0.785398163397448309 62...
-1.46035450880958681 2...
2.61237534868548834 3...
1.01467803160419205 46...
0.57463960715151959 27...

Self-avoidance, Sidon and autoconvolutions

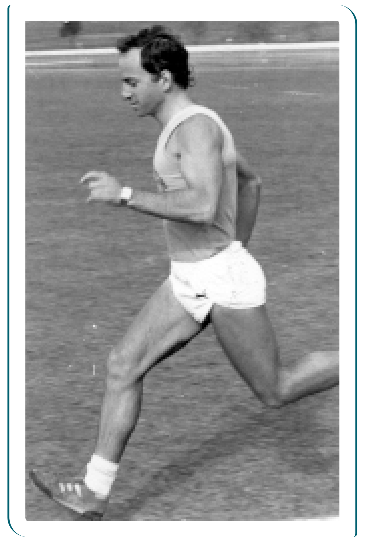
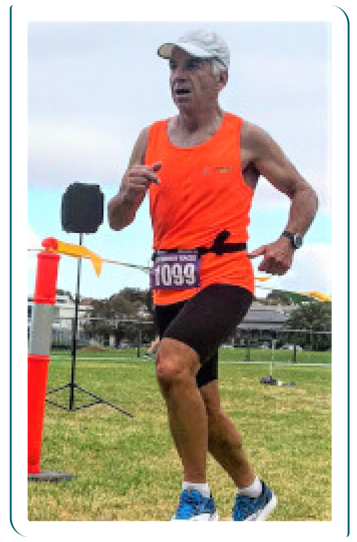
Adventures in numerology

and clumsy segways

Andrew Rechnitzer

80 years of Guttmannia, Unimelb 2025.

Tony Guttmann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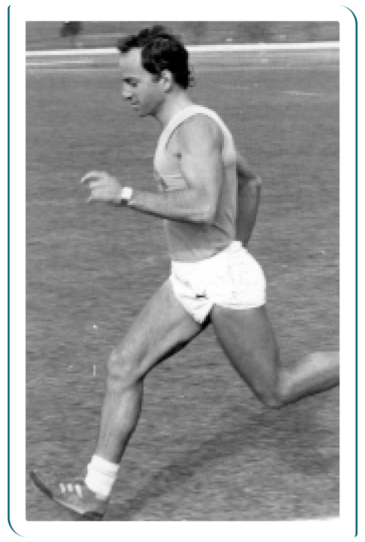
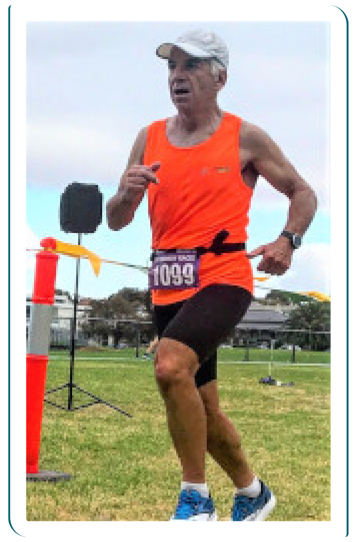

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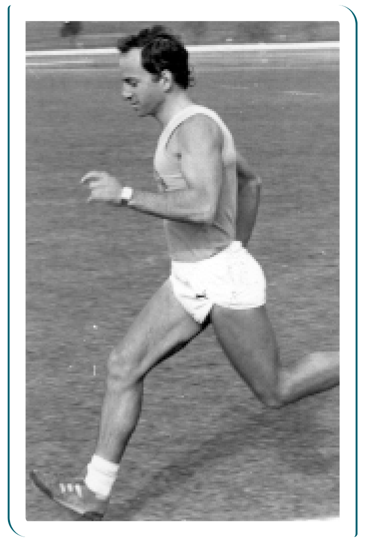

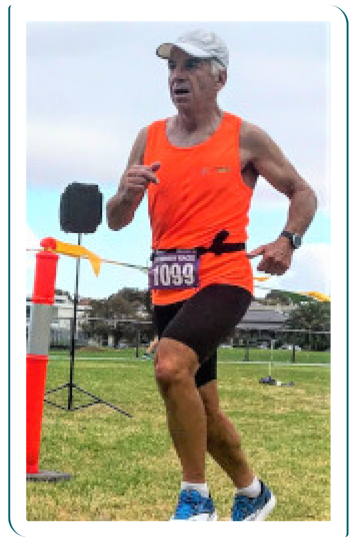

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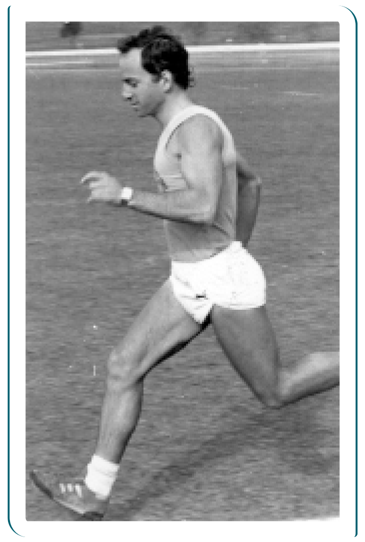
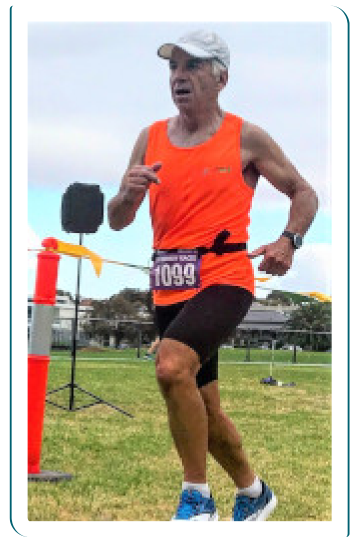


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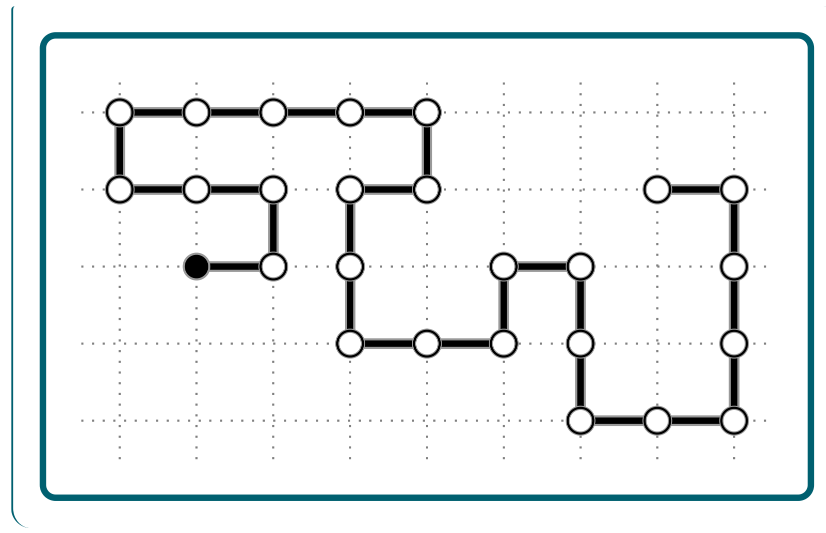
Don't repeat a vertex

The importance of self-avoidance



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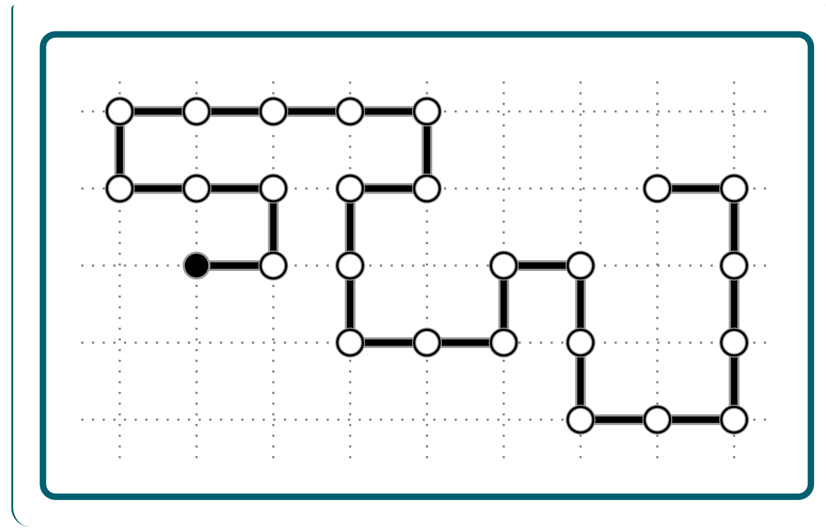

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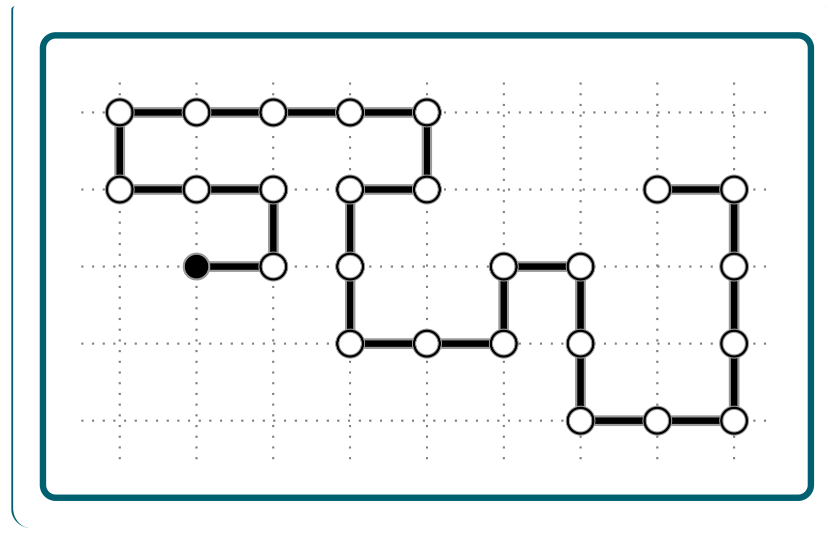

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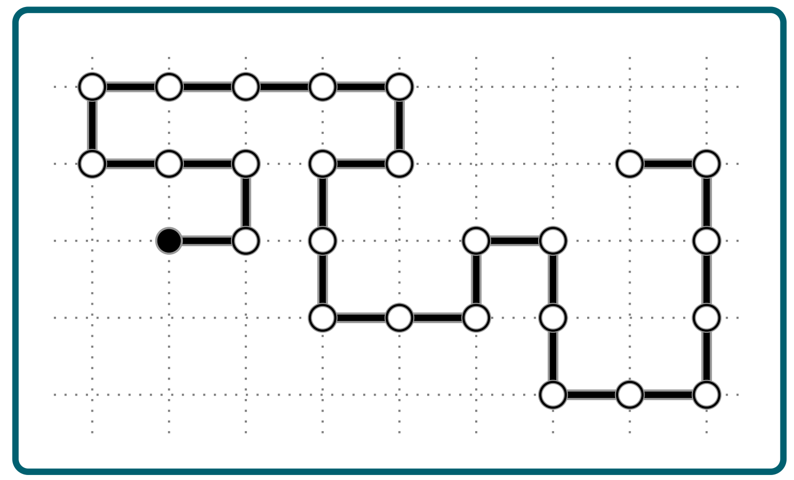

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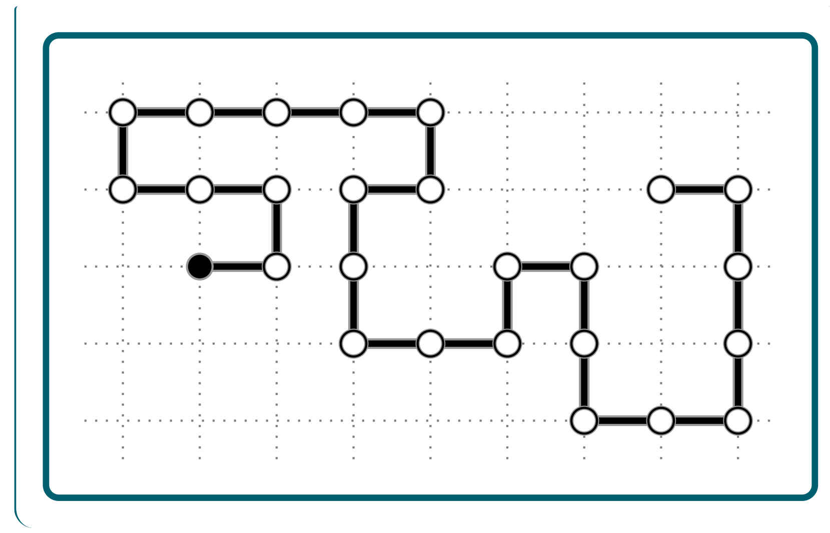

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Warning — clumsy segway ahead

Consider the sequence

$$1, 2, 4, 8, 13, 21, 31, 45, 66, 81, 97, 123, 148, \ldots$$

- 1:
- 2:
- 4:
- 8:
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This is the Mian–Chowla sequence Mian Chowla (1944) and is a Sidon sequence.

Sidon set

Don’t repeat your sums

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- Lots of number-theoretic fun to be had
- Excellent literature review by O'Bryant (2004)

Generalised Sidon set

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- Fix $\mathcal{A} \subseteq \mathbb{Z}$ then define

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- Want to understand the behaviour of $R_h(g,n)$

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- Open problem to compute the limit

$$\sigma_h(g) = \limsup_{n \to \infty} \frac{R_h(g, n)}{n^{1/h}}$$

the limiting maximal density of a $B_h(g)$ set.

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- Key to these bounds are autoconvolutions

Autoconvolution problem

Don't overlap too much

- Consider functions on $(-\frac{1}{2}, \frac{1}{2})$ with $\int f = 1$
- Minimise the autoconvolution norm: $\mu_2^2 = \inf \int_{-1}^{1} \left(\int_{-1/2}^{1/2} f(t) \cdot f(x-t) \mathrm{d}t \right)^2 \mathrm{d}x$
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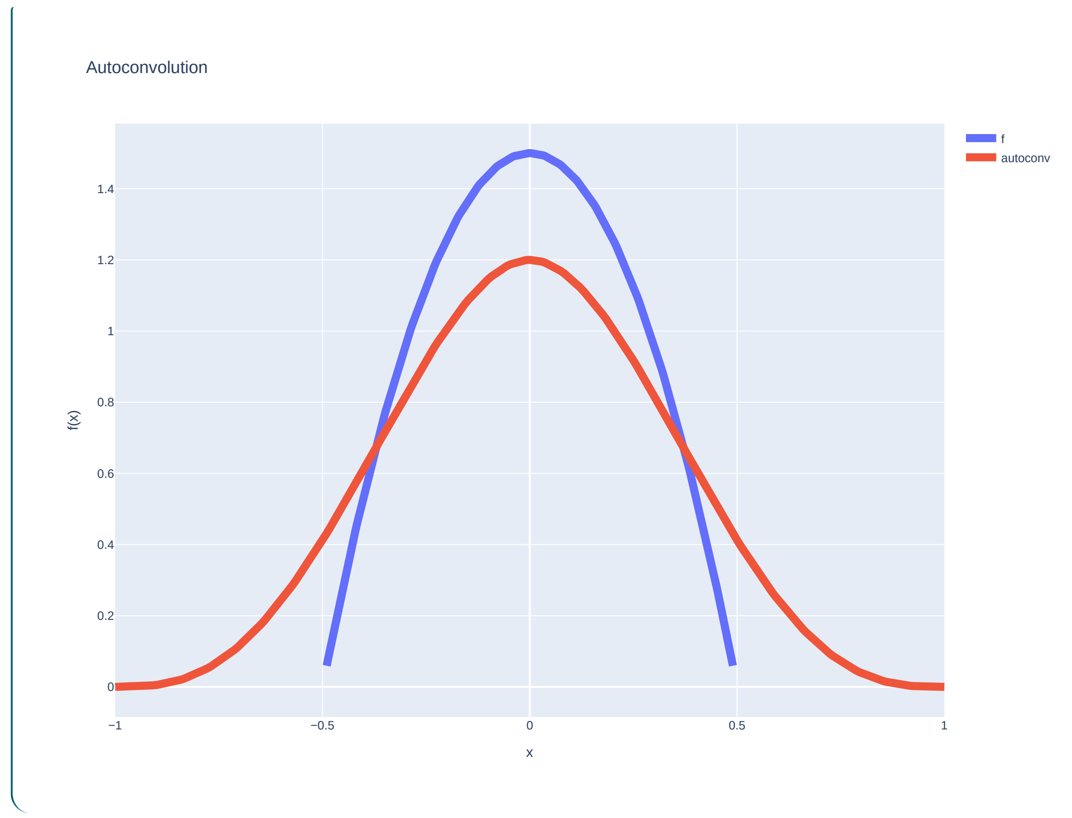

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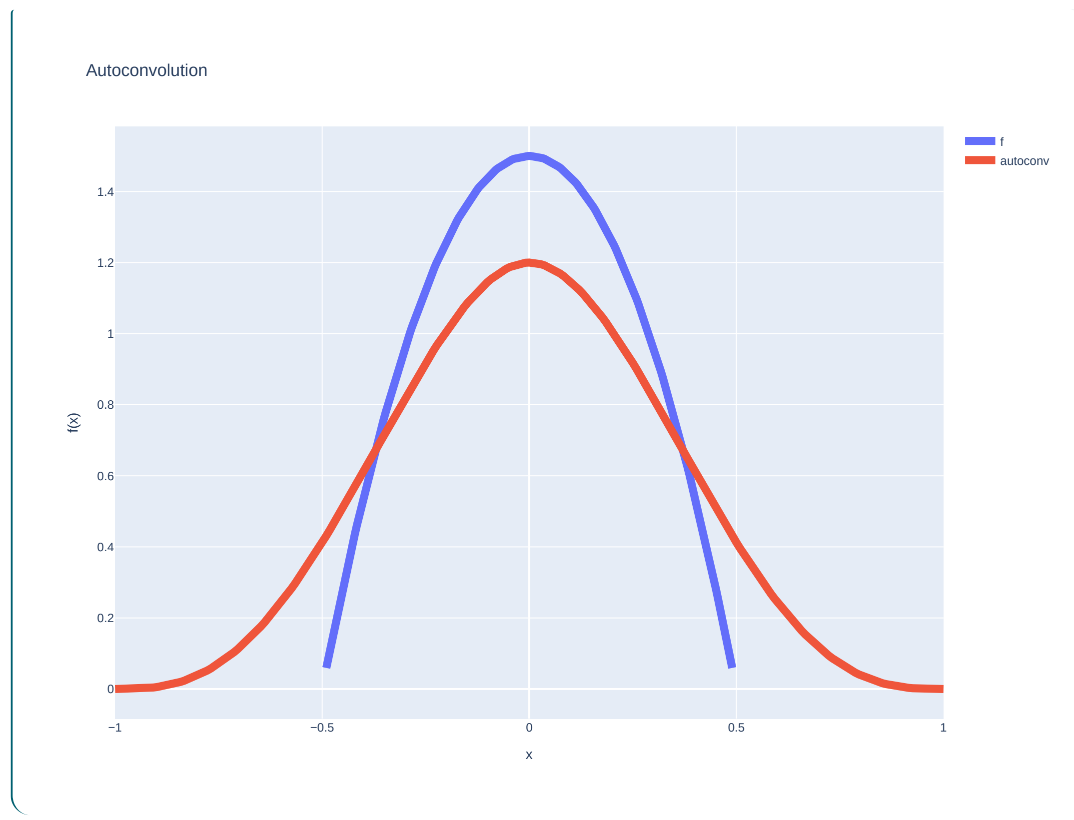

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- Eg: $f(x) = \frac{3}{2}(1-4x^2)$ gives $\mu_2^2 = 1$

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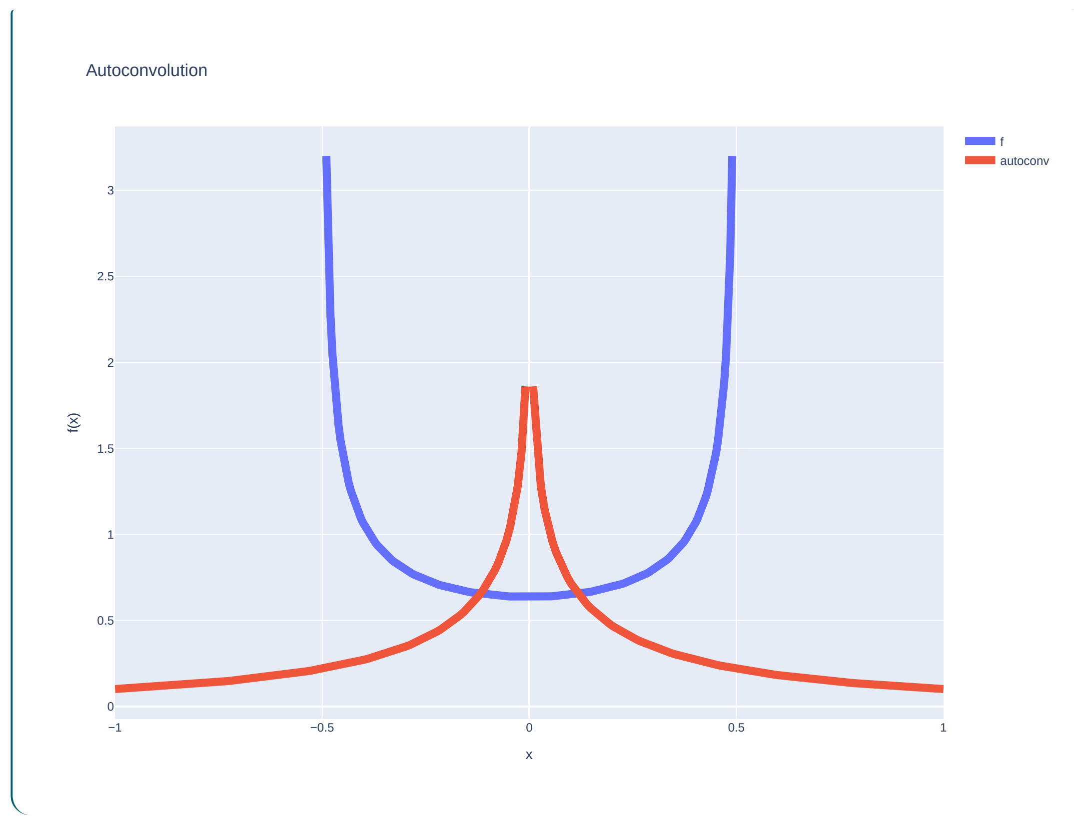

- Approximately — find a function that overlaps with itself as little as possible
- Eg: $f(x) = \frac{3}{2}(1 - 4x^2)$ gives $\mu_2^2 = 1$
- Eg: $f(x) = \frac{2}{\pi\sqrt{1-4x^2}}$ gives $\mu_2^2 = 0.574694862\ldots$

Autoconvolution and Sidon

Connect continuous and discrete worlds

- Connection was *"in the air"*, but made explicit by Martin & O'Bryant (2002)
- Bound Sidon density by autoconvolution norm

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- I learned about this problem as an examiner of White's thesis

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$$\mu_2^2 = \frac{1}{2} + \sum_{m=1}^{\infty} \hat{f}(m)^4 + \frac{16}{\pi^4} \sum_{m\geq 1}^{odd} \left(\frac{1}{m} + 2m \sum_{k=1}^{\infty} \frac{(-)^k \hat{f}(k)}{m^2 - 4k^2} \right)^4$$

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- By truncating sums White (2022) translated this to quadratically constrained linear program

Solution of program leads to bounds

- White computed near optimal $\hat{f}(k)$ for $0 \leq k \leq 30000$, giving 4 digits

$$0.574635 \leq \mu_2^2 \leq 0.574644$$

In thesis he lists first 20 coefficients.

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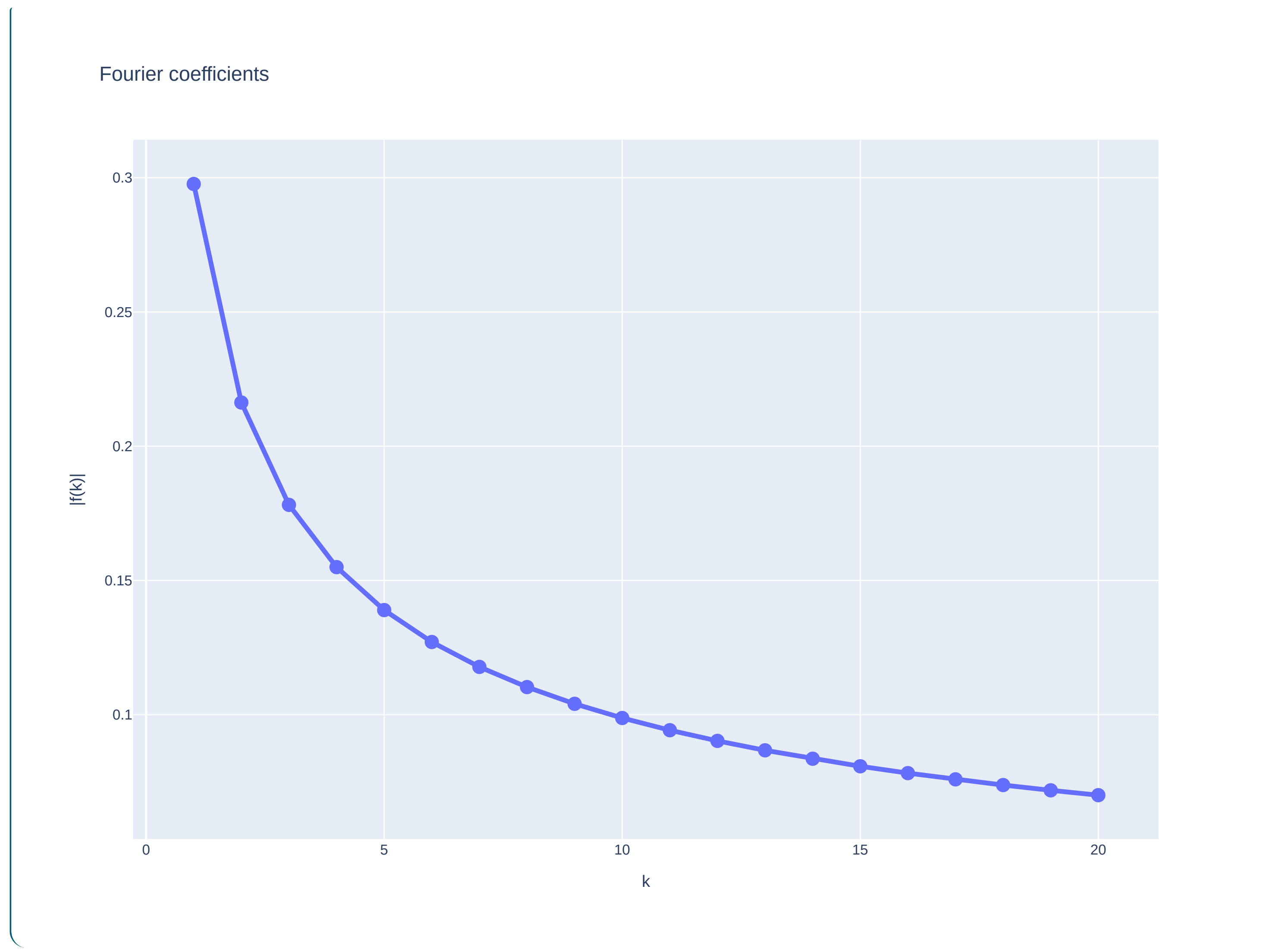

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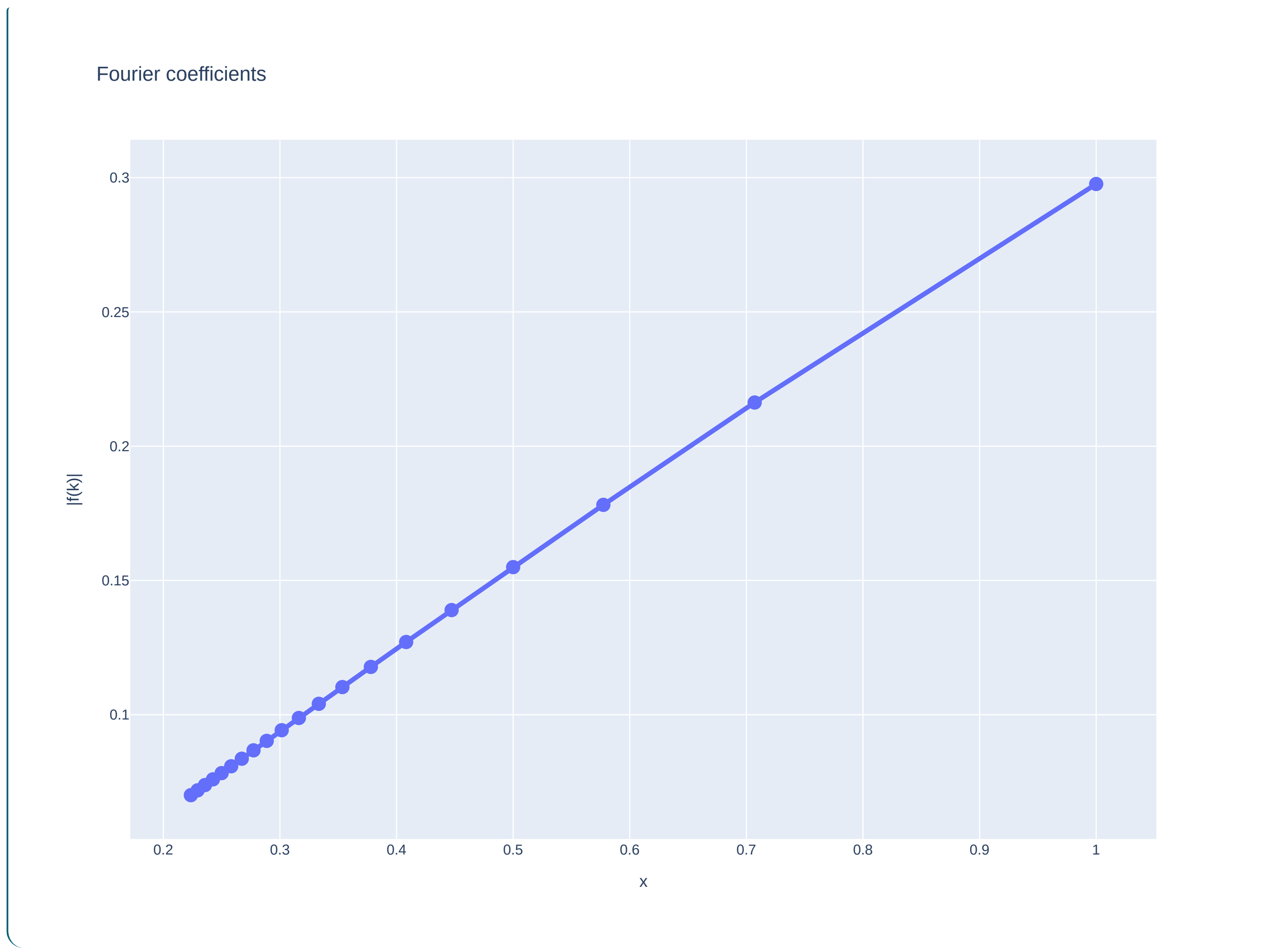

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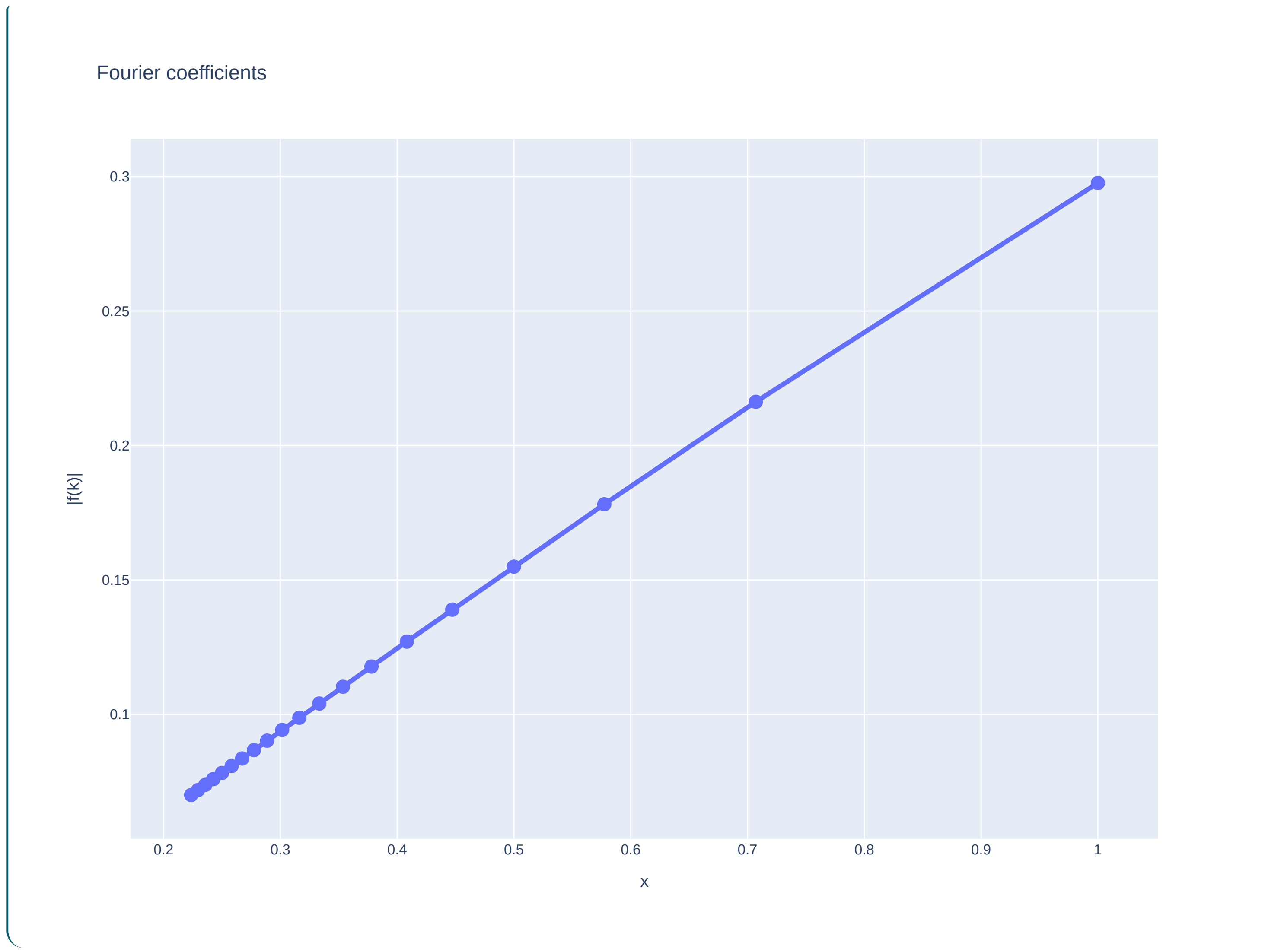

- Strongly suggests Ansatz: $(-1)^k \hat{f}(k) = \frac{a}{\sqrt{k}}$

Anstanz, series acceleration, numerics

Let the numerology begin

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$$A_m = \sum_k \frac{m}{(m^2 - 4k^2)\sqrt{k}}$$

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- Careful computation of slowly converging double-sum over k, m
- For fixed m, easy to compute A_m via Euler-Maclaurin
- The MPMATH python library is excellent — arbitrary precision numerics

Asymptotics of A_m

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$$c_1 = 0.78539816339 \quad c_2 = -0.730177254405 \quad c_3 = 0.196349540849$$
$$c_4 = -0.365088627202 \quad c_5 = 0.0736310778185 \quad c_6 = -0.185729963837$$

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$$c_1 = \pi/4 \qquad c_2 = \zeta(1/2)/2 \qquad c_3 = \pi/16$$
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- Use this to compute sums, and get

$$\mu_2^2(a) = \frac{2}{3} + \frac{\pi^2 a^4}{6} + \frac{16}{\pi^4}\left(-3.3099871a + 4.4489347a^2 - 2.088022a^3 + 0.97736a^4\right)$$

  Quick 1-variable minimisation gives $\mu_2^2 \leq 0.57469\ldots$
- Is $1.000085\times$ best upper bound (used 30k variables)

A better Ansatz

- Form Ansatz $(-1)^k \hat{f}(k) = \frac{1}{\sqrt{k}} \sum_p \frac{a_p}{k^p}$ and compute $A_{m,p} = \sum_k \frac{m}{(m^2-4k^2)k^{p+1/2}}$
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- Despite much numerology I have no idea what this is

Show me the function

a little tour of polylogarithm like functions

- Using nearly optimal a_p write $f(x)$ in terms of Claussen functions

$$f(x) = 1 + \sum_p a_p \sum_k (-1)^k \frac{\cos(2\pi k x)}{k^{p+1/2}} = 1 + \sum_p \mathrm{C}(p + 1/2; \pi + 2\pi x)$$

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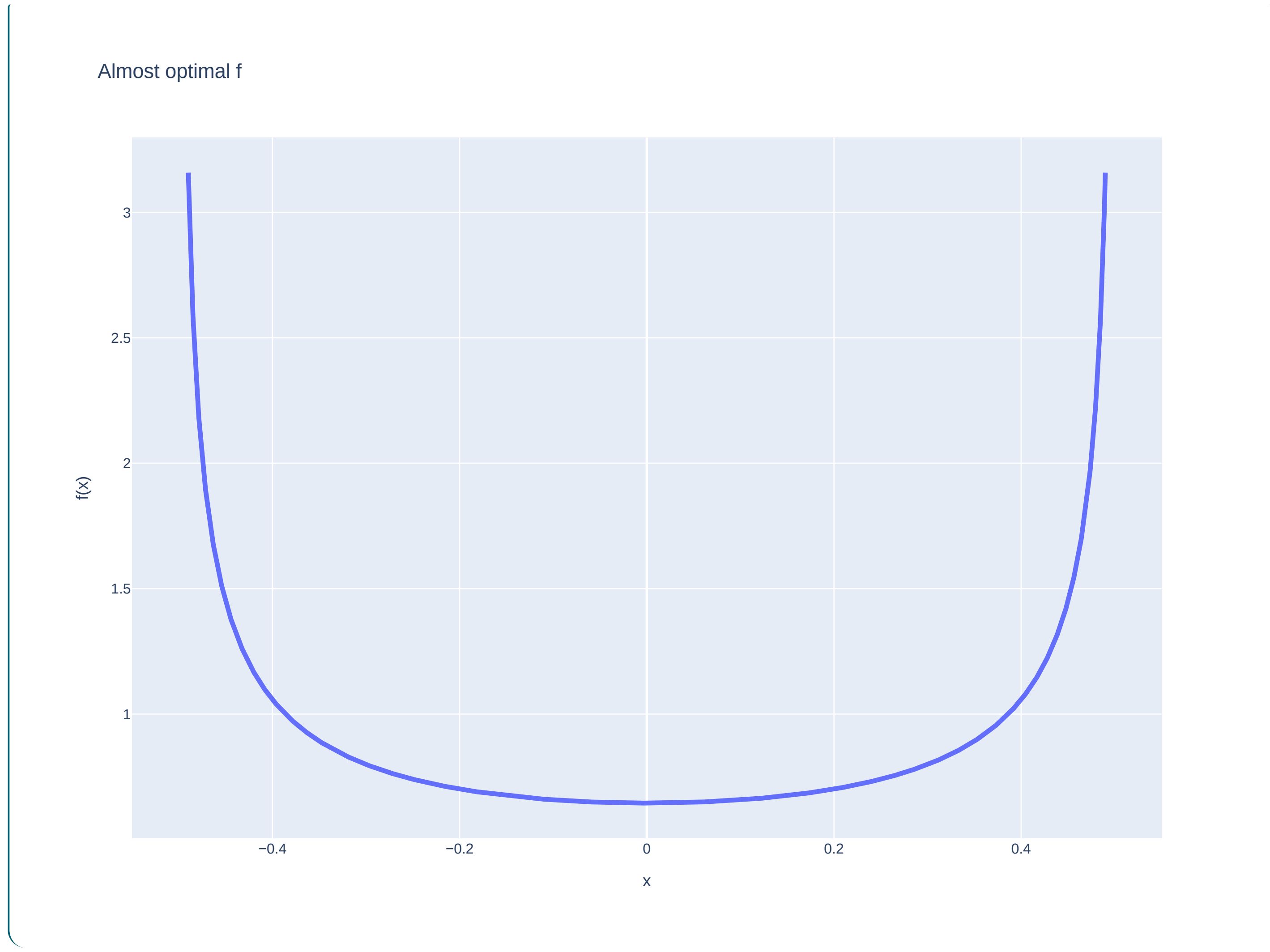

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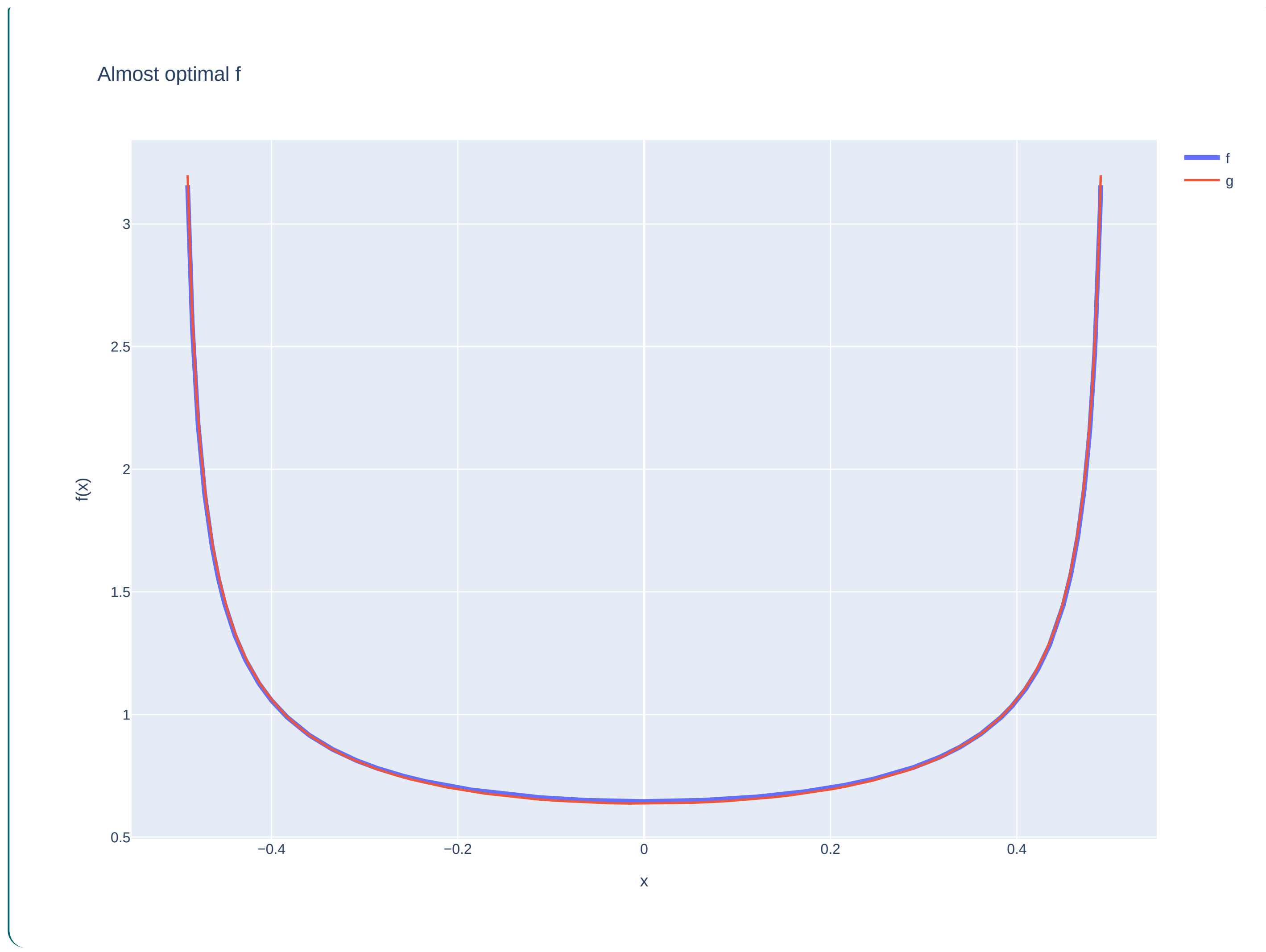

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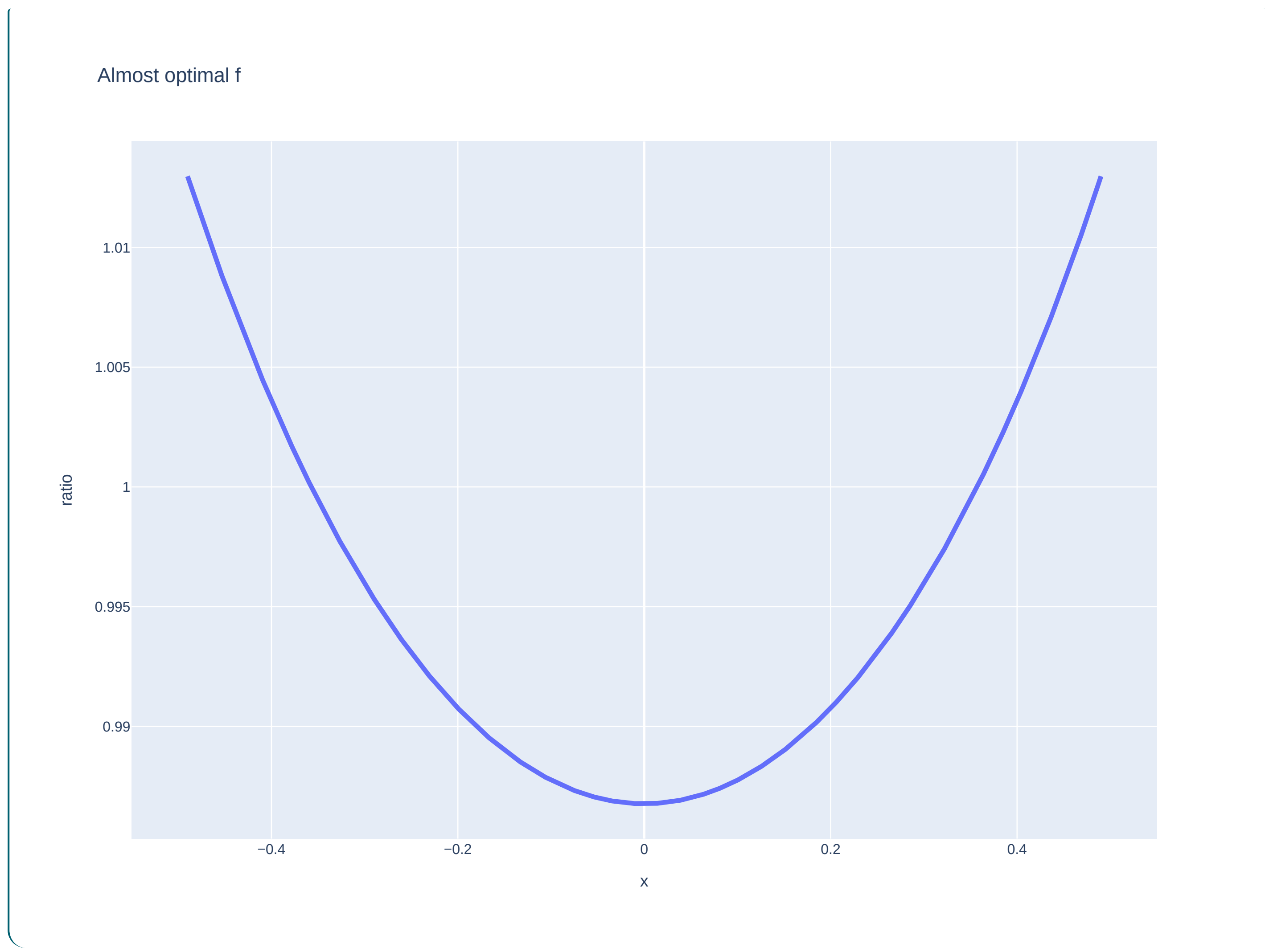

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$$\mathcal{J}(p;k) = \frac{4^p p!}{\pi^p k^p} \cdot J\left(p; \frac{\pi k}{2}\right)$$

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- And then $\mu_2^2(a_1, a_2 \ldots) = 8 \sum |\hat{F}(k)|^4$ – multivariate quartic subject to $\sum a_p = 1$

μ_2^2 Quartic polynomial with Besselly coefficients

Precomputation really helps

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- A little work with Hölder’s inequality gives bounds from both sides

$$\mu_2^2 = \begin{array}{r} 0.5746396071515195927272554275270529714370263 69373\ldots \\ \ldots 1566116308767489255216181789880(3) \end{array}$$

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- Not quite rigorous yet, but soon.

So what is this number?

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Guesses welcome

So what is the function

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I have some ideas

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- Can we extend these methods to the more general autoconvolution problem

$$\mu_p = \left(\int_{-1}^{1} \left| \int_{-1/2}^{1/2} f(t) f(x-t) \mathrm{d}t \right|^p \mathrm{d}x \right)^{1/p}$$

especially the $p \to \infty$ limit

Thanks to Nathan, Nick and Tim

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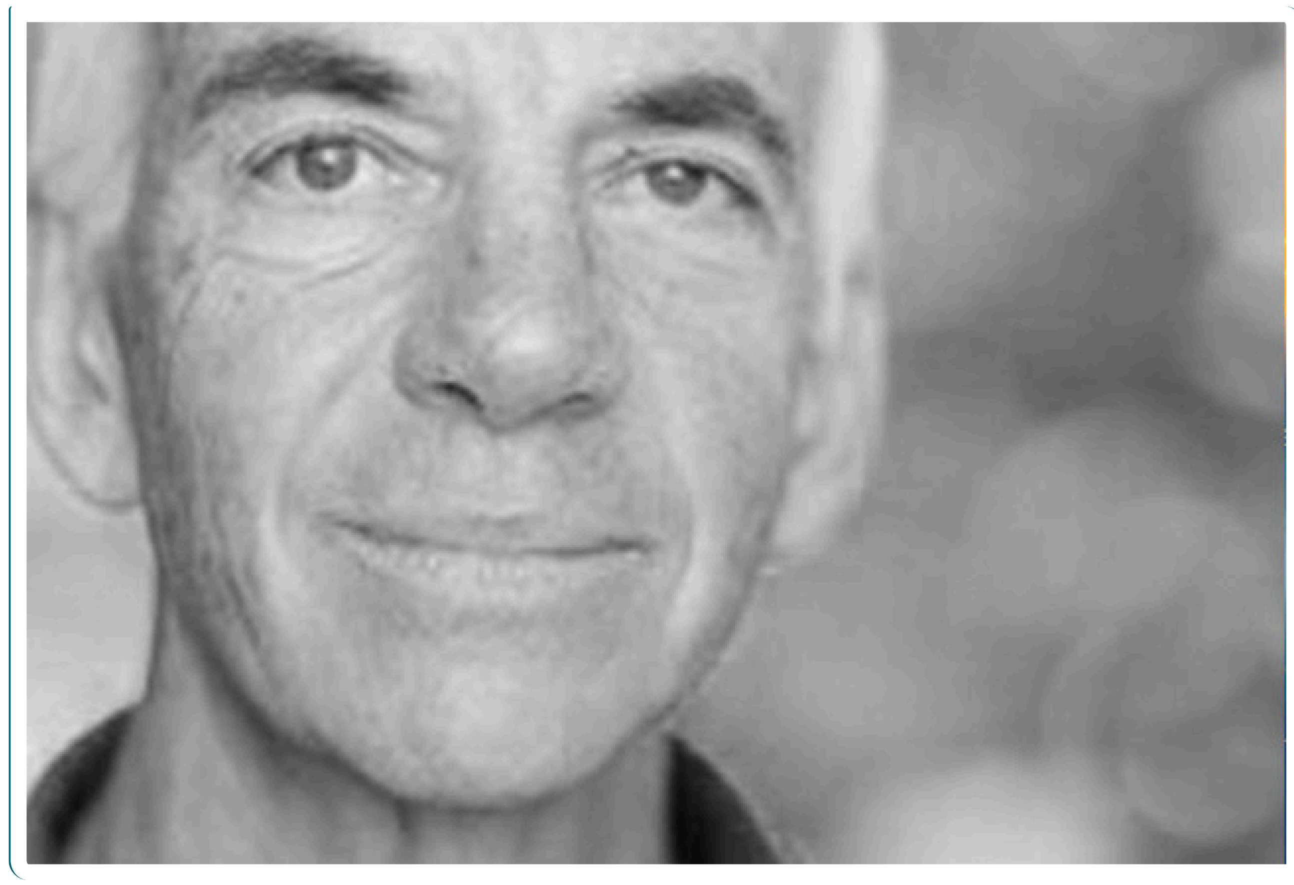

And, of course, thanks to Tony