i r' s} x_{(2r)\wedge r'}\left(\Lambda_{\frac{2r}{(2r)\wedge r'}}\right) \ .$$

1st proof: Compute the torus partition function twisted by a non-trivial group element of $O(n)$. This produces the character $\Lambda_{(r,s)}$, not just its dimension $L_{(r,s)}(n)$ as in [Read-Saleur, 2001] and [J-Richard, 2007].

2nd proof: The commutant of $O(n)$ on $\mathcal{S}_L^{O(n)}$ is the Brauer algebra, $\mathscr{B}_L(n)$, generated by e_i and p_i. But in $d = 2$, it reduces to $u\mathscr{JTL}_L(n)$.

Hence we must compute the branching rules $\mathscr{B}_L(n) \downarrow u\mathscr{JTL}_L(n)$.

This is a solvable combinatorial problem.

Combinatorial maps

A (connected) *combinatorial map* is a (connected) graph, together with a cyclic permutation of the half-edges around each vertex.
Monogons are forbidden.
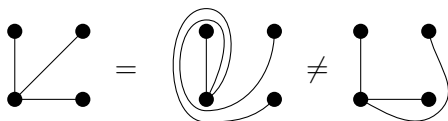
A map is *weakly connected* if it cannot be split into two non-trivial maps [trivial means: a sphere with 0 or 1 vertex].

This map is not weakly connected (it should have 'used' the handle).

Number of maps $\left|\mathscr{M}_{g,N}(r_i)\right|$ and of weakly connected maps $\left|\mathscr{M}_{g,N}^c(r_i)\right|$.
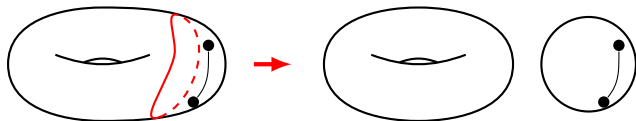Genus g, number of points N, vertex valencies $2r_i$.

$$\left|\mathscr{M}_{0,4}^c\right| = \left\lfloor \sum_{i=1}^4 r_i^2 - \frac{1}{2} \right\rfloor .$$

Signature of a planar map with four vertices:

$$\sigma_s = 1 \qquad \sigma_t = \frac{3}{2} \qquad \sigma_u = \frac{3}{2}$$
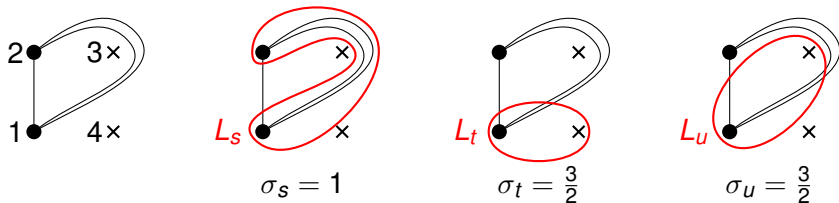
A map M is weakly connected iff $\forall x \in \{s, t, u\}, \sigma_x(M) > 0$.

For any *N*-point function of diagonal and non-diagonal fields, the dimension of the space of solutions of conformal bootstrap equation with spectra made only of non-diagonal fields is $\left| \mathcal{M}^c_{g,N}(r_i) \right|$.

The critical limit of a loop model correlation function is a solution of the conformal bootstrap equations.

The set of correlation functions is a basis of solutions of the corresponding conformal bootstrap equations.

Digression on the Barnes double gamma function

Recall $c = 13 - 6\beta^2 - 6\beta^{-2}$ and set $Q = \beta + \beta^{-1}$.

For $\Re x > 0$ define $\Gamma_\beta(x)$ through

$$\log \Gamma_\beta(x) = \int_0^\infty \frac{\mathrm{d}t}{t} \left[\frac{\mathrm{e}^{-xt} - \mathrm{e}^{-Qt/2}}{(1 - \mathrm{e}^{-\beta t})(1 - \mathrm{e}^{-t/\beta})} - \frac{(Q/2 - x)^2}{2\mathrm{e}^t} - \frac{Q/2 - x}{t} \right]$$

and the shift equations
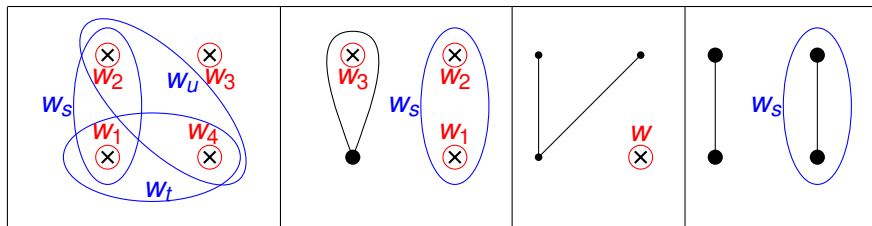
$$\frac{\Gamma_\beta(x + \beta)}{\Gamma_\beta(x)} = \sqrt{2\pi} \frac{\beta^{\beta x - \frac{1}{2}}}{\Gamma(\beta x)} \quad , \quad \frac{\Gamma_\beta(x + \beta^{-1})}{\Gamma_\beta(x)} = \sqrt{2\pi} \frac{\beta^{\frac{1}{2} - \beta^{-1} x}}{\Gamma(\beta^{-1} x)} \ .$$

Sometimes one defines also the upsilon function

$$\Upsilon_\beta(x) = \frac{1}{\Gamma_\beta(x)\Gamma_\beta(Q - x)} \ .$$

4-point functions of diagonal and non-diagonal fields

In addition to monomer weight K and bulk loop weight n, define *vertex weights* w_i (with $i = 1, 2, 3, 4$) and *channel weights* w_x (with $x = s, t, u$):

At most one of the loop types w_x can exist in a given configuration.

In the lattice model, define $C^{\text{loop}}(L, \ell | K, n, w_i, w_x)$, with L the size and ℓ the separation between z_1, z_2 and z_3, z_4 (in the s-channel).

We find the *s*-channel decomposition

$$C^{\text{loop}}(L, \ell | K, n, w_i, w_x) = \sum_{\omega \in S(L)} A_\omega(L | K, n, w_i, w_x) \left(\frac{\Lambda_\omega(L | K, n, w_s)}{\Lambda_{\max}(L | K, n, w_s)} \right)^\ell ,$$

with $(\Lambda_\omega)_{\omega \in S(L)}$ the $\mathscr{ATSL}_L(n)$ spectrum of transfer matrix eigenvalues.

Remarkable that only $\mathscr{ATSL}_L(n)$ eigenvalues participate here!

Define ratios wrt different values of the weights: $f(x : x') = \frac{f(x)}{f(x')}$.

Even more remarkably, we find that

$$A_{(r,s),\rho}\left(L \middle| K, n, w_i, w_x : w'_x\right) = D^{(s)}_{(r,s)}\left(n, w_i, w_x : w'_x\right) .$$

Here $\omega = (r, s), \rho$, where ρ labels states in the same module (r, s). There is no dependence on ρ, L and K. Hence the amplitudes have nothing to do with CFT and should be computable from $\mathscr{ATSL}_L(n)$.

Looks like a Wigner-Eckart theorem, but lifted from QM to $\mathscr{ATSL}_L(n)$.

Reference 2- and 3-point structure constants

Omitting the known coordinate dependence, define:

$$\langle V_1 V_2 \rangle = \delta_{12} B_1 \quad , \quad \langle V_1 V_2 V_3 \rangle = C_{123} \ .$$

For non-diagonal fields, set:

$$B^{\text{ref}}_{(r,s)} = \frac{(-)^{rs}}{2 \sin \left(\pi(\text{frac}(r) + s) \right) \sin \left(\pi(r + \beta^{-2}s) \right)} \prod_{\pm,\pm} \Gamma^{-1}_\beta \left(\beta \pm \beta r \pm \beta^{-1} s \right) \ ,$$

$$C^{\text{ref}}_{(r_1,s_1)(r_2,s_2)(r_3,s_3)} = \prod_{\epsilon_1,\epsilon_2,\epsilon_3 = \pm} \Gamma^{-1}_\beta \left(\frac{\beta + \beta^{-1}}{2} + \frac{\beta}{2} \left| \sum_i \epsilon_i r_i \right| + \frac{\beta^{-1}}{2} \sum_i \epsilon_i s_i \right) \ .$$

For diag fields, set $V_P = V_{(0,2\beta P)}$, so $C^{\text{ref}}_{(0,2\beta P_1)(0,2\beta P_2)(0,2\beta P_3)} = C_{P_1,P_2,P_3}$.

When $w_i \equiv 0$, C_{P_1,P_2,P_3} gives the probability that three points belong to the same FK cluster [Delfino-Viti, 2013]. The general C_{P_1,P_2,P_3} was found by [Ikhlef-J-Saleur, 2016] and proved by [Ang–Cai–Sun–Wu, 2024 preprint]. It is actually the imaginary Liouville version of the DOZZ formula.

Normalised 4-point structure constants

$$D_{(r,s)}^{(x)} = \frac{C_{(r_1,s_1)(r_2,s_2)(r,s)}^{\text{ref}} C_{(r,s)(r_3,s_3)(r_4,s_4)}^{\text{ref}}}{B_{(r,s)}^{\text{ref}}} d_{(r,s)}^{(x)}$$

Combining analytical arguments with numerical bootstrap and transfer matrices, we find that $d_{(r,s)}^{(x)}$ is a polynomial in $n = -2\cos(\pi\beta^2)$, with β-independent coefficients and $\deg_n d_{(r,s)}^{(x)} \leq r(r-1)$.

If the x-channel decomposition involves a diagonal field V_{P_x}, then $d_{(r,s)}^{(x)}$ is also polynomial in $w(P)$.

If some $V_i = V_{P_i}$ is diagonal, then $d_{(r,s)}^{(x)}$ is polynomial in $w_i = w(P_i)$.

The dependence on $w_x = w(P_x)$ becomes polynomial after we subtract a rational term that is needed for the 4-point function to be holomorphic in P_x.

This polynomial agrees between CFT and the off-critical finite-size lattice model.

Results on 3-point structure constants

Not clear how to factorise 4-point structure constants on 3-point ones, since several fields can have the same dimension.

But we find that 3-point structure constants of combinatorial maps are simply given by $C^{\text{ref}}_{(r_1,s_1)(r_2,s_2)(r_3,s_3)}$, by taking a ratio that is invariant under field renormalizations $V_{(r,s)} \to \lambda_{(r,s)} V_{(r,s)}$:
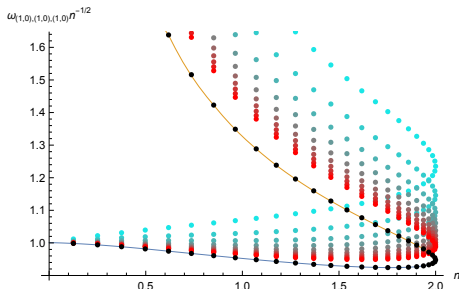
$$\omega_{123} = C^{\text{ref}}_{123} \sqrt{\frac{C^{\text{ref}}_{000}}{C^{\text{ref}}_{011} C^{\text{ref}}_{022} C^{\text{ref}}_{033}}}$$

Here $0 = \text{Id} = V_{(0,1-\beta^2)}$. The reference 2-point structure constant

$$C^{\text{ref}}_{(0,1-\beta^2)(r,s)(r,s)} = \prod_{\pm} \Gamma_{\beta}^{-2}\left(\beta^{\pm 1}\right) \prod_{\pm,\pm} \Gamma_{\beta}^{-1}\left(\beta^{\pm 1} + r\beta \pm s\beta^{-1}\right)$$

differ from the B^{ref} defined above only by simple factors.

E.g. $C^{\text{ref}}_{(1,0)(1,0)(1,0)}$ gives the probability that 3 points $\in$ same loop.

This case is the one proved by [Ang–Cai–Sun–Wu, 2024 preprint].
We checked a baker's dozen of cases: