Meanders: Improved algorithm and new statistics

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Thanks to: NCI Australia. Lots and lots of CPU time.
Australian Research Council. Financial support.

”The mind of man is beneficent and noble only when it obeys truth.
As soon as it betrays truth, as soon as it ceases to revere truth,
as soon as it sells out, it becomes intensely diabolical.”

Hermann Hesse, The Glass Bead Game

Outline

1. Efficient counting algorithms
   - Meanders
   - Transfer matrix algorithm
   - Pruning
2. Asymptotic series analysis
3. New statistics
   - Average height at the midpoint
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   - Average number of arcs at the midpoint
   - Average nesting level at the midpoint

Efficient counting algorithms

The vast majority of counting problem have exponential complexity.

Typically the number of objects we wish to count grows like

$$c_n \sim A\mu^n n^\alpha,$$

where A, μ and α are model dependent constants.

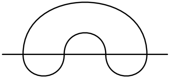
Objective: Design an efficient counting algorithm.

Direct counting always stuck with exponential complexity μ.

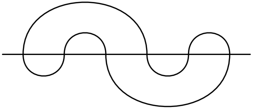
Can we find more efficient algorithms with complexity $\lambda < \mu$?

Meanders

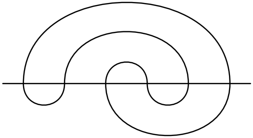
A closed plane meander of order n is a closed self-avoiding curve intersecting an infinite line $2n$ times.



$n = 2$



$n = 3$



$n = 3$

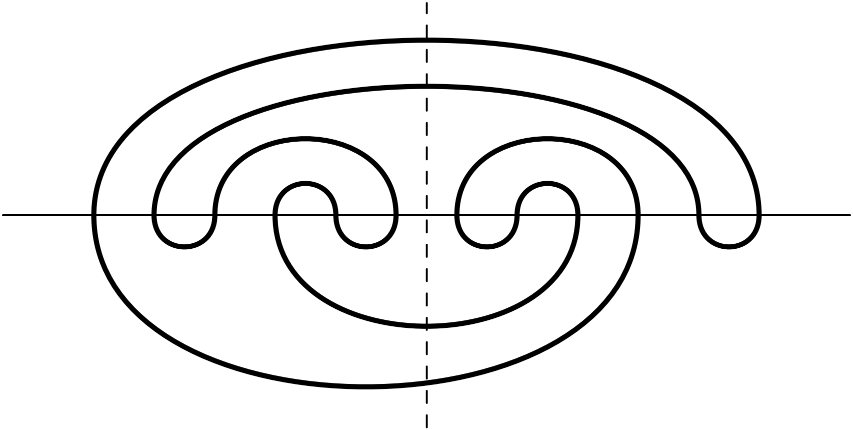
The number of closed meanders M_n is expected to grow exponentially, with a sub-dominant term given by a critical exponent,

$$M_n \sim C\mu^n n^{-\alpha},$$

where $\mu \approx 12.26$ and $\alpha = (29 + \sqrt{145})/12 = 3.4201328\ldots$.

Value for α from Di Francesco, Golinelli and Guitter (1997).

Transfer matrix algorithm



Signature: $\{2, 0011\}$ (Past)
Signature: $\{2, 0101\}$ (Future)

Computational complexity

The TM configurations are non-intersecting arcs on $2n$ points with a ‘dividing’ line placed in $2n+1$ possible positions.

Non-intersecting arcs is a well-known (much loved?) combinatorial problem.

The number of arc configurations is given by the Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}.$$

Hence the computational complexity is exponential with (at worst) $\lambda = 4$.

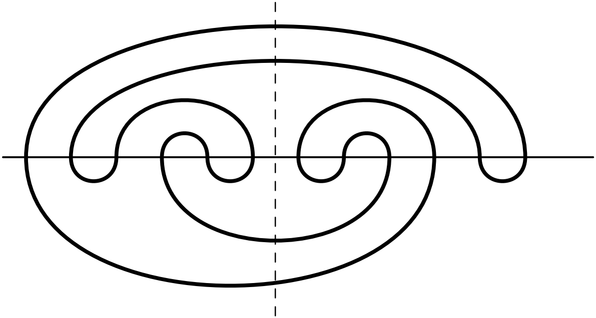
This a major improvement on direct counting which had $\lambda \approx 12.26$

The draw-back of the TM algorithm is that memory also grows exponentially.

But we can do even better!

Pruning

Not all possible signatures in a calculation with $2n$ crossings are needed.
Let us have a second look at a closed meander.



Any vertical line cutting the meander results in two sets of arcs.

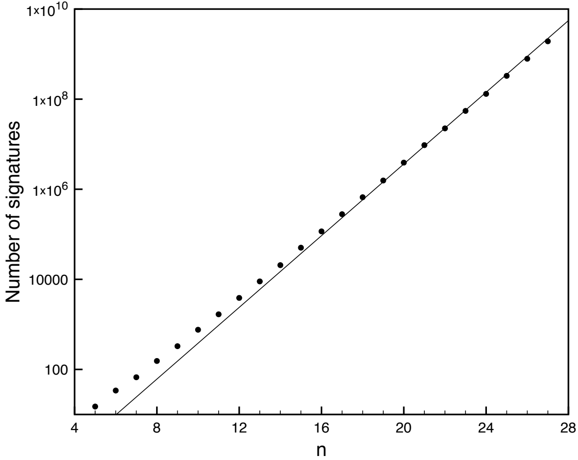
Those on right tells us how the arc-ends must be connected. $\{2, 0101\}$

Spanning arcs add 1 crossing other arcs add 2 crossings.

For any given signature we have to add at least n_a additional crossings.

If we inserted n_c crossings and $n_c + n_a > 2n$, we can discard the signature.

Computational complexity



The straight line has growth 2.5^n so it seems likely that $\lambda \approx 2.5$.

Asymptotic series analysis

When the number of objects we wish to count grows like

$$c_n \sim A\mu^n n^\alpha,$$

then the associated generating function will behave as

$$C(x) = \sum_{n=1}^{\infty} c_n x^n \sim A(x)(1-\mu x)^{-\alpha-1},$$

and hence has a singularity at $x_c = 1/\mu$ with exponent $\alpha - 1$.

Objective: Calculate accurate estimates for μ, α and A.

Differential equations for the generating function to estimate μ and α.

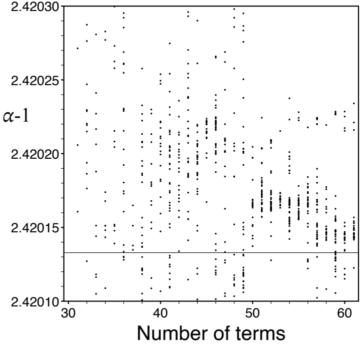
Direct fitting to the sequence c_n to estimate A.

Analysis of the closed meander series

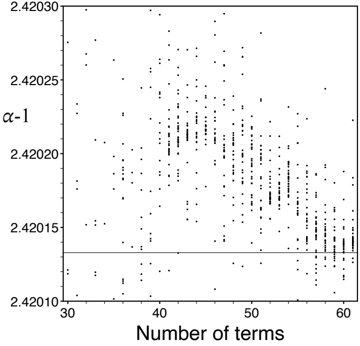
L	Second order DA		Third order DA	
	x_c^2	$\alpha - 1$	x_c^2	$\alpha - 1$
0	0.081547036(16)	2.420349(70)	0.0815470545(47)	2.42050(10)
1	0.081547045(13)	2.420340(20)	0.0815470570(61)	2.42037(36)
2	0.081547042(17)	2.42027(21)	0.0815470579(27)	2.420563(82)
3	0.0815470570(65)	2.42039(19)	0.0815470570(29)	2.42055(17)
4	0.0815470517(83)	2.420436(73)	0.0815470576(32)	2.420558(75)
5	0.0815470519(65)	2.42043(13)	0.0815470573(34)	2.420546(85)

Exact value: $\alpha = (29 + \sqrt{145})/12 = 3.4201328\ldots$.

Analysis of the open meander series



Third order DA



Fifth order DA

Average height at the midpoint

To every meander we can associate upper and lower Dyck paths $\left(D_n^1(i), D_n^2(i)\right)$ where $D_n^1(i)$ is the height of the upper path at step i.

Conjecture: If M_n is a uniform random meander of size n and $\left(D_n^1(i), D_n^2(i)\right)$ are the associated random Dyck paths, then

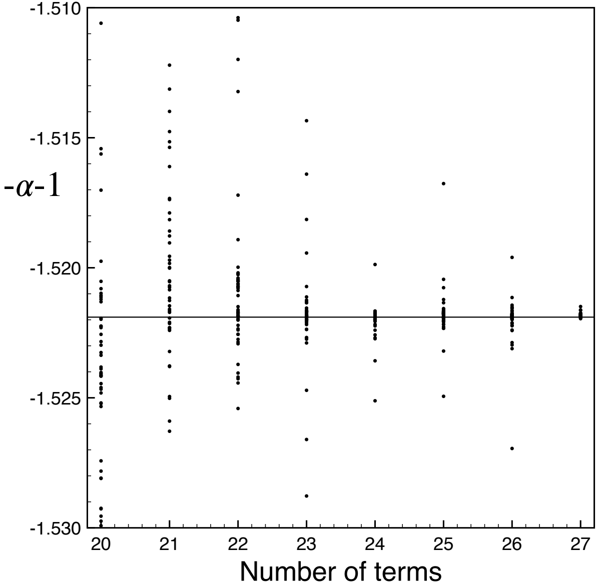
$$\mathbb{E}\left[D_n^1(n/2)\right] \sim C \cdot n^{\alpha} \quad \text{for} \quad n \to \infty,$$

where

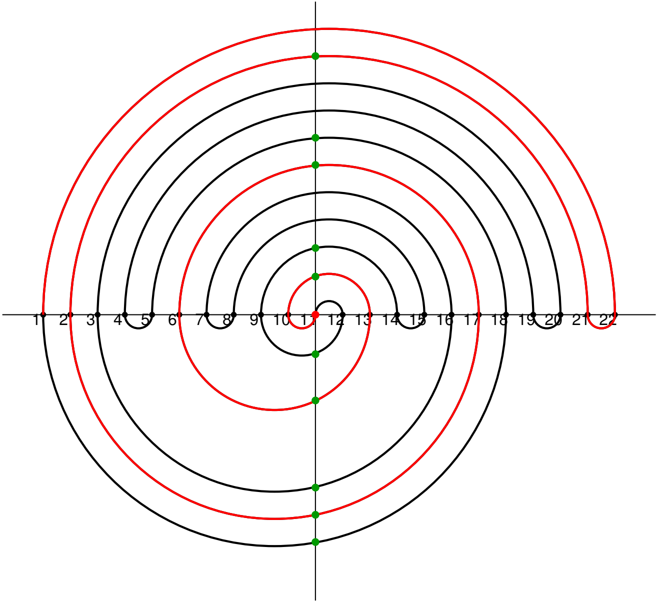
$$\alpha = \frac{\sqrt{29} - \sqrt{14}}{\sqrt{29} - \sqrt{5}} = 0.521898\ldots$$

Jacopo Borga, Ewain Gwynne and Xin Sun, Permutons, meanders, and SLE-decorated Liouville quantum gravity, arXiv:2207.02319, accepted for publication in J. Euro. Math. Soc.

Analysis of the series



Colour changes



Average number of colour changes

Conjecture: Set

$$\mathcal{M}_n := \{\text{Meanders of size } n\},$$

and for all $M_n \in \mathcal{M}_n$ denote by $\mathrm{CC}(M_n)$ the number of colour changes of M_n when marked at the mid point. Then

$$\sum_{M_n \in \mathcal{M}_n} \mathrm{CC}(M_n) \sim C \cdot |\mathcal{M}_n| \, n^{\beta} \quad \text{for } n \to \infty,$$

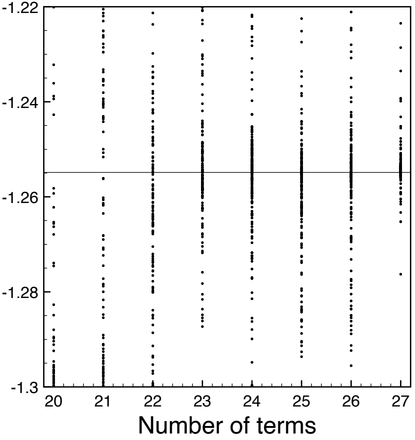
where

$$\beta = \frac{\sqrt{29} - \sqrt{23}}{\sqrt{29} - \sqrt{5}} = 0.187143\ldots$$

Average number of arcs at the midpoint

This sequence is identical to the one of the average height.

Average nesting level at the midpoint



Line is for exponent value $\frac{\sqrt{29}-\sqrt{21}}{\sqrt{29}-\sqrt{5}} = 0.254863\ldots$, but I take absolutely no responsibility for this guesstimate.

Final thought bomb

"An oasis of horror in a desert of boredom."

Charles Baudelaire

Some random quotes

“I would have been better off if I’d taken those six hundred yen and used them to set myself up as a milkman or some occupation like that instead of going to the Institute of Physical Sciences and learning something as useless as mathematics.”

Botchan, Natsume Sōseki

“. . . faint traces of the halitosis he had been unable to shake off for decades stung his nostrils. He was rather attached to his own bad breath by now. The nauseating smell was totally unconnected to any vital life force, and might be more closely described as the rotten stench of academia.”

Beautiful star, Yokio Mishima

“Something unignorable lurks in whatever passes our understanding, and there is something inherently noble in that which we cannot measure. For which reason laymen are loud in their praises of matters they do not understand and scholars lecture unintelligibly on points as clear as day. This lesson is daily demonstrated in our universities, where incomprehensible lectures are both deeply respected and popular, while those whose words are easily understood are shunned as shallow thinkers.”

I am a cat, Natsume Sōseki