Enumeration of planar Eulerian orientations

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Joint work with Mireille Bousquet-Mélou, Tony Guttmann and Paul Zinn-Justin

CNRS, Université de Tours, France

2025

Problem: How many planar Eulerian orientations have n edges? ([Bousquet-Mélou, Bonichon, Dorbec, Pennarun, 2017])

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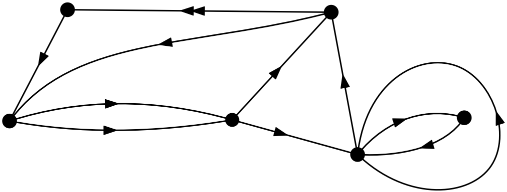
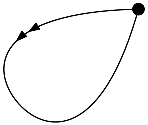
Planar Eulerian orientation:



Each vertex has equally many incoming as outgoing edges.

ROOTED PLANAR EULERIAN ORIENTATIONS

Let g_n be the number of rooted planar Eulerian orientations with n edges.



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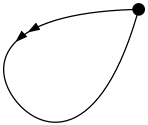

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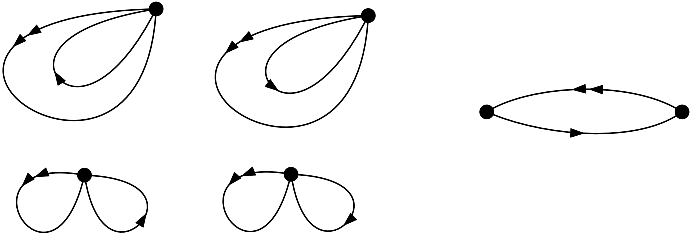

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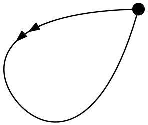

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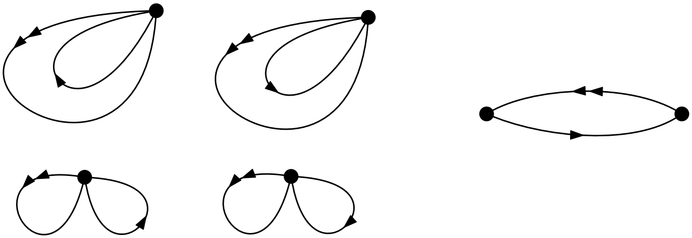

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Aim: Find a formula for g_n.

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Tony's asymptotic conjecture [E.P., Guttmann 2018]
$\rightarrow$ Exact conjecture $\rightarrow$ Guess and check proof [E.P., Bousquet-Mélou, 2020]

Since then:

- Rigorous exact solution to six vertex model on a planar map [E.P., Zinn-Justin, 2023], following [Kostov, 2000]
- Exact enumeration of Planar Eulerian orientations by edges *and* vertices [E.P., Bousquet-Mélou, 2025]
- distribution of height function related to six vertex model on random map [E.P. 2025+]

EXACT SOLUTION [E.P., BOUSQUET-MÉLOU, 2020]

Let $\mathsf{R}(t) = t - 2t^2 - \cdots$ be the unique series satisfying

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n}^2 \mathsf{R}(t)^{n+1}.$$

Theorem: The generating function of planar eulerian orientations is given by

$$\mathsf{G}(t) := g_1 t + g_2 t^2 + \cdots = \frac{1}{2t^2}(t - 2t^2 - \mathsf{R}(t)).$$

Asymptotically,

$$q_n \sim \kappa \frac{\mu^{n+2}}{n^2 (\log n)^2},$$

where $\kappa = 1/18$ and $\mu = 4\sqrt{3}\pi$.

New work: Refined enumeration of Eulerian (partial) orientations

[E.P., Bousquet-Mélou, 2025]

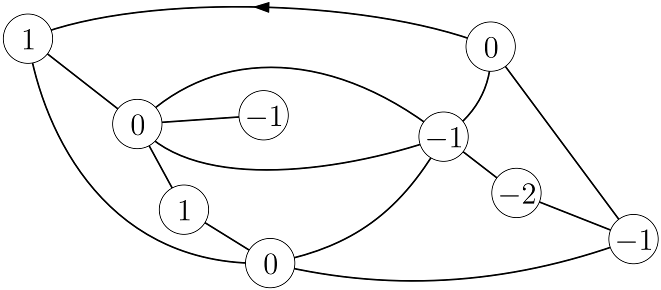
Related weighted model

Height-labelled quadrangulations:

- Each face has degree 4
- Adjacent labels differ by 1
- Root edge labelled from 0 to 1

Weights:

- A weight v per local minimum
- A weight ω per alternating face



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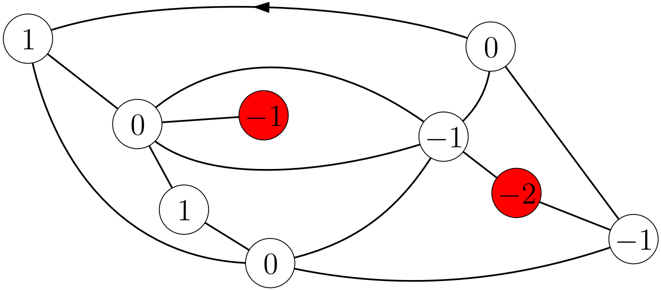

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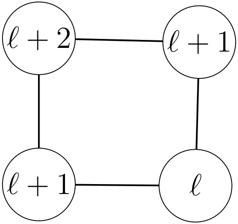
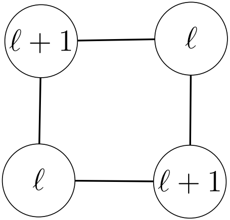


Non-alternating



Alternating (weight ω)

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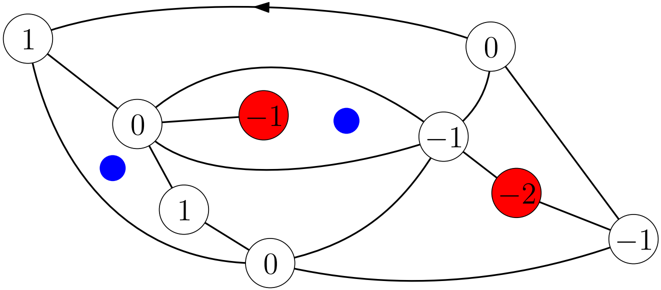

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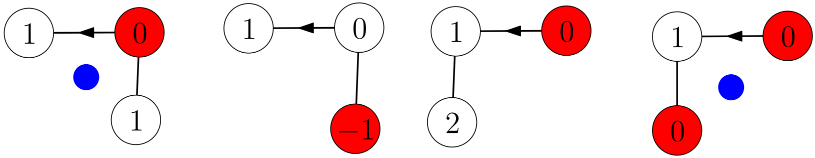
Aim: determine the refined generating function

$$\mathsf{Q}(t,\omega,v) = \left(2v + \omega v + \omega v^2\right) t + \cdots$$



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Claim: $2\mathsf{G}(t) = \mathsf{Q}(t,0,1)$

BIJECTIONS

Bijection: 1 to 1 correspondence between two types of objects
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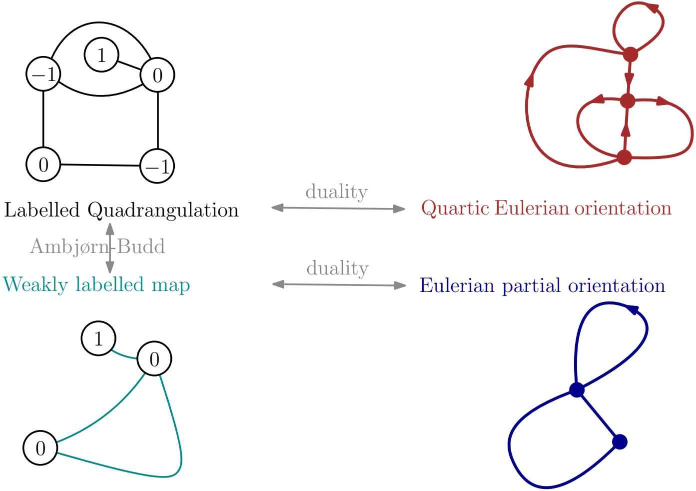
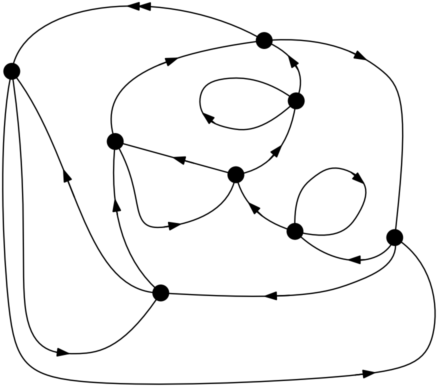
Our bijections:



Bijection 1: H-maps to Eulerian orientations (EO-maps)

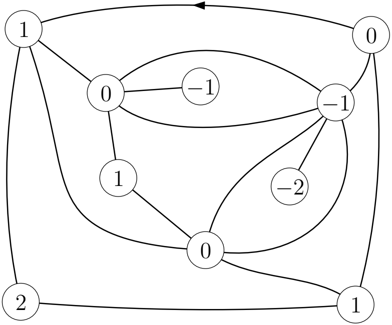
(EP and Guttmann (2018)).

EO-quarts: each vertex has two incoming and two outgoing edges.

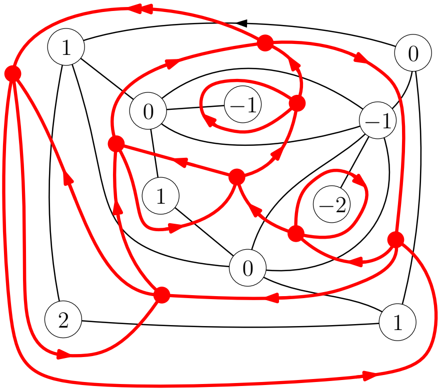


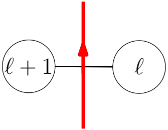
H-QUADS TO EO-QUARTS

Start with a height-labelled quadrangulation.

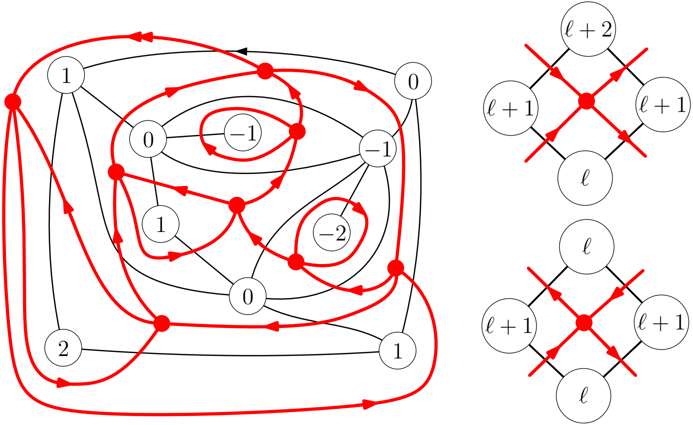


Draw the dual with edges oriented according to the rule.





Each red vertex has two incoming and two outgoing edges.



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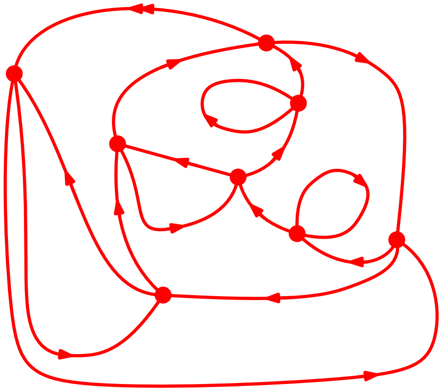

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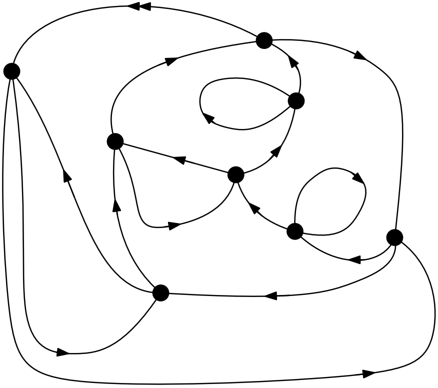
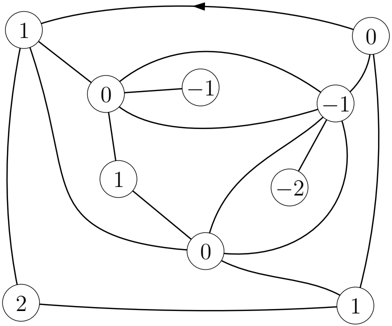


Bijection 2: height-labelled quadrangulations to weakly height-labelled maps

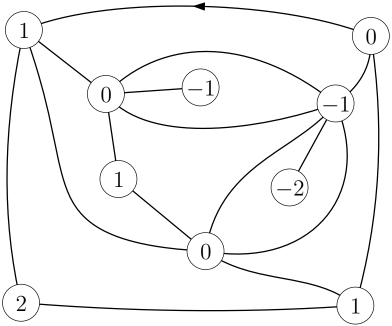
(Miermont (2009)/Ambjørn and Budd (2013)).

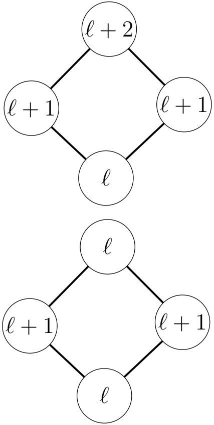
Start with a height-labelled quadrangulation.



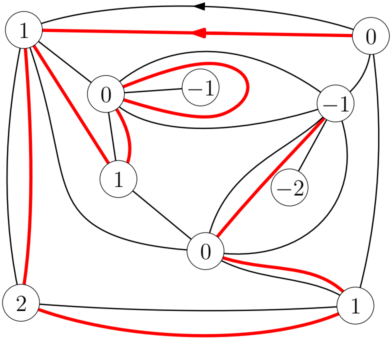
H-Quadrangulations to H-maps

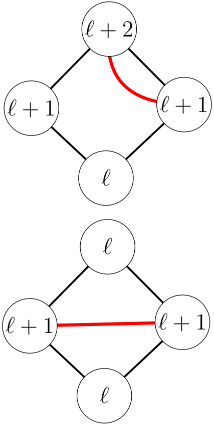
Start with a height-labelled quadrangulation.



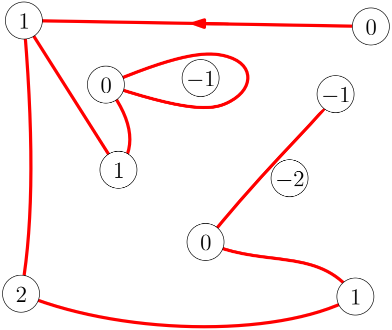


Draw a red edge in each face according to the rule.

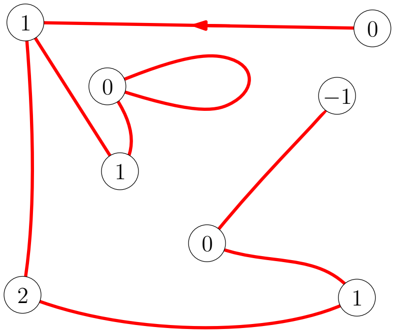




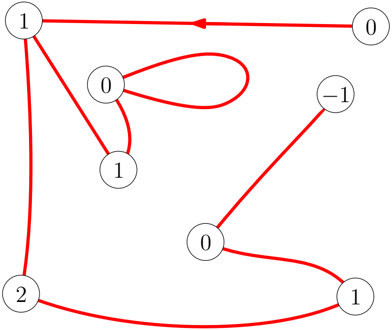
Remove all of the original edges.



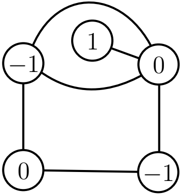
Remove any isolated vertices.



The new map is a weakly height-labelled map (adjacent labels differ by *at most* 1).

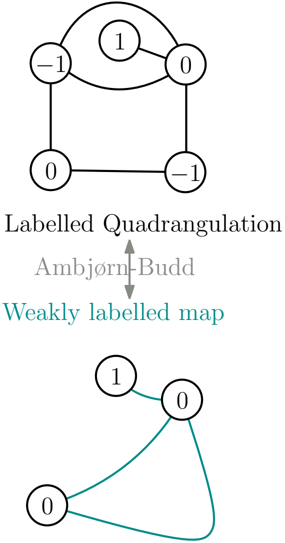


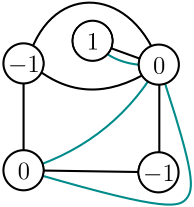
TWO MORE STATISTICS

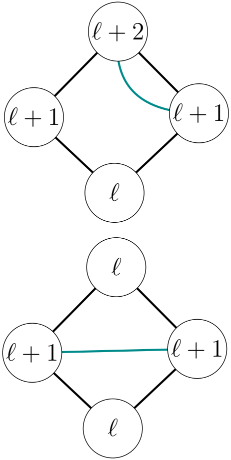


Labelled Quadrangulation

Two more statistics







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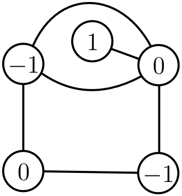
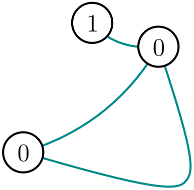


Labelled Quadrangulation

Ambjørn-Budd

Weakly labelled map



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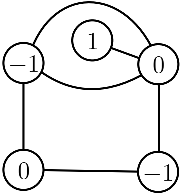
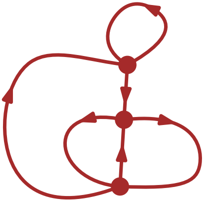
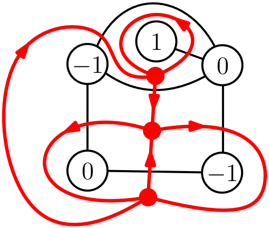


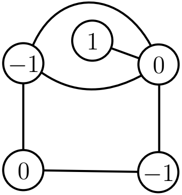
Labelled Quadrangulation

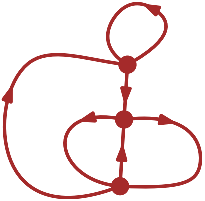
duality



Quartic Eulerian orientation



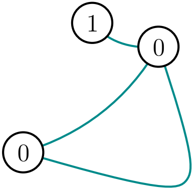




Labelled Quadrangulation ⟷ duality ⟷ Quartic Eulerian orientation

↕ Ambjørn-Budd

Weakly labelled map



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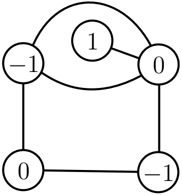
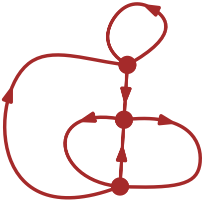
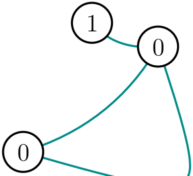


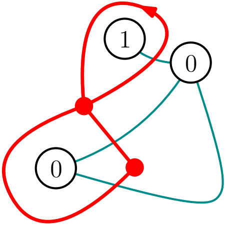


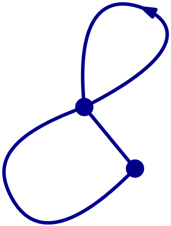
Labelled Quadrangulation ⟷ (duality) ⟷ Quartic Eulerian orientation

↕ Ambjørn-Budd

Weakly labelled map ⟷ (duality) ⟷ Eulerian partial orientation







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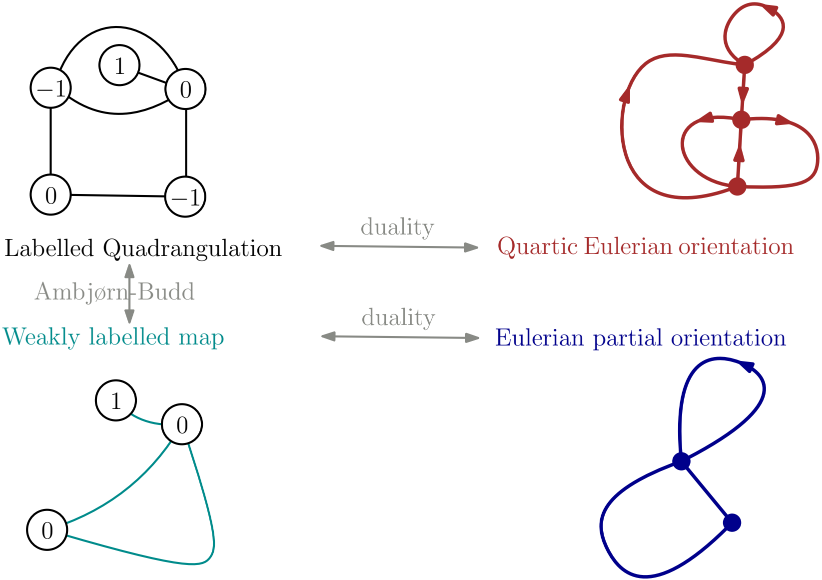

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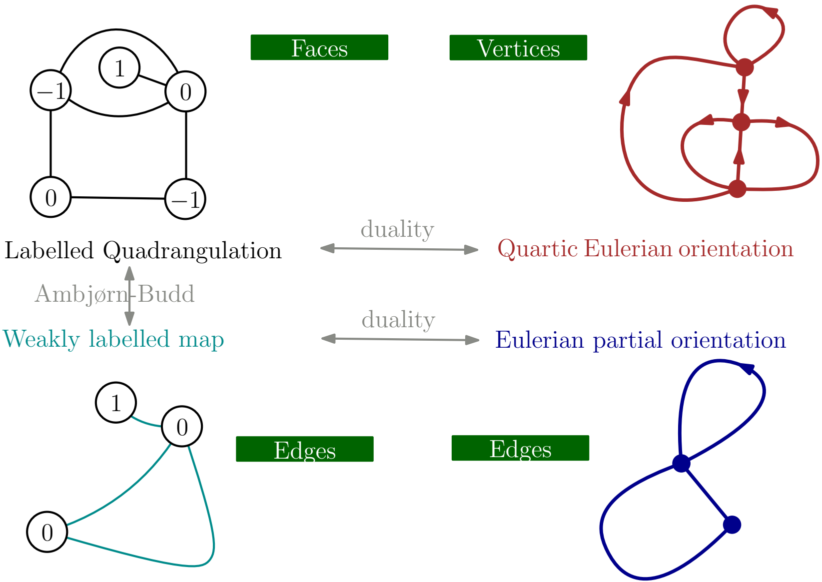

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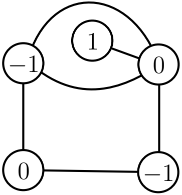
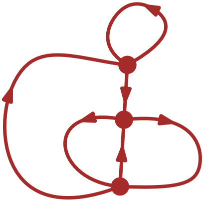


Faces

Alternating faces

Vertices

Alternating vertices



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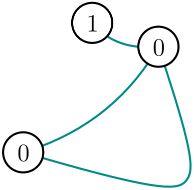
Weakly labelled map ⟵ duality ⟶ Eulerian partial orientation

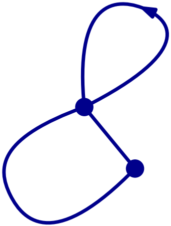


Edges

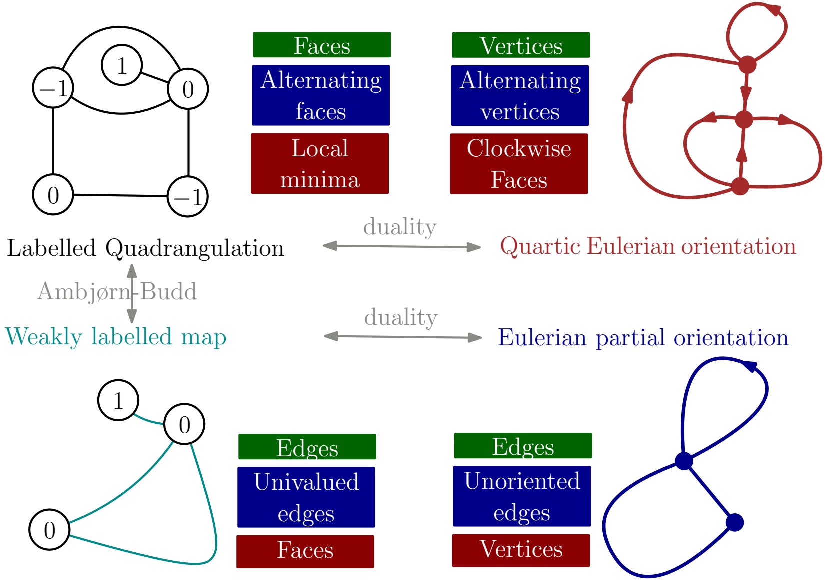
Univalued edges

Edges

Unoriented edges



Two more statistics



WEIGHTED MODEL BACKGROUND

- 2000: Quartic Eulerian orientation problem non-rigorously "solved" with weight ω [Kostov]
- 2013: Bijection between Height-labelled quadrangulations and Height-labelled maps [Ambjørn and Budd]
- 2017: Eulerian orientation enumeration problem posed [Bousquet-Mélou, Bonichon, Dorbec, Pennarun]
- 2018: Bijective link H-maps to EO-maps and H-quads to EO-quarts [E.P., Guttmann], conjectured Asymptotics
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New work: [E.P., Bousquet-Mélou, 2025]

- Exact solution for all ω (using algebraic methods)
- Exact solution for $\omega = 0, 1$ with new weight v
- Functional equations for all ω, v.

Solution part 1: Combinatorics $\rightarrow$ Functional equations for $\mathsf{Q}(t, \omega, v)$

Characterisation 1: There are series $\mathsf{P}(y)$, $\mathsf{D}(x,y)$ and $\mathsf{E}(x,y)$, uniquely defined by:

$$\mathsf{D}(x,y) = v + \frac{y}{v}\mathsf{D}(x,y)[z^1]\mathsf{D}(x,z) + y[x^{\geq 0}]\left(\frac{1}{x}\mathsf{D}(x,y)\mathsf{P}\left(\frac{t}{x}\right)\right),$$

$$(1-x)(\mathsf{D}(x,y)-v) = [y^{>0}]\mathsf{D}(x,y)\left(y\mathsf{P}(y) + y - vy + \omega\frac{t}{y} + \frac{t}{v}[z^1]\mathsf{D}\left(\frac{t}{y},z\right)\right),$$

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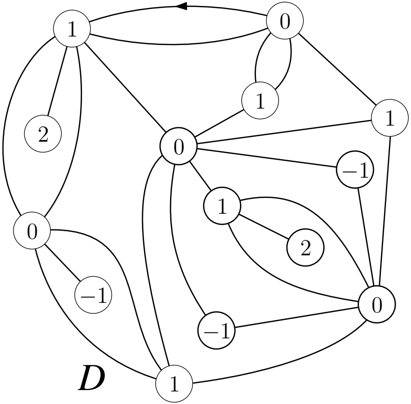
I will show one element of the proof.

D-PATCHES

D-patch: Digons are allowed next to the root vertex and the outer face may have any degree.



Restrictions:

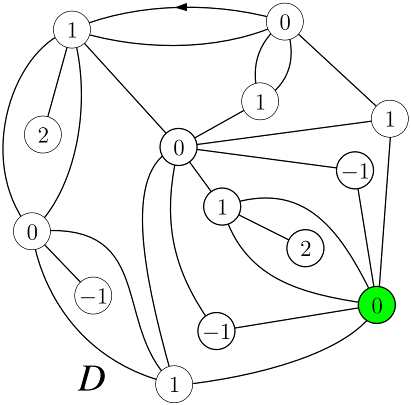
- outer labels must be 0 or 1.
- vertices adjacent to the root must be labelled 1.

In $\mathsf{D}(x, y)$:

- x counts digons.
- y counts the degree of the outer face (halved)
- t, ω, v same as before.

DECOMPOSITION OF D-PATCHES

Colour the vertex two places clockwise from the root vertex around the outer face.



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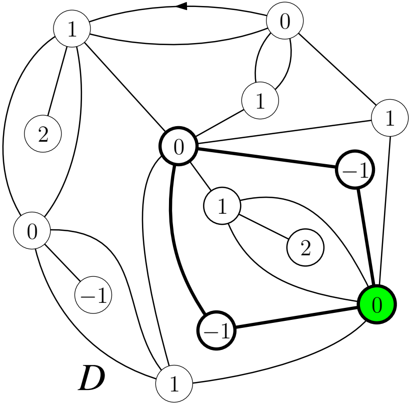
Decomposition of D-patches

Highlight the maximal connected subgraph of nonpositive labels, containing the coloured vertex.



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In $\mathsf{D}(x, y)$:
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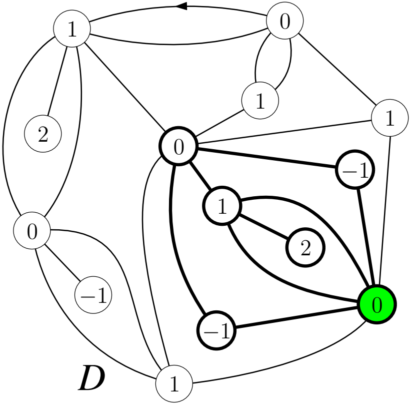
Decomposition of D-patches

Add to the subgraph all vertices and edges contained in its inner face(s).



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- vertices adjacent to the root must be labelled 1.

In $\mathsf{D}(x, y)$:
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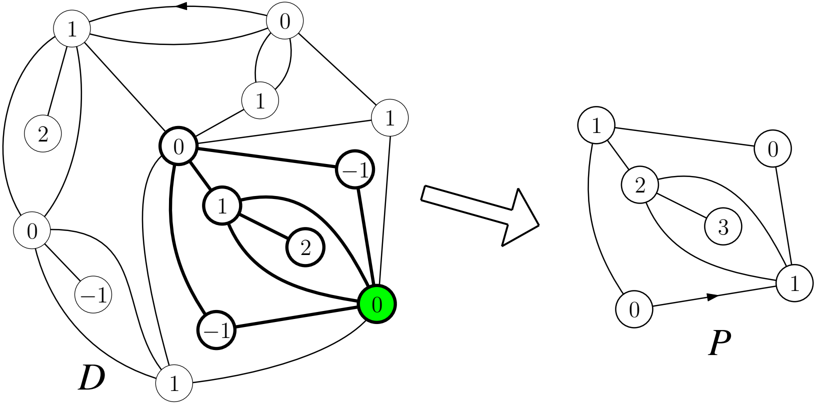
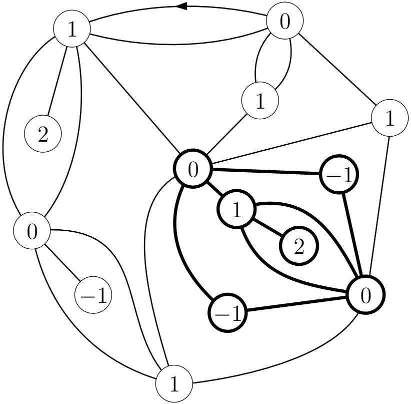
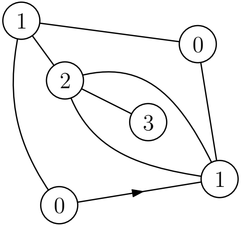
- t, ω, v same as before.

Record the subgraph with labels increased by 1.



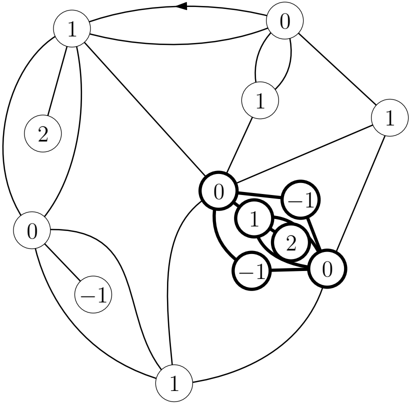
Contract the highlighted map to a single vertex (labelled 0).

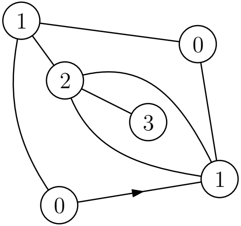




P

Contract the highlighted map to a single vertex (labelled 0).





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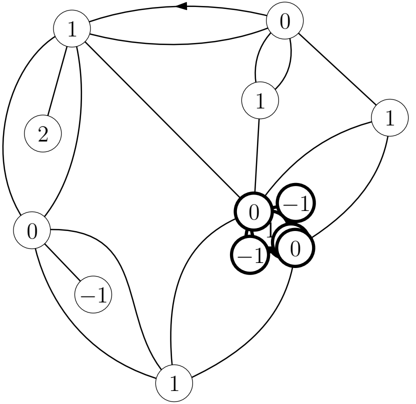
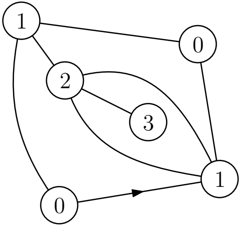
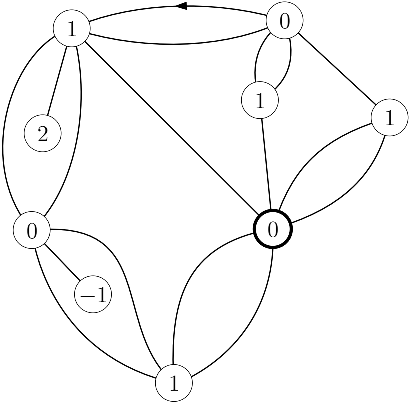


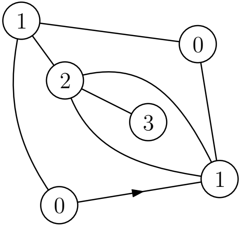


P

Decomposition of D-patches

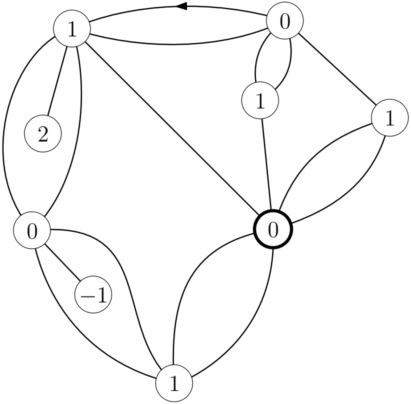
Contract the highlighted map to a single vertex (labelled 0).

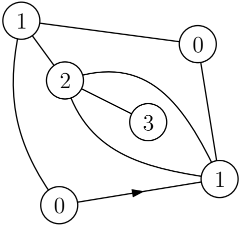




P

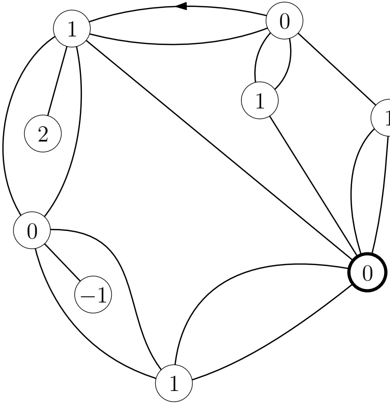
Contract the highlighted map to a single vertex (labelled 0). The new vertex may be adjacent to digons.

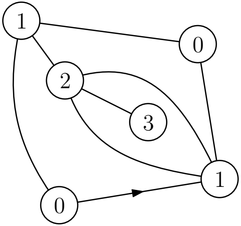




P

Merge the new vertex with the root vertex.





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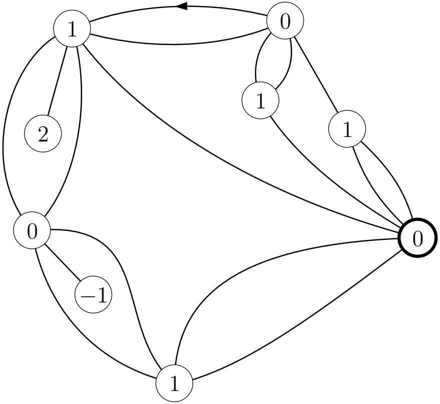
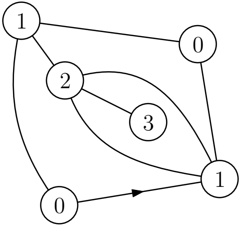
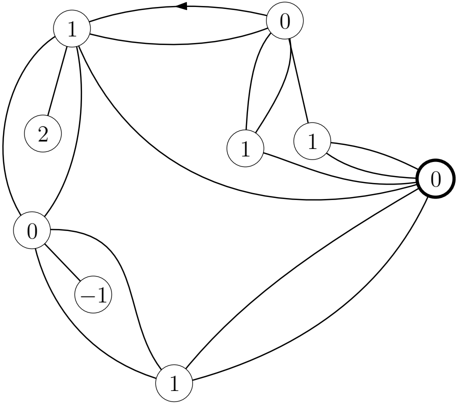


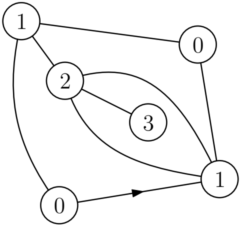


P

DECOMPOSITION OF D-PATCHES

Merge the new vertex with the root vertex.





P

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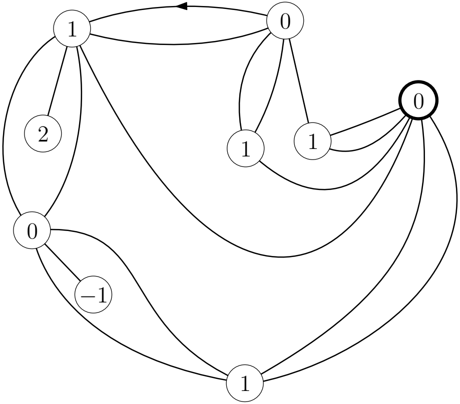

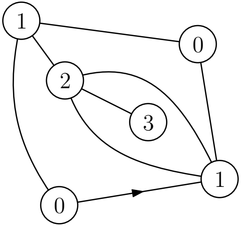

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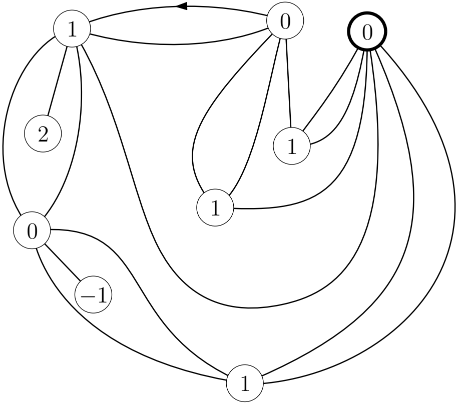

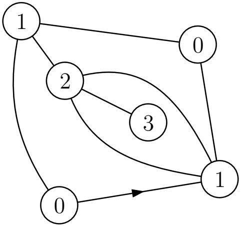

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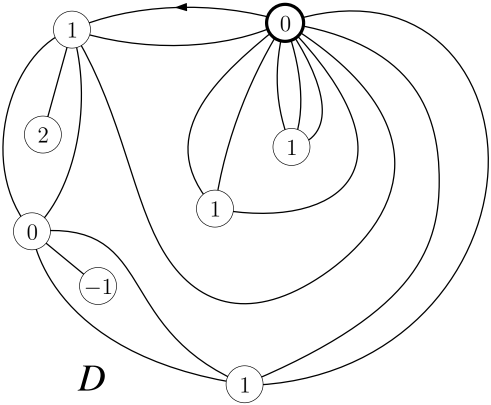
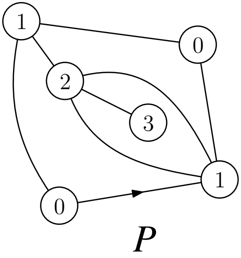
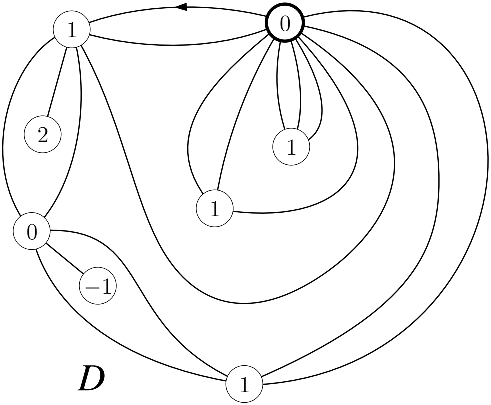
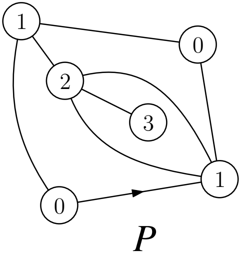




DECOMPOSITION OF D-PATCHES

Merge the new vertex with the root vertex. This new map is a D-patch!





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Simplification: Define $\mathcal{M}(x)$ by

$$\mathcal{M}(x) = \frac{t}{x}\mathsf{P}\left(\frac{t}{x}\right) + \frac{t}{v}[z^1]\mathsf{D}(x,z),$$

CHARACTERISATION OF $\mathcal{M}(x)$

Theorem: Fix $v, \omega \in \mathbb{C}$. There is a unique series

$$\mathcal{M}(y) = \sum_{n=1}^{\infty} \sum_{k=-n}^{\infty} m_{n,k} t^n y^k,$$

with $[y^{-1}]\mathcal{M}(y) = tv$ such that

$$y\mathcal{M}(y)\left(1 - \mathcal{M}(y) - \frac{(1-v)t}{y} - \omega y\right)$$

has only positive powers of y and

$$\mathcal{M}(\mathcal{M}(x)) = x.$$

The series $\mathsf{Q}(t, \omega, v)$ is given by

$$\mathsf{Q}(t, \omega, v) = t^{-2}[y^{-2}]\mathcal{M}(y) - v.$$

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The series $\mathsf{Q}(t, \omega, v)$ is given by

$$\mathsf{Q}(t, \omega, v) = t^{-2}[y^{-2}]\mathcal{M}(y) - v.$$

Next section: Solution for $\omega = 0, 1$
Following section: Solution for $v = 1$
Still open: General solution

Part 2: Solution for $\omega = 0, 1$

(Eulerian (partial) orientations by edges and vertices).

SOLUTION FOR $\omega = 0$

Theorem: Let $R(t, v)$ be the unique series with constant term 0 satisfying

$$t = \sum_{n,k\geq 0} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{2n+k}{n} t^k (v-1)^k R^{n+1}.$$

The generating function $\mathsf{Q}(t, 0, v)$ for height-labelled quadrangulations (with no alternating faces) counted by faces and local minima is given by

$$\mathsf{Q}(t,0,v) = -v + \frac{1}{t^2}\sum_{n,k} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{2n+k-1}{n} t^k (v-1)^k R^{n+1}.$$

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The generating function $\mathsf{Q}(t, 0, v)$ for Eulerian orientations counted by edges and vertices is given by

$$\mathsf{Q}(t, 0, v) = -v + \frac{1}{t^2} \sum_{n,k} \frac{1}{n+1} \binom{2n}{n} \binom{2n+k}{k} \binom{2n+k-1}{n} t^k (v-1)^k R^{n+1}.$$

SOLUTION FOR $\omega = 1$

Theorem: Let $R(t,v)$ be the unique series with constant term 0 satisfying

$$t = \sum_{n,k\geq 0} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{3n+2k}{n+k} t^k (v-1)^k R^{n+1}.$$

The generating function $\mathsf{Q}(t,1,v)$ for height-labelled quadrangulations counted by faces and local minima is given by

$$\mathsf{Q}(t,1,v) = -v + \frac{1}{t^2}\sum_{n,k} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{3n+2k-1}{2n+k} t^k (v-1)^k \mathsf{R}_1^{n+1}.$$

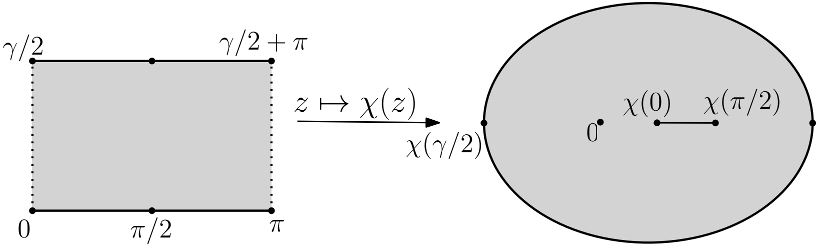
Theorem: Let $R(t, v)$ be the unique series with constant term 0 satisfying

$$t = \sum_{n,k \geq 0} \frac{1}{n+1} \binom{2n}{n} \binom{2n+k}{k} \binom{3n+2k}{n+k} t^k (v-1)^k R^{n+1}.$$

The generating function $\mathsf{Q}(t, 1, v)$ for Eulerian partial orientations counted by edges and vertices is given by

$$\mathsf{Q}(t,1,v) = -v + \frac{1}{t^2} \sum_{n,k} \frac{1}{n+1} \binom{2n}{n} \binom{2n+k}{k} \binom{3n+2k-1}{2n+k} t^k (v-1)^k \mathsf{R}_1^{n+1}.$$

Part 3: Analytic functional equations



ANALYTIC FUNCTIONAL EQUATIONS

Recall: There is a unique series $\mathcal{M}(y) \in \frac{t}{y}\mathbb{Z}[\omega, v][[y, t/y]]$ with $[y^{-1}]\mathcal{M}(y) = tv$ satisfying

$$y\mathcal{M}(y)\left(1 - \mathcal{M}(y) - \frac{(1-v)t}{y} - \omega y\right) \in K[[y]],$$

$$\mathcal{M}(\mathcal{M}(x)) = x,$$

Claim: For sufficiently small t, there is an even meromorphic function χ on $\mathbb{C}$ and some $\gamma \in i\mathbb{R}_{>0}$ satisfying $\chi(z+\pi) = \chi(z)$

$$\mathcal{M}(\chi(z)) = \chi(\gamma - z),$$

and

$$1 + \frac{t(v-1)}{\chi(z)} = \chi(\gamma + z) + \omega\chi(z) + \chi(z - \gamma).$$

Analytic functional equations

Recall: There is a unique series $\mathcal{M}(y) \in \frac{t}{y}\mathbb{Z}[\omega, v][[y, t/y]]$ with $[y^{-1}]\mathcal{M}(y) = tv$ satisfying

$$y\mathcal{M}(y)\left(1 - \mathcal{M}(y) - \frac{(1-v)t}{y} - \omega y\right) \in K[[y]],$$

$$\mathcal{M}(\mathcal{M}(x)) = x,$$

Claim: For sufficiently small t, there is an even meromorphic function χ on $\mathbb{C}$ and some $\gamma \in i\mathbb{R}_{>0}$ satisfying $\chi(z+\pi) = \chi(z)$

$$\mathcal{M}(\chi(z)) = \chi(\gamma - z),$$

and

$$1 + \frac{t(v-1)}{\chi(z)} = \chi(\gamma + z) + \omega\chi(z) + \chi(z - \gamma).$$

Last section: Solved for $\omega = 0, 1$.
Next section: Solution for $v = 1$.
Still open: All other values ω, v.

Part 4: Six vertex model $(\nu = 1)$

(Previous solution: Kostov (2000)/EP and Zinn-Justin (2019)).

Recall: Solutions at $\omega = 0, 1$

The generating function $\mathsf{Q}(t, 0, 1)$ is given by

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n}^2 \mathsf{R}_0(t)^{n+1},$$

$$\mathsf{Q}(t, 0, 1) = \frac{1}{2t^2}(t - 2t^2 - \mathsf{R}_0(t)).$$

The generating function $\mathsf{Q}(t, 1, 1)$ is given by

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} \binom{3n}{n} \mathsf{R}_1(t)^{n+1},$$

$$\mathsf{Q}(t, 1, 1) = \frac{1}{3t^2}(t - 3t^2 - \mathsf{R}_1(t)).$$

Solution for $\mathsf{Q}(t, \omega, 1)$

Define

$$\vartheta(z,q) = \sum_{n=0}^{\infty}(-1)^n(e^{(2n+1)iz} - e^{-(2n+1)iz})q^{(2n+1)^2/8}.$$

Let $q = q(t,\alpha)$ be the unique series satisfying

$$t = \frac{\cos\alpha}{64\sin^3\alpha}\left(-\frac{\vartheta(\alpha,q)\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)^2} + \frac{\vartheta''(\alpha,q)}{\vartheta'(\alpha,q)}\right).$$

Define $\mathsf{R}(t,\omega)$ by

$$\mathsf{R}(t,-2\cos(2\alpha)) = \frac{\cos^2\alpha}{96\sin^4\alpha}\frac{\vartheta(\alpha,q)^2}{\vartheta'(\alpha,q)^2}\left(-\frac{\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)} + \frac{\vartheta'''(0,q)}{\vartheta'(0,q)}\right).$$

Then

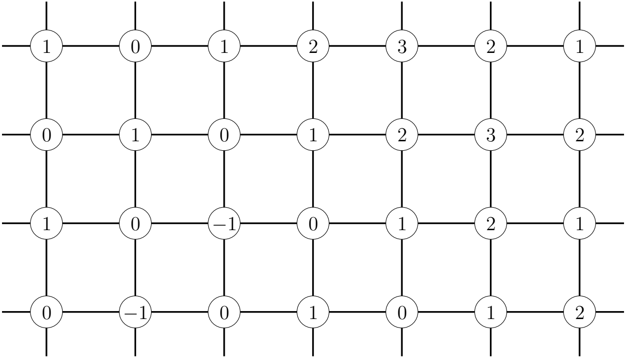
$$\mathsf{Q}(t,\omega) = \frac{1}{(\omega+2)t^2}\left(t - (\omega+2)t^2 - \mathsf{R}(t,\omega)\right).$$

Part 5: Height distribution

(To appear eventually EP 2025+)

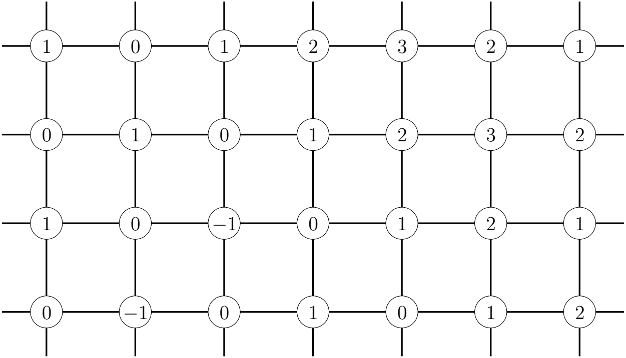
Height function on lattice

Height function: Labelling of vertices of square lattice where adjacent labels differ by 1, origin is labelled 0.



HEIGHT FUNCTION ON LATTICE

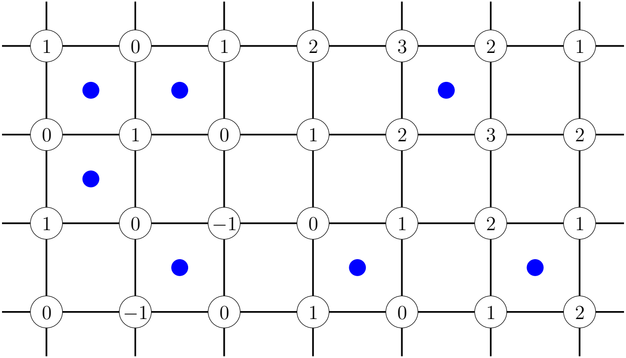
Height function: Labelling of vertices of square lattice where adjacent labels differ by 1, origin is labelled 0.

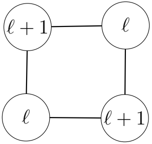


Claim: Uniform random height function well defined.
Theorem: For large n the height of $(n,0)$ has variance $\sim \log(n)$.
[Duminil-Copin, Harel, Laslier, Raoufi, Ray, 2019]
Conjecture: Converges to *Gaussian free field* (GFF).

Weighted height function on lattice

Height function: Labelling of vertices of square lattice where adjacent labels differ by 1, origin is labelled 0.





Alternating (weight ω)

Boltzmann weight: $\omega \geq 0$ per alternating face.

Weighted height function on lattice

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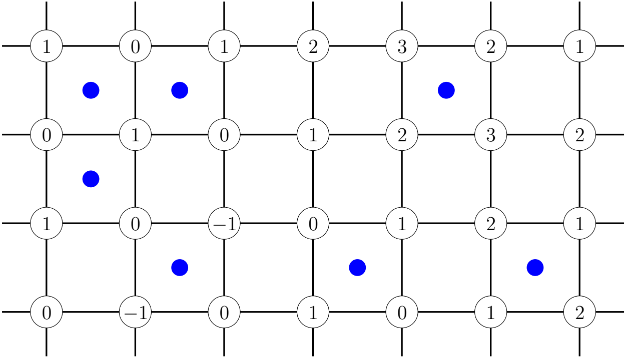
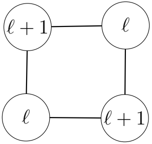




Alternating (weight ω)

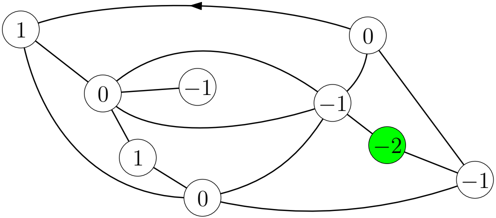
Boltzmann weight: $\omega \geq 0$ per alternating face.

Theorem: If $\omega \in [1,2]$, the height of $(n,0)$ has variance $\sim \log(n)$. [Duminil-Copin, Karrila, Manolescu, Oulamara, 2022]

Theorem: If $\omega > 2$, height variance bounded. [Glazman, Peled, 2019]

Height distribution in height labelled map

We now count height-labelled quadrangulations with a highlighted vertex v which gets weight $\delta^{\text{height of } v}$.
New generating function: $\hat{\mathsf{Q}}(t, \omega, \delta)$.



This example contributes $t^7\omega^2\delta^{-2}$ to $\hat{\mathsf{Q}}(t, \omega, \delta)$

We have now found the exact form of $\hat{\mathsf{Q}}(t, \omega, \delta)$, using theta functions.

Recall: Solution for $\mathsf{Q}(t,\omega)$

Define

$$\vartheta(z,q)=\sum_{n=0}^{\infty}(-1)^n\left(e^{(2n+1)iz}-e^{-(2n+1)iz}\right)q^{(2n+1)^2/8}.$$

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Then

$$\mathsf{Q}(t,\omega,1)=\frac{1}{(\omega+2)t^2}\left(t-(\omega+2)t^2-\mathsf{R}(t,\omega)\right).$$

SOLUTION FOR $\hat{\mathsf{Q}}(t, \omega, \delta)$

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Define $\hat{\mathsf{R}}(t, \omega, \delta)$ by

$$\hat{\mathsf{R}}(t, -2\cos(2\alpha), e^{2i\beta}) = \frac{\cos\alpha\sin\beta}{\sin\alpha\cos\beta}\frac{\vartheta(\alpha, q)\vartheta'(\beta, q)}{\vartheta'(\alpha, q)\vartheta(\beta, q)}.$$

Then

$$\hat{\mathsf{Q}}(t, \omega, \delta) = (\delta + 1)\frac{1 - 2t(\omega + \delta + \delta^{-1}) + \hat{\mathsf{R}}(t, \omega, \delta)}{2t(\omega + \delta + \delta^{-1})}.$$

HEIGHT DISTRIBUTION

From $\hat{\mathsf{Q}}(t, 1, \delta)$ we get the exact distribution of vertex heights in height-labelled quadrangulations with n faces.

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$$V_n \sim \frac{3}{2\pi^2} \log(n)^2.$$

HEIGHT DISTRIBUTION

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$$V_n \sim \frac{3}{2\pi^2} \log(n)^2.$$

- After rescaling by multiplying each height by $\frac{\pi^2}{3\log(n)}$, the limiting distribution has density function

$$4\frac{(x-1)e^x + (x+1)e^{-x}}{(e^x - e^{-x})^3}.$$

- Similar for any $\omega \in [0, 2)$.
- Appears that heights localised when $\omega > 2$.

Further questions

- Bijective interpretations for nice formulas (at $\omega = 0$ and $\omega = 1$)
- Understand local minima and maxima simultaneously
- prove maps converge to critical Liouville quantum gravity.
- More reasonably: Prove that other *observables* behave as they should according to the description above.

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Thank you!

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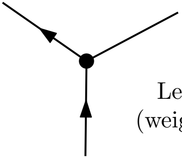
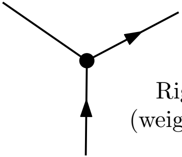
Let $\mathsf{C}(t,\omega)$ be the generating function for partially oriented cubic maps in which each vertex is one of the following types.





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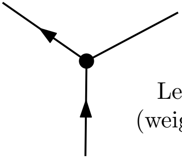
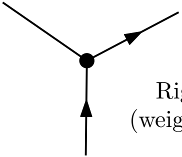


Left turn
(weight $\omega\sqrt{t}$)



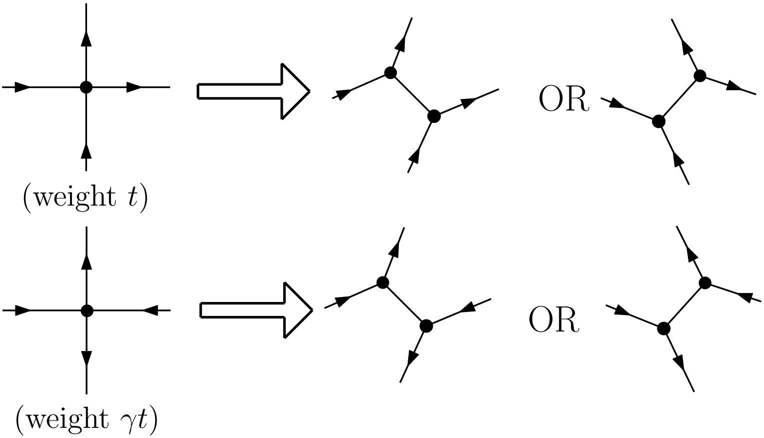
Right turn
(weight $\omega^{-1}\sqrt{t}$)

Theorem: $\mathsf{Q}(t,\omega^2+\omega^{-2}) = \mathsf{C}(t,\omega)$.

Bijection 3: A loop model

(Kostov (2000)).

Theorem: $\mathsf{Q}(t,\omega^2+\omega^{-2}) = \mathsf{C}(t,\omega)$



BONUS SLIDE: BIJECTION TO A LOOP MODEL

Theorem: $\mathsf{Q}(t, \omega^2 + \omega^{-2}) = \mathsf{C}(t, \omega)$