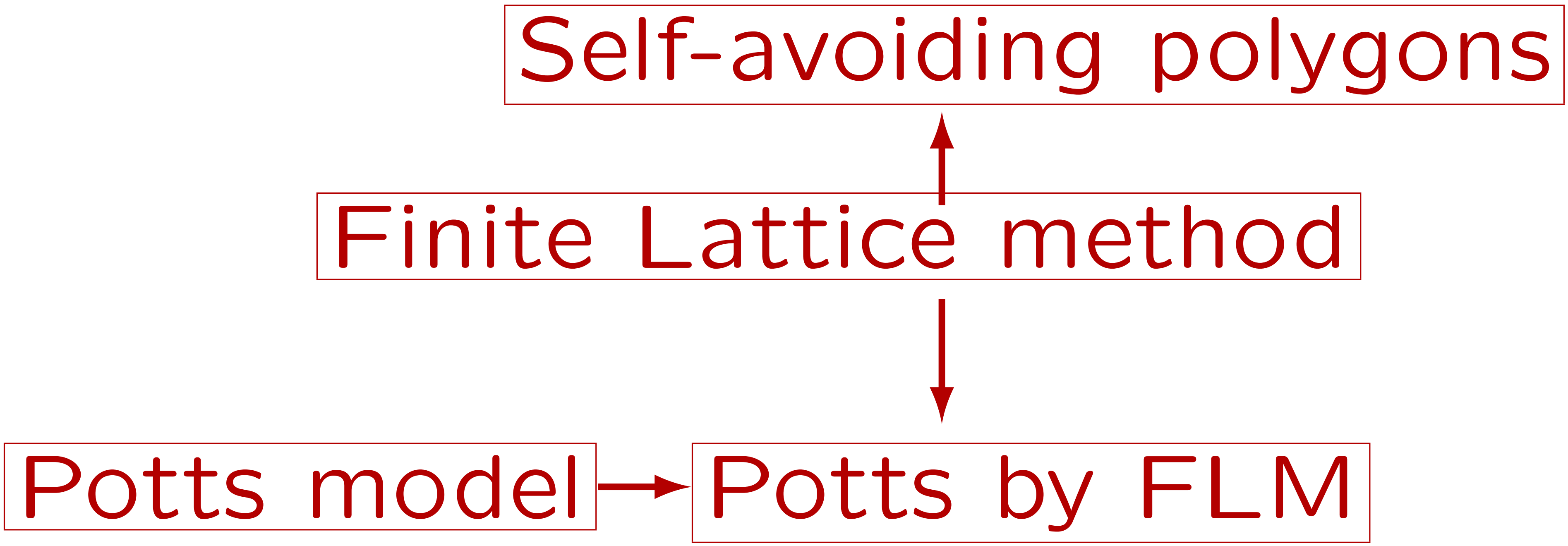
A Long and Winding Road

FLM: Finite lattice method
CTM: Corner transfer matrix
SAP: Self-avoiding polygons
PGF: Partial generating functions

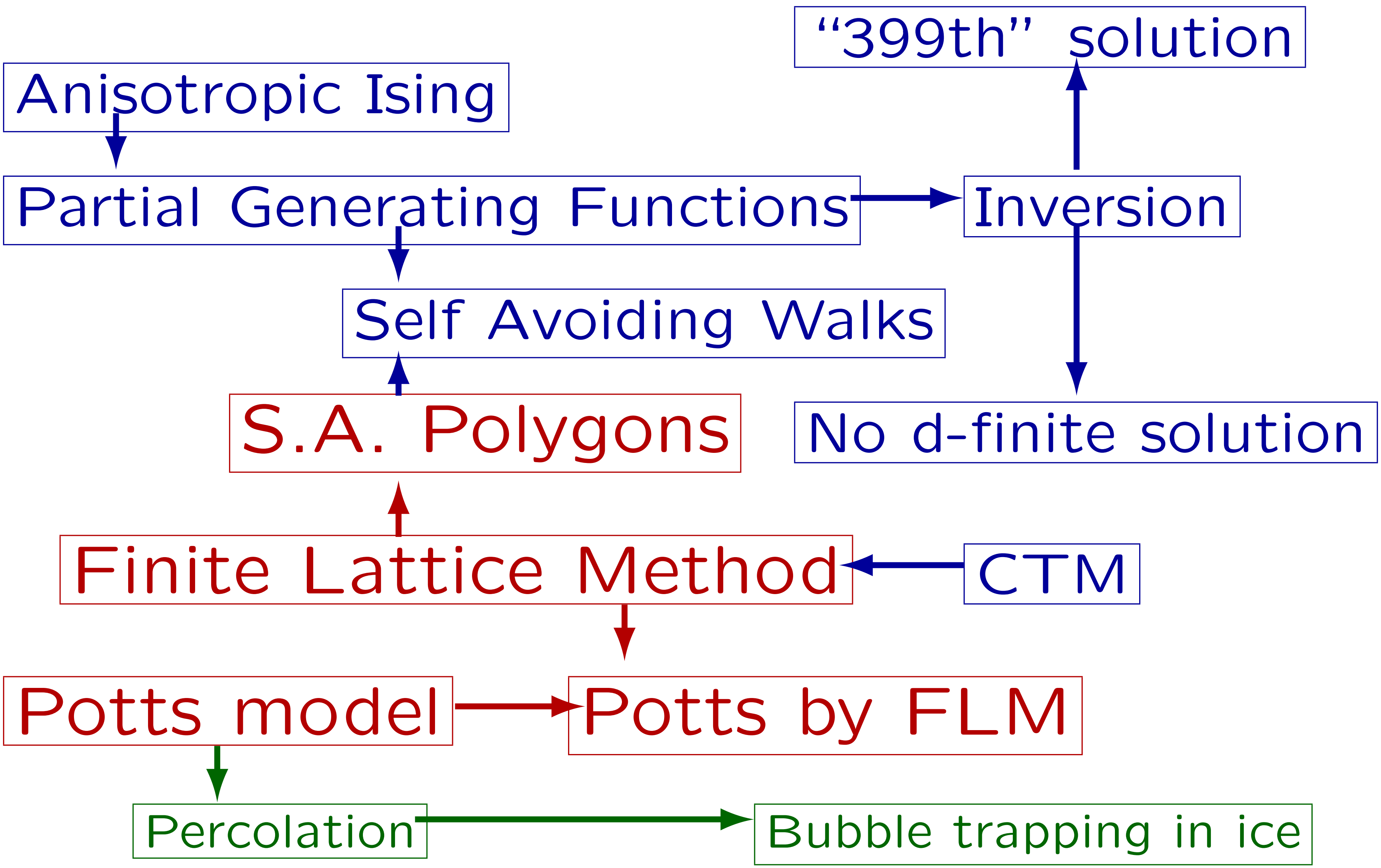
Ian G. Enting

For the 80th birthday of A.J. Guttmann

Main themes


Self-avoiding polygons
Finite Lattice method
Potts model
Potts by FLM

Connections (including CSIRO life)



Acknowledgements

Tony

Coauthors with Tony:

R. Brak, L. Richmond, N. Wormald, A. Conway, K. Briggs, I. Jensen, D. Bennett-Wood, D. Gaunt, A. Owczarek, S. Whittington, J. Cardy, J. Leask, L. Wong

Other coauthors:

J. Oitmaa, M. Watts, C. Domb, T. de Neef, R. Baxter, D. Kim. M. Barber, J. Adler, V. Privman, G. Baker, L. Benofy, N. Clisby

And also:

H. Bolton, C. Thompson, A. Rechnitzer

Getting started

Anisotropic Ising model PhD at Monash (Jaan Oitmaa)

$\chi(v_x, v_y, v_z) = \sum_{k,m,n} c_{k,m,n} v_x^k v_y^m v_z^n$

Partial generating functions:

$C_k(v) = \sum_n c_{k,0,n} v^n$ can be calculated for small k

2D Ising susceptibility functions have a symmetry, that

reflects $\chi(v_x, v_y) = -\chi(1/v_x, -v_y)$

Corresponding PGFs for self-avoiding walks do not.

In 1973 I was blissfully unaware of the significance.

–

Potts model at Kings College London

Mainly using the 'code' method, for bi-partite lattices,

i.e. sum over all configurations on one sub-lattice.

Also at KCL, met Tom de Neef when he visited Domb.

Symmetries and Inversion

Yes

Baxter: Geometric symmetry $F(v_x, v_y) = F(v_y, v_x)$ and inversion symmetry relating $F(v_x, v_y)$ to $F(1/v_x, -v_y)$ allows complete solution of F

"399th solution of the Ising model"

No

Maillard conjectured checkerboard Potts was invariant under all perturbations of interactions (applies for Ising). Tested with FLM $q = 3$ series comparing cases not related by geometric symmetries:

```
            a                      b
            !                      !
b   –   o   –   c  ↔  a   –   o   –   c
            !                      !
            d                      d
```

Failed (at next term after Maillard tests).

AJG-80

Finite Lattice Method (FLM)

Series for free energy is given by linear combination of $r \times s$ finite lattice free energies:

$$F \approx \sum_{r,s}{}' a_{r,s} F_{r,s}$$

- Summation cutoff determines length of series.
- Pointed out to me by Tom de Neef.
- $a_{r,s}$ is inverse of incidence matrix
- Triangular incidence matrix: diagonal elements $= 1$
- For some cases, (including square lattice surface corrections) the inverse is known explicitly.

Comments on FLM:

Everyone knows that: Martin Sykes

That's obvious: Donald Betts

Surface series

FLM gives bulk free energy (per site) as $F \approx \sum_{r,s}' a_{r,s} F_{r,s}$

Surface corrections are a similar linear combination (IGE 1978), but only for high-T series with free boundary or low-T series with fixed boundary.

FLM used to extend high-T $\dfrac{\partial^2 F}{\partial H \partial H_{\text{surf}}}$ and $\dfrac{\partial^2 F}{\partial H_{\text{surf}}^2}$

for square lattice Ising (IGE & AJG 1980).

Began a long collaboration combining series from FLM (IGE) and analysis by differential approximants (AJG). (As described for Tony's 70th birthday).

Potts model surface series, including more general boundary conditions, presented for Tony's 60th.

PGFs and Self-avoiding walks

FLM enumeration of SAW (and high-T χ) hard as series to order k requires $r \times s$ lattices $r + s \leq k$

Generalising to anisotropic case allows various ‘tricks’ :

- $C(u, v) = \sum_{m,n} c_{m,n} u^m v^n$ vs $C(x) = \sum c_m x^m$ so $c_n = \sum_k c_{n-k,k}$ so for $n \leq 2M + 1$, only need $c_{m,k}$ for $m \leq M$ using $r \times s$ lattices for $r \leq M$
- Partial generating functions $\sum_n c_{m,n} x^n$ can be factored into products of irreducible PGFs, that are enumerated by lattices with smaller r

All this was pulled together by Andrew Conway, enumerating square lattice SAW to 39 steps.

Transition in 3D Potts model

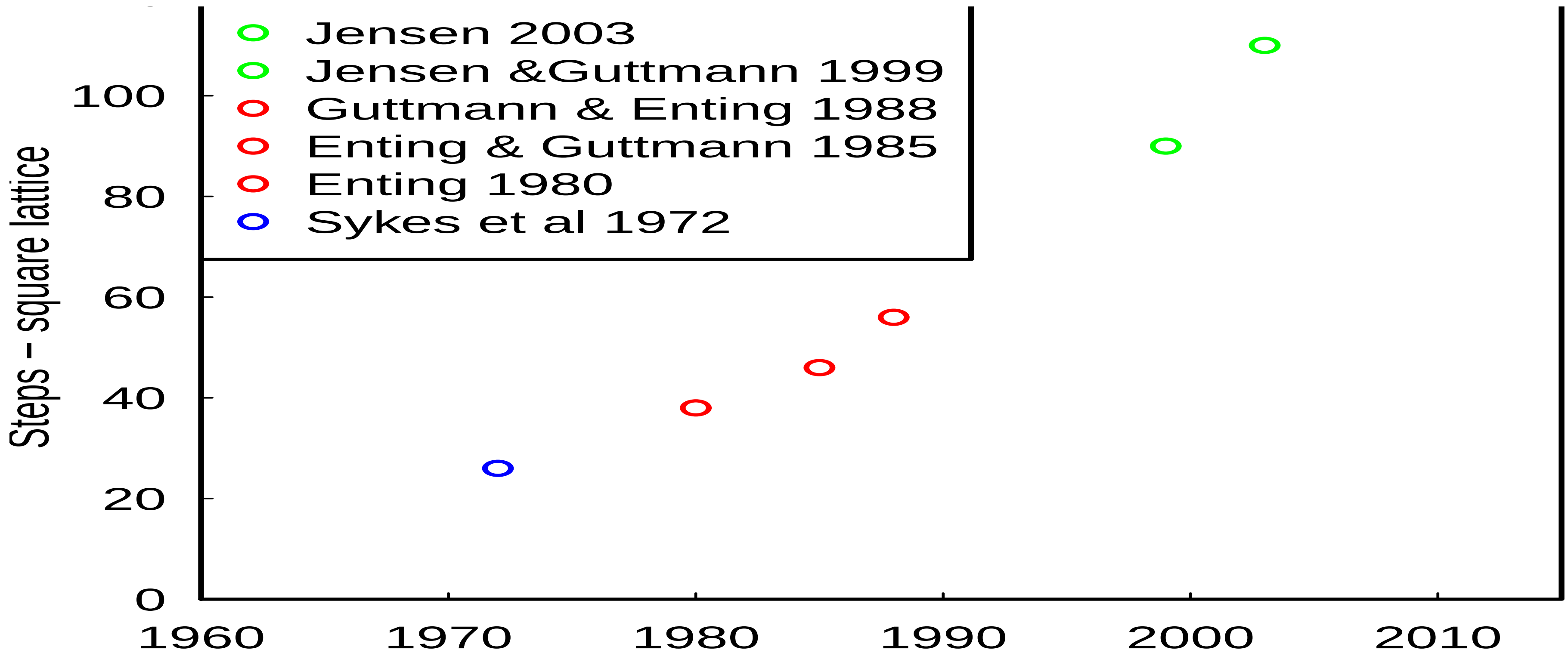
Is $q = 3$ 3D Potts transition continuous or 1st -order?

Four steps:

- Generate square lattice low-T series of F, M, χ for q-state Potts model for $q = 2, 3, 4, 5, 6, 7, 8, 9, 10$. High-$T$ F follows from duality,
- Test if series analysis, especially using differential approximants, could reliably determine known order of transition (continuous for $q \leq 4$, 1st order for $q > 4$).
- For simple cubic $q = 3$, generate low-T F, M, χ and high-T F.
- Estimate latent heat and ΔM, confirming 1st order.

Self-Avoiding Polygons

. . . a conventional algorithm would have to run for a time comparable to the lifetime of the universe on the world's faster computer to achieve the same results as those presented here. — AJG and IGE 1988.



Extensions

Other self-avoiding polygon statistics

Spanning moments, area-weighted moments.

Other lattices:

Triangular, honeycomb & Manhattan (directed lattice)

Convex and almost convex polygons:

(Initially as next-order correction to FLM)

Fitting differential approximants found exact solutions.

Self-avoiding polygons:Towards an exact solution?

(AJG & IGE, Nucl Phys B (proc supp) 1988.)

The known critical point for the honeycomb lattice suggested that fitting differential approximants to long series might reveal exact solution

The **?** turned out to be justified.

No solution (as ‘standard‘ functions)

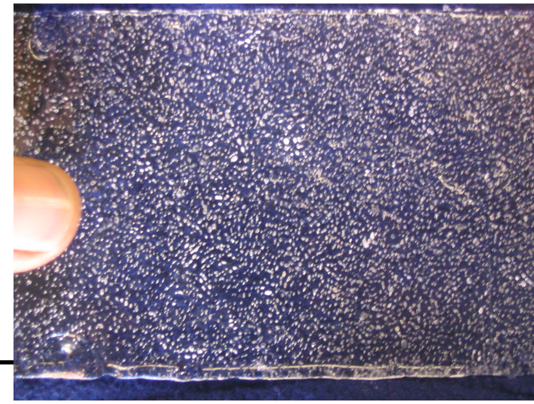
Partial generating functions: $X(x,y) = \sum_n Y_n(x) y^n$

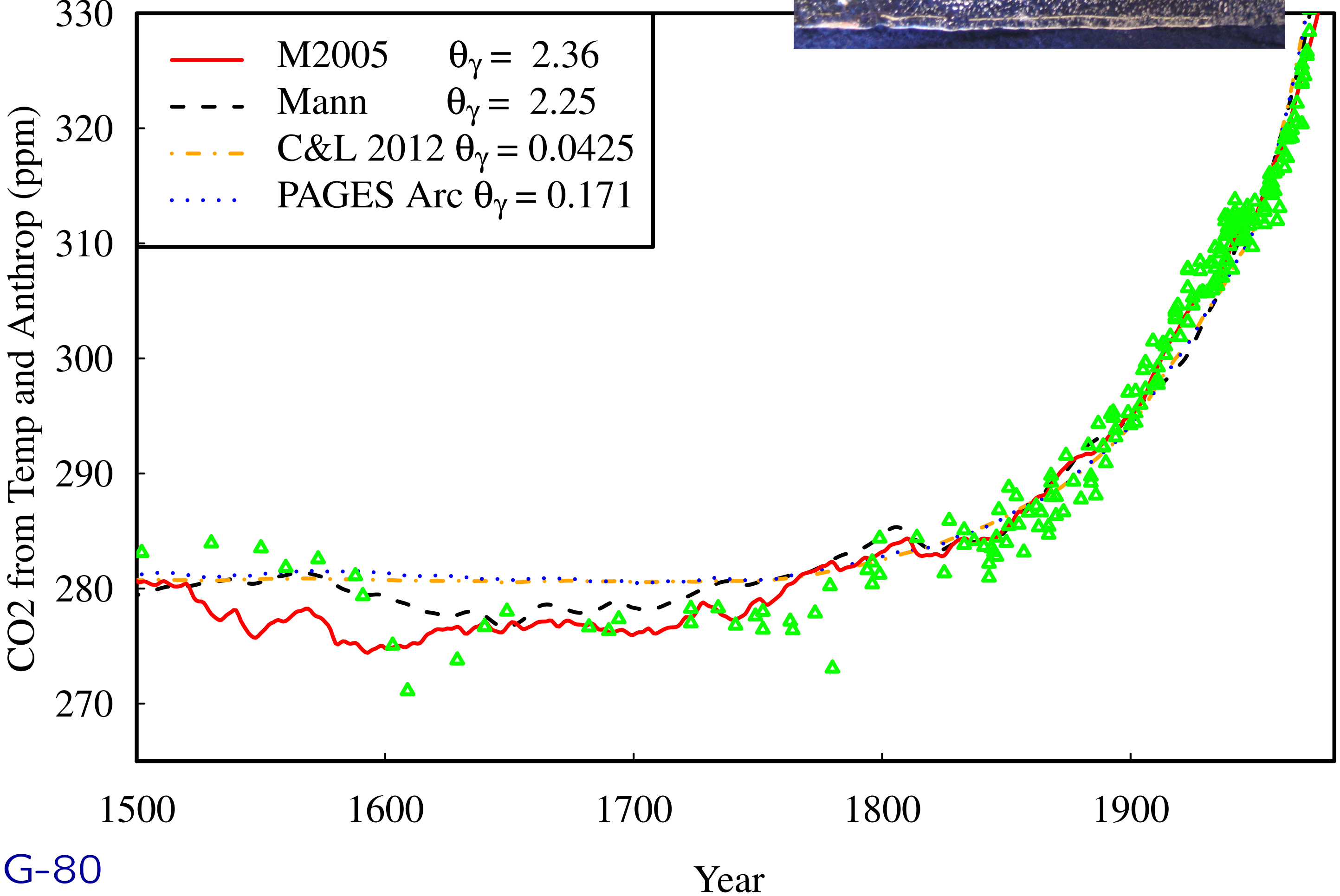
- The $Y_n(v_x)$ are rational functions $P_n(v_x)/Q_n(v_x)$
- For known soluble models, $Q_n(x)$ have zeroes at $x = \pm 1$
- For other models, the $Q_n(x)$ have factors $(1 - x^k)$ where k increases as n increases.
- This behaviour was found empirically for square lattice Ising χ, SAP and SAW
- This implies such functions $X(x,y)$ cannot be solution of any DE of finite order and polynomial coefficients.

In some cases, these arguments, from AJG and IGE, have been given rigorous proofs by Andrew Rechnitzer.

CSIRO life -bubbles in polar ice

CO_2 from ice cores and atmosphere $\triangle$ and estimates from paleotemperature data with feedback θ_γ

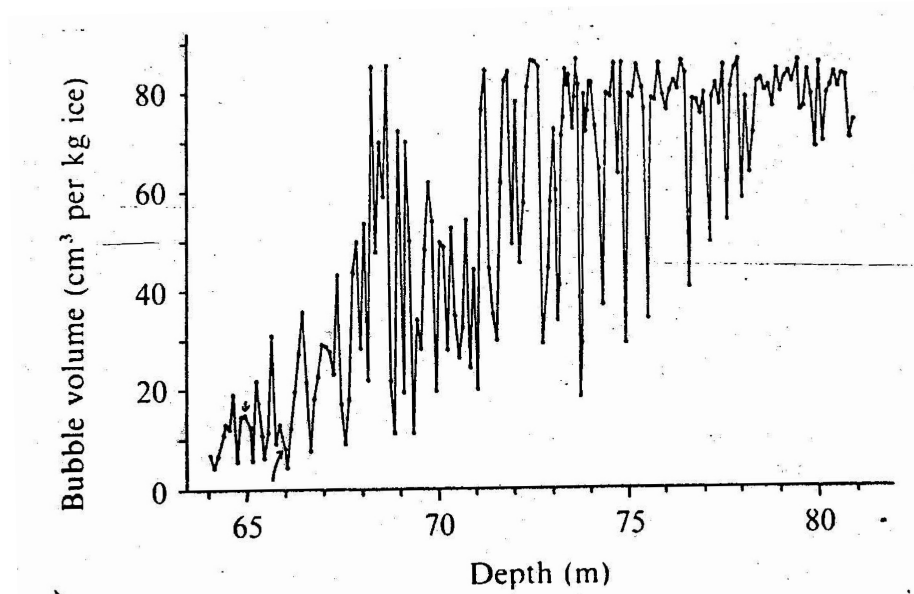




Trapping air bubbles in polar ice (2)

Schwander and Stauffer 1984

Bubble trapping looks like percolation.
Porosity data look like critical fluctuations.
Bond percolation as $q \to 1$ limit of q-state Potts



But trapping dominated by diffusion which seems to go to zero before close-off.
But analogue is resistance of random networks - exponent is ≈ 2, so maybe percolation is relevant.
Not yet explored:
— Surface corrections
— Amplitude relations (Corresponding states):
Betts, Guttmann and Joyce, 1971)

Other life: Padé-Laplace

- Padé approximants to $\frac{df}{dx}/f$ identify singularities
- (Equivalent to 1st-order DE – generalised DSG AJG.)
- Applied to Laplace transforms, it can approximate transform and inverse as sum of exponentials.
- (Applied in electrical/electronic engineering.)
- For CO_2 and climate, Padé-Laplace gives a way of combining and inverting components of linear response in the carbon cycle and the coupled carbon-climate system (IGE 2022)
- Can possibly be generalised to response to bomb ^{14}C.
- All a bit late – tipping points are taking (or have taken) earth out of the linear response regime.