From Tony's LGF to diagonals of rational functions and differentially algebraic globally bounded series.

Tony's 80th birthday conference 2025, Melbourne University

Dedicated to Anthony Guttmann: a marathon runner of enumerative combinatorics

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Tony has studied so many LGF. The simplest is the simple hyper-cubic LGF:

$$G(t) \;=\; \frac{1}{(2\pi)^n} \int \frac{d\phi_1\, d\phi_2\, d\phi_3\, d\phi_4\, d\phi_n}{1 \;-t \cdot \lambda},$$
$$\lambda \;=\; c_1 + c_2 + c_3 + \cdots + c_n,$$

where $c_j = \cos\phi_j$. Let us consider the example introduced by Tony (no lattice !):

$$G(t) \;=\; \frac{1}{(2\pi)^4} \int \frac{d\phi_1\, d\phi_2\, d\phi_3\, d\phi_4}{1 \;-t \cdot \lambda},$$
$$\lambda \;=\; c_1c_2c_3 + c_1c_2c_4 + c_1c_3c_4 + c_2c_3c_4,$$

The (minimal order) linear differential operator annihilating $G(t)$ is irreducible, of order 8, and has a $Sp(8, \mathbb{C})$ differential Galois group. All the LGF are simple examples of diagonals of rational functions (just change ϕ_j into $z_j = \exp(i \cdot , \phi_j)$).

The well-suited framework: diagonal of rational functions

We also found in enumerative combinatorics, lattice statistical mechanics, many other solutions of selected linear differential operators, which have **special differential Galois groups**. All these linear differential operators are **globally nilpotent**: they are not only **Fuchsian**, they are such that their p-curvatures are nilpotent, and all their **critical exponents are rational numbers**, ... They are "**Derived from Geometry**": they annihilate n-fold integrals of **algebraic integrands** (in mathematician's wording "**Periods**"). These n-fold integrals are (or can be recast into) series with **integer coefficients** (globally bounded series). These two set of properties are, in fact, the consequence of the fact that these holonomic functions are actually **diagonal of rational functions**. *As Monsieur Jourdain talked prose, without knowing it*, n-fold integrals in physics are, without knowing it, **diagonal of rational functions**, which corresponds to a **quite remarkable set**.

Definition of the diagonal of series of several complex variables

Definition:

$$\mathcal{F}\Big(z_1,\, z_2,\, \ldots,\, z_n\Big) \;=\; \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} F_{m_1, m_2, \ldots, m_n} \cdot z_1^{m_1}\, z_2^{m_2} \;\cdots\; z_n^{m_n},$$

$$Diag\Big(\mathcal{F}\Big(z_1,\, z_2,\, \ldots,\, z_n\Big)\Big) \;=\; \sum_{m=0}^{\infty} F_{m, m, \ldots, m} \cdot z^m.$$

The result: if the **algebraic, or rational, integrand** of a n-fold integral has a **multi-Taylor expansion**, then this n-fold integral **is the diagonal of a rational function**.

Two by-products: Diagonal of rational functions are (or can be recast into) series with **integer coefficients**, which **actually reduce modulo any prime to algebraic functions !!**

Ising n-fold integrals : the $\chi^{(n)}$'s

The full susceptibility of the two-dimensional Ising model can be written as an **infinite sum** of n-folds integrals **holonomic functions** ($w = s/2/(1+s^2), k = s^2$):

$$\chi(w) \;=\; \sum_{n=1}^{\infty} \chi^{(n)}(w).$$

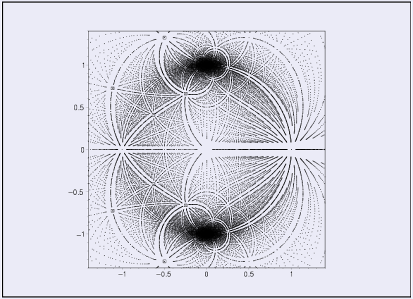
All these n-fold integrals $\chi^{(n)}$ are actually **diagonals of rational functions**.

The **magnetic susceptibility** χ is **not a holonomic function**, it is **not D-finite**: χ is **not solution of a linear differential equation**. It is much more involved.

The full susceptibility χ has a (unit circle) **natural boundary**, in the complex k-plane.

$$|k| \;=\; 1 \;\text{ is a natural boundary of }\; \chi(k).$$

Accumulation of the singularities of the linear ODEs for the $\chi^{(n)}$ in the k complex plane



the full susceptibility is clearly a quite involved function !
Remark: for a holonomic function, there is a difference between the **singularities of that function**, and the **singularities of the linear differential operator** annihilating the function !!

Is the full susceptibility of the Ising model differentially algebraic ?

We also considered the full susceptibility of the square Ising model, in order to see if it could be **differentially algebraic**:

Automata and the susceptibility of the square lattice Ising model modulo powers of primes, A.J. Guttmann and JMM, 2015 J. Phys. A: Math. Theor. 48 474001

Lacunary series mod. 32, 64 and thus reduce to algebraic series mod. 2^5, 2^6.

$$L(u) = 1 + u + u^2 + u^4 + u^8 + u^{16} + u^{32} + u^{64} + u^{128} + u^{256} + u^{512} + u^{1024} + \cdots$$

More generally, the full susceptibility series reduces to algebraic series mod. 2^r.

Diagonals solutions of high order linear differential operators

As seen in "*Experimental mathematics on the magnetic susceptibility of the square lattice Ising model*", or "*High order Fuchsian equations for the square lattice Ising model:* $\chi^{(5)}$", by A J Guttmann et. al, the $\chi^{(n)}$'s are solutions of linear differential operators of quite large order, which factorize into **products** and **direct sums** of **many** factors:

$$\Big(\big(\ \big) \cdot \big(\ \big) \cdots \big(\ \big) \Big) \oplus \Big(\big(\ \big) \cdot \big(\ \big) \cdots \big(\ \big) \Big) \oplus \ \cdots$$

where each factor has highly selected function solutions: **elliptic functions**, **modular forms**, derivatives of modular forms, and other remarkable functions with modularity properties (Calabi-Yau but that's another story, ...).

At this step: Modular forms are just snob, posh elliptic functions.

In the following, we will focus on modular forms, modular curves, modular correspondences ... At this step, just see a modular form as an "automorphic function" $\Phi(x)$ for a "symmetry" $x \to y(x)$:

$$\Phi\Big(y(x)\Big) \;=\; \mathcal{A}(x) \cdot \Phi(x).$$

Z_2 in $\chi^{(3)}$ or $\chi^{(5)}$: a modular form

The solution of the linear differential operator Z_2 can be expressed in terms of the ${}_2F_1$ hypergeometric function **up to a modular invariant pull-back**:

$$\mathcal{S} \, = \, \Big(\Omega \cdot \mathcal{M}_x\Big)^{1/12} \, \times \, {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}]; [1]; \, \mathcal{M}_x\Big), \qquad \text{where:}$$

$$\Omega \, = \, \frac{1}{1728} \, \frac{(1-4\,x)^6 \, (1-x)^6}{x \cdot (1\,+3\,x\,+4\,x^2)^2 \, (1+2\,x)^6},$$

$$\mathcal{M}_x \, = \, 1728 \, \frac{x \cdot (1\,+3\,x\,+4\,x^2)^2 \, (1\,+2\,x)^6 \, (1-4\,x)^6 \, (1-x)^6}{(1\,+7\,x+4\,x^2)^3 \cdot P^3},$$

$$P \, = \, 1\,+237\,x\,+1455\,x^2+4183\,x^3+5820\,x^4+3792\,x^5+64\,x^6.$$

It is a **modular form**.

Be careful not any ${}_2F_1([\alpha, \, \beta], \, [1], \, p(x))$ corresponds to a modular form ...

Simple (automorphy) covariance is too simple: the full susceptibility of the Ising model

Remarkably long series expansion (5000 **coefficients !!!**) were obtained for the **low-temp. full susceptibility** of the Ising model

$$\tilde{\chi}_L(w) \;=\; 4\,w^4 \,+ 80\,w^6 \,+ 1400\,w^8 \,+ 23520\,w^{10} \,+ 388080\,w^{12} \\ +6342336\,w^{14} \,+ 103062976\,w^{16} \,+ 1668639424\,w^{18} \\ +26948549680\,w^{20} \;+\; \cdots$$

to be compared with the series for $\tilde{\chi}^{(2)}(w)$ namely :

$$\tilde{\chi}^{(2)}_L \;=\; 4\,w^4 \cdot {}_2F_1\Big([\frac{3}{2},\, \frac{5}{2}], [3],\, 16\,w^2\Big) \\ =\; 4\,w^4 \,+ 80\,w^6 \,+ 1400\,w^8 \,+ 23520\,w^{10} \,+ 388080\,w^{12} \\ +6342336\,w^{14} + 103062960\,w^{16} \,+ 1668638400\,w^{18} \\ +26948510160\,w^{20} \;+\; \cdots$$

Simple (automorphy) covariance is too simple: the full susceptibility of the Ising model

The hypergeometric function $\tilde{\chi}_L^{(2)}$ is not exactly of the automorphic form

$$\Phi\Big(y(x)\Big) \;=\; \mathcal{A}(x) \cdot \Phi(x).$$

The hypergeometric function $\tilde{\chi}_L^{(2)}$ can, in fact, be written as an order-one linear diff. operator $\mathcal{L}_1$ acting on a **modular form**:

$$\tilde{\chi}_L^{(2)} \;=\; -\frac{1}{12} \cdot \mathcal{L}_1 \,\cdot\, {}_2F_1\Big([\frac{1}{2}, \,\frac{1}{2}], [1], \, 16\, w^2\Big).$$

$$\mathcal{L}_1 \;=\; w \cdot (8\, w^2 \,-1) \cdot \frac{d}{dw} \,+\, 8\, w^2.$$

and we have a rather simple generalization of the previous automorphy relation:

Simple (automorphy) covariance is too simple: Renormalization group

$$\tilde{\chi}^{(2)}\Big(\frac{2\sqrt{k}}{1\,+\,k}\Big) \;=\; 4\cdot\,\frac{1\,+\,k}{k}\cdot\,\frac{d\,\tilde{\chi}^{(2)}(k)}{dk},$$

where: $$\tilde{\chi}^{(2)}(k) \;=\; \frac{k^4}{4^3}\cdot\,{}_2F_1\Big([\frac{3}{2},\,\frac{5}{2}],\;[3],\;k^2\Big).$$

Conversely, this relation can also be written as

$$\tilde{\chi}^{(2)}(k) \;=\; \frac{1}{4}\cdot\,\Big(k\cdot\,(k-1)\cdot\,\frac{d}{dk}\,+\,\frac{k^2\,+k\,+2}{k\,+1}\Big)\cdot\,\tilde{\chi}^{(2)}\Big(\frac{2\sqrt{k}}{1\,+\,k}\Big),$$

or, introducing the *inverse (descending)* **Landen transformation**:

$$\tilde{\chi}^{(2)}\Big(\frac{1\,-\sqrt{1\,-k^2}}{1\,+\sqrt{1\,-k^2}}\Big) = \Big(\frac{(k^2\,-2)\cdot\sqrt{1\,-k^2}\,+\,2}{4\,k^2}\Big)\cdot\tilde{\chi}^{(2)}(k)$$
$$+\;\frac{k^2\,-1}{4\,k}\cdot\,\Big(1\,-\,\sqrt{1\,-k^2}\Big)\cdot\,\frac{d\tilde{\chi}^{(2)}(k)}{dk}.$$

Landen transformation and renormalization group

The Landen transformation, or the inverse Landen transformation, is an exact generator of the renormalization group. An exact generator of the renormalization group must be compatible with the elliptic parametrization of the Ising (resp. Baxter) model. It must have the critical point, $k = 1$, as a fixed point, but, beyond must have $k = 0, 1, \infty$ preserved by such a generator. At this step, an infinite number of functions can be generator of the renormalization group. However one must impose that **the lattice of periods is actually compatible with such generator** of the renormalization group. The only such transformations are **isogenies of the elliptic curves**: they are algebraic transformations, corresponding to modular correspondences. We are going to study these **modular correspondences** in detail, in the following.

The Landen transformation is the simplest example of such a transformation. Naively one expects covariance like

$$\Phi\Big(\frac{2\sqrt{k}}{1\,+\,k}\Big) \;=\; \mathcal{A}(k)\cdot\,\Phi(k),$$

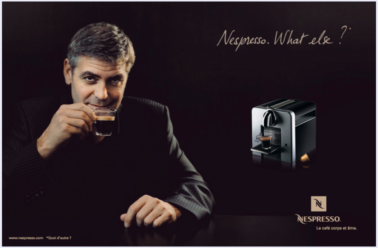
like, for instance, in the simplest example of elliptic function (and modular form ...), namely the (complete elliptic integral) EllipticK function:

$$K\Big(\frac{2\sqrt{k}}{1\,+\,k}\Big) \;=\; (1\,+k)\cdot\,K(k),$$

With $\tilde{\chi}^{(2)}$ we see that we have a slight generalization of these automorphy relations.

Modular forms

The Ising model seems to be nothing but the theory of **elliptic curves** and other modular forms, and also derivatives of modular forms, **what else ?**



Let us focus on **modular forms**, modular curves, modular equations, **modular correspondences**.

We need to understand modular forms, modular correspondences.

Modular Forms

Let us consider the second order linear differential operator

$$\frac{d^2}{dt^2} + \frac{(t^2 + 56\,t + 1024)}{t\cdot(t+16)\,(t+64)}\cdot\frac{d}{dt} - \frac{240}{t\cdot(t+16)^2\,(t+64)}.$$

which has the (modular form) solution:

$$\begin{aligned} &{}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \ 1728\,\frac{t}{(t+16)^3}\Big) \\ &= \ 2\cdot\Big(\frac{t+256}{t+16}\Big)^{-1/4}\cdot \ {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \ 1728\,\frac{t^2}{(t+256)^3}\Big). \end{aligned}$$

This looks like **one** identity: in fact it is an **infinite number of identities**.

Fundamental modular curve.

The **two pull-backs** in the previous modular form

$$x \;=\; \frac{t}{(t+16)^3}, \qquad y \;=\; \frac{t^2}{(t+256)^3} \;=\; x\Big(\frac{2^{12}}{t}\Big),$$

are related by a simple *involution* $t \quad \longleftrightarrow \quad 2^{12}/t$, and correspond to a *rational parametrization* of the **modular curve**:

$$\begin{aligned}
&1574640000000000 \cdot x^3 y^3 \;-8748000000 \cdot x^2 y^2 \cdot (x+y) \\
&\quad +10125 \cdot x\,y \cdot (16\,x^2 - 4027\,x\,y + 16\,y^2) \\
&\qquad -(x+y)\cdot(x^2 + 1487\,x\,y + y^2) \;+ x\,y \;=\; 0.
\end{aligned}$$

Let us introduce **another** *rational parametrization* where the elliptic function parametrization of the Ising (resp. Baxter model) plays a crucial role, thus underlining the Landen transformation as an exact generator of the renormalization group.

Isogenies, Landen transformations, modular curve.

We will denote k the modulus of the elliptic functions in the parametrization of the Ising (resp. Baxter model), and $j(k)$ the j-invariant of the corresponding elliptic curve.
The previous modular curve has **another rational parametrization**

$$x \, = \, \frac{1}{j(k)}, \qquad y \, = \, \frac{1}{j(k_L)} \qquad \text{where} \qquad k_L \, = \, \frac{2\sqrt{k}}{1\, +k}$$

$$j(k) = \, 256 \cdot \frac{(1-k^2+k^4)^3}{(1-k^2)^2 \cdot k^4}, \quad j\Big(\frac{2\sqrt{k}}{1\,+k}\Big) \, = \, 16 \cdot \frac{(1+14\,k^2+k^4)^3}{(1-k^2)^4 \cdot k^2}$$

These two *rational parametrizations* are actually related by the following change of variables:

$$t \, = \, 256 \cdot \frac{k^2}{(k^2-1)^2} \quad \text{or:} \quad 16 \cdot \frac{(k^2-1)^2}{k^2} \quad \text{i.e:} \quad k \rightarrow \frac{1\, -k}{1\, +k}$$

Isogenies, Landen transformations, Modular curve.

The **modular curve** is thus an **algebraic representation** of the **Landen transformation** $k \rightarrow 2\sqrt{k}/(1+k)$, **and in the same time**, since this curve is $x \leftrightarrow y$ symmetric, of its **compositional inverse**, the **inverse Landen transformation**. The algebraic function $y = y(x)$ is a **multivalued** function, but we can single out the **series expansions**:

$$y_2 = x^2 + 1488\,x^3 + 2053632\,x^4 + 2859950080\,x^5 + \cdots$$

and its compositional inverse series (with $\omega^2 = 1$):

$$y_{1/2} = \omega \cdot x^{1/2} - 744 \cdot x^{2/2} + 357024 \cdot \omega \cdot x^{3/2} + \cdots$$

More correspondences.

$$
{}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \, 1728 \, \frac{t}{(t+27)\cdot(t+3)^3}\Big)
$$

$$
= \, A(t)\cdot \, {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \, 1728 \, \frac{t^3}{(t+27)\cdot(t+243)^3}\Big),
$$

where $A(t)$ is an involved algebraic function. The elimination of t between the two pullbacks

$$
x \, = \, \frac{t}{(t+27)\cdot(t+3)^3}, \quad y \, = \, \frac{t^3}{(t+27)\cdot(t+243)^3} \, = \, x\Big(\frac{3^6}{t}\Big),
$$

gives **another modular curve** $P_3(x, y) \, = \, P_3(y, x) \, = \, 0$

$$
y_3 \, = \, x^3 \, + 2232\,x^4 \, + 3911868\,x^5 \, + 6380013816\,x^6 \, + \, \cdots
$$

and its compositional inverse series (with $\omega^3 = 1$):

$$
y_{1/3}(\omega, x) \, = \, \omega\cdot x^{1/3} - 744\cdot\omega^2\cdot x^{2/3} + 356652\cdot x^{3/3} \, + \, \cdots
$$

Simple covariance: modular form.

Revisiting the previous ${}_2F_1$ identities, corresponding to the modular correspondence series, one can write:

$$ {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728\, y\Big) \; = \; \mathcal{A}(x) \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728\, x\Big), $$

where $\mathcal{A}(x)$ is an *algebraic function*. The relation $P(y, x) = 0$ is one of the previous **modular equations**. Introducing

$$ F(x) \; = \; x \cdot (1 - 1728 \cdot x)^{1/2} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot x\Big)^2, $$

the previous covariance relation on ${}_2F_1$ can, in fact, be written

$$ \lambda \cdot F(y) = F(x) \cdot \frac{dy}{dx} \qquad \text{or:} \qquad \lambda \cdot \frac{dx}{F(x)} = \frac{dy}{F(y)}. $$

Modular form: Schwarzian condition.

Eliminating $\mathcal{A}(x)$ one gets the following Schwarzian **differentially algebraic equation** condition:

$$W(x) \quad - W(y(x)) \cdot y'(x)^2 \quad + \{y(x), x\} \quad = \quad 0,$$

where

$$W(x) \;=\; -\frac{1}{2} \cdot \frac{1 \,-1968\,x \,+2654208\,x^2}{x^2 \cdot (1 \,-1728\,x)^2},$$

and where $\{y(x), x\}$ denotes the **Schwarzian derivative**:

$$\{y(x), x\} \;=\; \frac{y'''(x)}{y'(x)} \,-\, \frac{3}{2} \cdot \Big(\frac{y''(x)}{y'(x)}\Big)^2$$

This non-trivial condition coincides exactly with one of the conditions G. Casale obtained in a classification of Malgrange's $\mathcal{D}$-envelope and $\mathcal{D}$-groupoids on $\mathbb{P}_1$.

The Schwarzian equation **encapsulates all the modular equations of the theory of elliptic curves**: the **infinite number of correspondences**, $x \to x^n + \cdots$, are **actually solutions of the same Schwarzian equation**.

Are the solutions of the Schwarzian equations only modular correspondences ?

Beyond the modular correspondence $x \to x^n + \cdots$, a one-parameter series.

A **one-parameter** series is actually solution of the Schwarzian equation:

$$y(a, x) = a \cdot x \quad - 744 \cdot a \cdot (a-1) \cdot x^2$$
$$+36 \cdot a \cdot (a-1) \cdot (9907\, a - 20845) \cdot x^3$$

$$-32 \cdot a \cdot (a-1) \cdot (4386286\, a^2 - 20490191\, a + 27274051) \cdot x^4$$
$$+6 \cdot a \cdot (a-1) \cdot (8222780365\, a^3 - 61396351027\, a^2$$
$$+171132906629\, a - 183775457147) \cdot x^5 \quad + \cdots$$

One-parameter family of commuting series:

$$y\Big(a, y(a', x)\Big) = y(a\, a', x).$$

In the $a \to 0$ limit

$$\tilde{Q}(x) = \lim_{a \to 0} \frac{y(a, x)}{a} = x + 744\,x^2 + 750420\,x^3 + 872769632\,x^4 + 1102652742882\,x^5 + \cdots$$

In the $a \to \infty$ limit

$$\tilde{X} = \lim_{a \to \infty} y\Big(a, \frac{x}{a}\Big) = x - 744\,x^2 + 356652\,x^3 - 140361152\,x^4 + 49336682190\,x^5 + \cdots$$

$$y(a, x) = \tilde{X}\Big(a \cdot \tilde{Q}(x)\Big) \quad \text{or:} \quad \tilde{Q}\Big(y(a, x)\Big) = a \cdot \tilde{Q}(x).$$

Since $y(1, x) = x$, one deduces that $\tilde{X}$ **must be the compositional inverse** of $\tilde{Q}$. $\tilde{X}$ and $\tilde{Q}$ are differentially algebraic: they are solutions of some Schwarzian equations.

Similarly the two previous algebraic series, y_2 and $y_{1/2}$, can be written respectively:

$$\tilde{X}\Big(\tilde{Q}(x)^2\Big) \qquad \text{and:} \qquad \tilde{X}\Big(\omega \cdot \tilde{Q}(x)^{1/2}\Big).$$

$$2 \cdot y_2 \cdot (1 - 1728 \cdot y_2)^{1/2} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot y_2\Big)^2$$
$$= \quad x \cdot (1 - 1728 \cdot x)^{1/2} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot x\Big)^2 \cdot \frac{dy_2}{dx}.$$

$$2 \cdot F(y_2) \; = \; F(x) \cdot \frac{dy_2}{dx}.$$

More generally, the modular correspondence series read:

$$\tilde{X}\Big(\tilde{Q}(x)^N\Big) \qquad \text{and:} \qquad \tilde{X}\Big(\omega \cdot \tilde{Q}(x)^{1/N}\Big).$$

N-th root values of the parameter

Note that $y(a, x)$ for $a^N = 1$ is such that its N-th compositional iterate is the identity. Such a series must be "special". Let us consider the modular curve Γ_N having $\tilde{X}\Big(\tilde{Q}(x)^N\Big)$ and $\tilde{X}\Big(\omega \cdot \tilde{Q}(x)^{1/N}\Big)$ as solution series. In the nome $\tilde{Q}(x)$ Γ_N amounts to writing in the same time $\tilde{Q} \rightarrow \tilde{Q}^N$ and $\tilde{Q} \rightarrow \omega \cdot \tilde{Q}^{1/N}$. Performing the resultant of Γ_N with itself, in order to get Γ_{N^2}, amounts to performing $\tilde{Q} \rightarrow \tilde{Q}^N \rightarrow \tilde{Q}^{N^2}$, $\tilde{Q} \rightarrow \omega \cdot \tilde{Q}^{1/N} \rightarrow \tilde{Q}^{1/N^2}$ but also $\tilde{Q} \rightarrow \tilde{Q}^N \rightarrow \omega \cdot (\tilde{Q}^N)^{1/N}$, namely $\tilde{Q} \rightarrow \omega \cdot \tilde{Q}$ with $\omega^N = 1$. In other words $y(a, x)$ for $a^N = 1$ is not only a series of order-N with respect to the composition of function, it is an algebraic series, solution of a modular curve: it is a correspondence. We thus have an **infinite number of algebraic series** solutions of the Schwarzian equation.

More generally

In the $a \to 1$ limit, let us denote $\epsilon = a - 1$. The one-parameter series $y(x) = y(a, x)$ can, thus, be seen as an ϵ-expansion:

$$y(a, x) = x + \sum_{n=1}^{\infty} \epsilon^n \cdot B_n(x),$$

where $B_1(x) = F(x)$

$$B_2(x) = \frac{1}{2} \cdot F(x) \cdot \Big(\frac{dB_1(x)}{dx} - 1\Big).$$

$$B_3(x) = \frac{1}{3} \cdot F(x) \cdot \Big(\frac{dB_2(x)}{dx} - \frac{dB_1(x)}{dx} + 1\Big),$$

$$B_4(x) = \frac{1}{4} \cdot F(x) \cdot \Big(\frac{dB_3(x)}{dx} - \frac{dB_2(x)}{dx} + \frac{dB_1(x)}{dx} - 1\Big),$$

More generally.

$$(n+1) \cdot \ B_{n+1} \ \ + n \cdot \ B_n \ = \ \ F(x) \cdot \ \frac{dB_n(x)}{dx},$$

$$\frac{\partial \sum_n \ B_{n+1} \ \cdot \ \epsilon^{n+1}}{\partial \epsilon} \ + \epsilon \cdot \ \frac{\partial \sum_n \ B_n \ \cdot \ \epsilon^n}{\partial \epsilon}$$
$$= \ \ F(x) \cdot \ \Big(\frac{\partial \sum_n \ B_n(x) \cdot \ \epsilon^n}{\partial x} \Big),$$

$$a \cdot \ \frac{\partial y(a, x)}{\partial a} \ = \ \ F(x) \cdot \ \frac{\partial y(a, \ x)}{\partial x}.$$

The series $y(a, \ x)$ is solution of the Schwarzian equation with:

$$W(x) \ = \ \ \frac{F''(x)}{F(x)} \ - \frac{1}{2} \cdot \ \Big(\frac{F'(x)}{F(x)}\Big)^2.$$

This remains valid **for any** function $F(x)$.

Polynomial example

$$F(x) \; = \; x \cdot (1 - 373 \cdot x) \cdot (1 - 371 \cdot x).$$

$$W(x) \; = \; -\frac{1}{2} \cdot \frac{1 - 830298\, x^2 + 411827808\, x^3 - 57449564067\, x^4}{x^2 \cdot (1 - 373\, x)^2 \cdot (1 - 371\, x)^2}.$$

$$y(a,\, x) \; = \; a \cdot x \; - 744 \cdot a \cdot (a-1) \cdot x^2$$
$$+\frac{1}{2} \cdot a \cdot (1245455\, a - 968689) \cdot (a-1) \cdot x^3 \; + \; \cdots$$

$$\tilde{Q}(x) \; = \; x \cdot \frac{(1 - 371\, x)^{371/2}}{(1 - 373\, x)^{373/2}}$$

$$a \cdot \tilde{Q}(x) \; = \; \tilde{Q}\Big(y(a,\, x)\Big), \qquad y(a,\, x) \; = \; \tilde{X}\Big(a \cdot \tilde{Q}(x)\Big).$$

Finding the (simple) algebraic expressions of $\tilde{Q}(x)$ and $\tilde{X}(x)$ from large series expansions is quite hard !

Polynomial truncation of the hypergeometric.

$$F(x) = x - 744\,x^2 - 393768\,x^3 = x \cdot \Big(1 - p \cdot x\Big) \cdot \Big(1 - q \cdot x\Big),$$

with:

$$p = 372 + 6 \cdot 14782^{1/2}, \qquad q = 372 - 6 \cdot 14782^{1/2},$$

$$\tilde{Q}(x) = \frac{x \cdot (1 - p \cdot x)^{p/(q-p)}}{(1 - q \cdot x)^{q/(q-p)}} = x + 744\,x^2 + 750420\,x^3$$

$$+753621408\,x^4 + 782312864472\,x^5 + \frac{4097211834177216}{5}\,x^6 + \cdots$$

$\tilde{Q}(x)$ is *D-finite*, but the linear differential operator is **not globally nilpotent** and the series for $\tilde{Q}(x)$ is **not globally bounded**.

Differentially algebraic series.

With $y(a, x)$ associated with canonical correspondences, we had an infinite number of algebraic functions for $y(a, x)$ with $a^N = 1$, and an infinite number of differentially algebraic series with integer coefficients for $y(a, x)$ with $a \in \mathbb{Z}$.

The λ-extensions of the two-point correlation functions of the square Ising model have very similar properties. These series are solutions of **(sigma-form of) Painlevé equations**, they are, thus, differentially algebraic. For selected values ($\lambda = \cos(\pi\, m/n)$, which can also be written as N-th root of unity) these series become algebraic series, and for integer values of λ we have differentially algebraic series with integer coefficients.

We thus have the **same remarkable properties** with **different kinds** of differentially algebraic series (Schwarz versus Painlevé, Replicable functions versus isomonodromy).

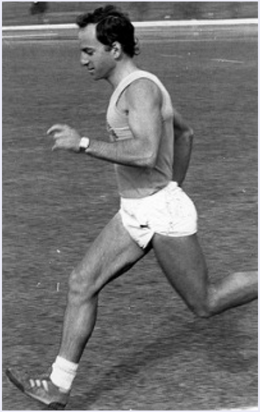
So many people have a defeatist attitude towards non-linear differential equations: they think nothing can be done on non-linear differential equations.

This is defeatist nonsense

As far as **differentially algebraic functions** are concerned:

Making further progress in differentially algebraic series will be a huge challenge. It will require many acts of great courage or skill, almost a marathon feat.

But we are strong !



The road is hard, but I am strong (Jean-Paul Sartre's Roads to Freedom trilogy, sung by Georgia Brown).
La route est dure mais qu'elle est belle. Le but est difficile mais qu'il est grand ! Allons ! Le départ est donné. Allocution radiodiffusée du Général de Gaulle (13 mai 1958).
THE END (of this talk)

Additional slides to answer the questions the public did not ask.

Modular form, Eisenstein series

$$E_4 = 1 + 240 \sum_{n=1}^{\infty} n^3 \cdot \frac{q(\tau)^n}{1 - q(\tau)^n} = {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{1728}{j(\tau)}\Big)^4$$

In terms of k the modulus of the elliptic functions, the E_4 **Eisenstein series** can also be written as:

$$\begin{aligned} &{}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{27}{4} \frac{k^4 \cdot (1 - k^2)^2}{(k^4 - k^2 + 1)^3}\Big)^4 \\ &\quad = (1 - k^2 + k^4) \cdot {}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], [1], k^2\Big)^4. \end{aligned}$$

$$\begin{aligned} E_6 &= (1 + k^2) \cdot (1 - 2k^2) \cdot \Big(1 - \frac{k^2}{2}\Big) \cdot {}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], [1], k^2\Big)^6 \\ &= (1 + k^2) \cdot (1 - 2k^2) \cdot \Big(1 - \frac{k^2}{2}\Big) \\ &\quad \times (1 - k^2 + k^4)^{-3/2} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{27}{4} \frac{k^4 \cdot (1 - k^2)^2}{(k^4 - k^2 + 1)^3}\Big)^6. \end{aligned}$$

A pedagogical example of diagonal of rational functions.

Let us consider the **rational function of three complex variables** $\mathcal{F} = 1/(1 - z_2 - z_3 - z_1 z_2 - z_1 z_3)$. Its diagonal reads:

$$1 \;\; + 4z \;\; + 36z^2 \;\; + 400z^3 \;\; + 4900z^4 + 63504z^5 \;\; + \;\; \cdots$$

which is nothing but the **complete elliptic integral** (first kind):

$$\sum_{m \geq 0} \binom{2m}{m}^2 \cdot z^m \;\; = \;\; {}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], [1], 16\, z\Big)$$

This diagonal **modulo any prime** reduces to an **algebraic function**, for instance:

$$Diag(\mathcal{F}) \mod 7 \; =$$
$$= \;\; 1 \;\; + 4\, z \; + z^2 + z^3 + 4z^7 + 2z^8 + 4z^9 \;\; + \cdots$$
$$= \;\; \frac{1}{\sqrt[6]{1 + 4z + z^2 + z^3}} \mod 7.$$

Another example of diagonal of rational functions.

A less obvious example corresponds to the **modular form**:

$$\Big(\frac{1}{1 - z_1 - z_2 - z_3 - z_1 z_2 - z_2 z_3 - z_3 z_1 - z_1 z_2 z_3}\Big)$$
$$= \quad \frac{1}{1-z} \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1]; \frac{54\,z}{(1-z)^3}\Big).$$

Such **diagonals of rational functions** are **highly selected functions**: modulo **any prime** they reduce to **algebraic functions**.

They can be seen as the **simplest (transcendental) generalisations of algebraic functions**.

The integrands of the $\chi^{(n)}$ n-fold integral of the Ising model have a **multi-Taylor expansion** and are, thus, **diagonals of a rational function**.

Ising n-fold integrals : $\chi^{(5)}$

The five-particle contribution $\tilde{\chi}^{(5)}$ of the susceptibility of the Ising model is solution of an order-33 linear differential operator which has a **direct-sum** factorization (DFactorLCLM in Maple): the selected linear combination

$$\tilde{\chi}^{(5)} \; - \frac{1}{2}\,\tilde{\chi}^{(3)} \; + \frac{1}{120}\,\tilde{\chi}^{(1)},$$

is solution of an order-29 (globally nilpotent) linear differential operator

$$L_{29} \; = \quad L_5 \cdot L_{12} \cdot \tilde{L}_1 \cdot L_{11},$$

where:

$$L_{11} \; = \quad (Z_2 \cdot N_1) \oplus V_2 \oplus (F_3 \cdot F_2 \cdot L_1^s).$$

Ising n-fold integrals : $\chi^{(6)}$

Similarly $\tilde{\chi}^{(6)}$ is solution of an order-52 linear differential operator which has a **direct-sum** factorization: the selected linear combination

$$\tilde{\chi}^{(6)} \; - \frac{2}{3}\tilde{\chi}^{(4)} \; + \frac{2}{45}\tilde{\chi}^{(2)},$$

is solution of an order-46 (globally nilpotent) linear differential operator

$$L_{46} \;\; = \;\; L_6 \cdot L_{23} \cdot L_{17},$$

where:

$$L_{17} \; = \;\; \tilde{L}_5 \oplus L_3 \oplus \, (L_4 \cdot \tilde{L}_3 \cdot L_2),$$

$$\tilde{L}_5 \; = \;\; \left(\frac{d}{dx} - \frac{1}{x} \right) \oplus \Big(L_{1,3} \cdot \, (L_{1,2} \oplus L_{1,1} \oplus D_x) \Big).$$

The "Quarks" in $\chi^{(5)}$ and $\chi^{(6)}$

Quasi-trivial order-one (globally nilpotent) linear differential operators: $\tilde{L}_1, \ N_1, \ L_1^s, \ L_{1,n} \quad \longrightarrow \quad D_x \ - \frac{1}{N} \cdot \frac{d\ln(R(x))}{dx}$

$V_2, \ L_2, \ L_3, \ L_5$ and L_6 are respectively equivalent (homomorphic) to L_K, to the symmetric square of L_K and to the *symmetric fourth and fifth power* of L_K, where L_K is the second order linear differential operator annihilating the **complete elliptic integral** $K \, = \, _2F_1([1/2, \, 1/2], \, [1], \, k^2)$.

$F_2, \ F_3, \ \tilde{L}_3$ do correspond to **modular forms**: F_3 and $\tilde{L}_3$ are homomorphic to the symmetric square of order-two operators associated with the (fundamental) **modular curve** $X_0(2)$, and F_2 is related to Z_2 (and thus h_6, Apéry, ...).

Remains to understand the "very nature" of:

L_4 and: $L_{12}, \ L_{23}$

L_4 is a Hadamard product of two elliptic curves:

it is a Calabi-Yau operator !

Seeking for ${}_4F_3$ hypergeometric functions up to homomorphisms, and assuming an **algebraic pull-back** with the *square root extension*, $(1 - 16 \cdot w^2)^{1/2}$, we actually found that the solution of L_4 *can be expressed in terms of a selected* ${}_4F_3$

$$
{}_4F_3\Big([1/2,\, 1/2,\, 1/2,\, 1/2], [1, 1, 1];\; z\Big)
$$

$$
= \quad {}_2F_1\Big([1/2,\, 1/2], [1];\; z\Big) \star {}_2F_1\Big([1/2,\, 1/2], [1];\; z\Big),
$$

where: $$z \;=\; \Big(\frac{1 \,+\, \sqrt{1 \,-\, 16 \cdot w^2}}{1 \,-\, \sqrt{1 \,-\, 16 \cdot w^2}}\Big)^4 \;=\; k^4$$

where the pull-back z *is nothing but* the fourth power of the **modulus** k of the elliptic functions !

The $\chi^{(n)}$'s are diagonal of rational functions.

Let us consider the series of $\tilde{\chi}^{(3)}/8/w^9$

$$1 \; + 36\, w^2 \; + 4\, w^3 \; + 884\, w^{13} \; + 196\, w^5 \; + 18532\, w^6 \; + \; \cdots$$

Let us now consider this very series **modulo the prime** $p \,=\, 2$. It reads the **lacunary** series

$$1 \; + w^8 \; + w^{24} \; + w^{56} \; + w^{120} \; + w^{248} \; + w^{504} \; + w^{1016} \; + \cdots \, ,$$

In fact,*modulo the prime* $p \,=\, 2$, $H(w) \,=\, \tilde{\chi}^{(3)}/8$ is, actually, an **algebraic function**, solution of the quadratic equation:

$$H(w)^2 \; + w \cdot H(w) \; + w^{10} \quad = \quad 0 \qquad \mod 2.$$

Modulo $p = 3$. Indeed, $H(w)$ satisfies a polynomial equation of degree nine (the p_n are polynomials of degree less that 63):

$$p_9 \cdot H(w)^9 \; + \; w^6 \cdot p_3 \cdot H(w)^3 \; + \; w^{10} \cdot p_1 \cdot H(w) \; + p_0.$$

Elimination of the automorphic prefactor $\mathcal{A}(x)$

$$\mathcal{A}(x) \cdot {}_2F_1\Big([\alpha,\, \beta],\, [\gamma],\, x\Big) \,=\, {}_2F_1\Big([\alpha,\, \beta],\, [\gamma],\, y(x)\Big),$$

The Gauss hypergeometric function ${}_2F_1([\alpha,\, \beta],\, [\gamma],\, x)$ is solution of the second order linear differential operator of wronskian $w(x)$:

$$\Omega \,=\, \frac{d^2}{dx^2} \,+\, A(x) \cdot \frac{d}{dx} \,+\, B(x), \qquad B(x) \,=\, \frac{\alpha\, \beta}{x \cdot (x\, -1)},$$

$$A(x) \,=\, \frac{(\alpha+\beta+1) \cdot x \,\, -\gamma}{x \cdot (x\, -1)} \,=\, -\frac{w'(x)}{w(x)},$$

A straightforward calculation gives the algebraic function $\mathcal{A}(x)$ in terms of the **algebraic function pullback** $y(x)$:

$$\mathcal{A}(x) \,=\, \Big(\frac{w(y(x))}{w(x)} \cdot y'(x)\Big)^{-1/2}$$

The set of solutions of the Schwarzian condition has a closure property for composition of functions

$$\mathcal{A}(x) \cdot {}_2F_1\Big([\alpha,\, \beta],\, [\gamma],\, x\Big) \;=\; {}_2F_1\Big([\alpha,\, \beta],\, [\gamma],\, y(x)\Big),$$

$$\mathcal{B}(x) \cdot {}_2F_1\Big([\alpha,\, \beta],\, [\gamma],\, x\Big) \;=\; {}_2F_1\Big([\alpha,\, \beta],\, [\gamma],\, z(x)\Big),$$

$$\mathcal{B}(y(x)) \cdot {}_2F_1\Big([\alpha,\, \beta],\, [\gamma],\, y(x)\Big) \;=\; {}_2F_1\Big([\alpha,\, \beta],\, [\gamma],\, z(y(x))\Big)$$

$$\;=\; \mathcal{B}(y(x)) \cdot \mathcal{A}(x) \cdot {}_2F_1\Big([\alpha,\, \beta],\, [\gamma],\, x\Big)$$

The set of solutions of the Schwarzian condition *must have a closure property for composition of functions*. It works: see the Schwarzian derivative of a composition of function:

$$\{z(y(x)),\, x\} \;=\; \{z(y),\, y\}_{y=y(x)} \cdot y'(x)^2 \;+\; \{y(x),\, x\}$$

Non-holonomic functions ratio of holonomic functions

Along this line it is fundamental to recall that the **ratio** (not the product !) of **two holonomic** functions is **non-holonomic**

$$\frac{d^2y}{dx} + R(x) \cdot y \; = \; 0, \qquad \tau(x) \; = \; \frac{y_1}{y_2}, \qquad \{\tau(x), \, x\} \; = \; 2\, R(x).$$

The **Chazy III equation** is a third-order **non-linear** differential equation (it can also be rewritten using a **Schwarzian derivative**) that has a **natural boundary** for its solutions:

$$\frac{d^3y}{dx^3} \; = \; 2\, y \frac{d^2y}{dx^2} - 3 \left(\frac{dy}{dx}\right)^2.$$

It has the **quasi-modular form** Eisenstein series E_2 has a solution

$$y \; = \; \frac{1}{2} \cdot \frac{\Delta'}{\Delta} \; = \; \frac{1}{2} \cdot E_2$$

where Δ is a selected holonomic function: a **modular form**.

Schwarzian derivative and natural boundary

It can be rewritten in terms of a Schwarzian derivative:

$$f^{(4)} \;=\; 2\, f'^2 \cdot \{f,\, x\} \;=\; 2\, f'\, f''' - 3\, f''^{\,2} \qquad \text{with:} \quad y \;=\; \frac{df}{dx}.$$

It was introduced by Jean Chazy (1909, 1911) as an example of a third-order differential equation with a movable singularity that has a **natural boundary** for its solutions. It is also worth recalling the **Halphen-Ramanujan differential system**:

$$P' \;=\; \frac{P^2 - Q}{12}, \qquad Q' \;=\; \frac{P\,Q - R}{3}, \qquad R' \;=\; \frac{P\,R - Q^2}{2},$$

where $P = E_2$, $Q = E_4$, $R = E_6$ and X' denotes here the homogeneous derivative $q \cdot \frac{dX}{dq}$, and E_n the Eisenstein series.