# Stieltjes moment sequences and coefficientwise Hankel-total positivity in enumerative combinatorics

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Tony Guttmann 2025 — 80 and (still) counting
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Overview of a big project in collaboration with
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- From numbers to polynomials: Coefficientwise Hankel-TP
- Combinatorics of classical continued fractions
- Application: Some examples of coefficientwise Hankel-TP
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- (Tentative) conclusion

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- (Contrast with positive semidefiniteness.)
- But . . . In many areas of mathematics, there is a preferred basis.

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**Applications of total positivity:**

- Mechanics of oscillatory systems
- Zeros of polynomials and entire functions
- Numerical linear algebra
- Approximation theory
- Stochastic processes
- Lie theory and cluster algebras
- Representation theory of the infinite symmetric group
- Planar discrete potential theory and the planar Ising model
- Stieltjes moment problem
- Enumerative combinatorics

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$$H_\infty(\boldsymbol{a}) \;=\; (a_{i+j})_{i,j \geq 0} \;=\; \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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- We say that the sequence $\boldsymbol{a}$ is *Hankel-totally positive* if its Hankel matrix $H_\infty(\boldsymbol{a})$ is totally positive.
- This implies that the sequence is *log-convex*, but is much stronger.

# Hankel-total positivity and the moment problem

## Main Characterization (Stieltjes 1894, Gantmakher–Krein 1937)

For a sequence $\boldsymbol{a} = (a_n)_{n \ge 0}$ of real numbers, the following are equivalent:

(a) $\boldsymbol{a}$ is Hankel-totally positive.

(b) There exists a positive measure $\mu$ on $[0, \infty)$ such that $a_n = \int x^n \, d\mu(x)$ for all $n \ge 0$.

[That is, $\boldsymbol{a}$ is a **Stieltjes moment sequence**.]

(c) There exist numbers $\alpha_0, \alpha_1, \ldots \ge 0$ such that

$$\sum_{n=0}^{\infty} a_n t^n \;=\; \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}}$$

in the sense of formal power series.

[**Stieltjes-type continued fraction** with nonnegative coefficients]

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[or, From counting to counting-with-weights]

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   Counting permutations of $[n]$ by number of descents:

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An industry in combinatorics: cf. Sokal–Zeng 2020 and Deb–Sokal 2022

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Coefficientwise Hankel-TP is **combinatorial**, not analytic.

Coefficientwise Hankel-TP *implies* that $(P_n(\mathbf{x}))_{n\geq 0}$ is a Stieltjes moment sequence for all $\mathbf{x} \geq 0$, but it is *stronger*.

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- More general approach: *production matrices* — still *sufficient but far from necessary*.

# Classical continued fractions

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- Stieltjes-type continued fractions (S-fractions):

$$\sum_{n=0}^{\infty} \underbrace{S_n(\boldsymbol{\alpha})}_{\text{Stieltjes–Rogers polynomial}} t^n \quad = \quad \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}}$$

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- Jacobi-type continued fractions (J-fractions):

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- This is combinatorialists' notation. Analysts take $t^n \to \dfrac{1}{z^{n+1}}$

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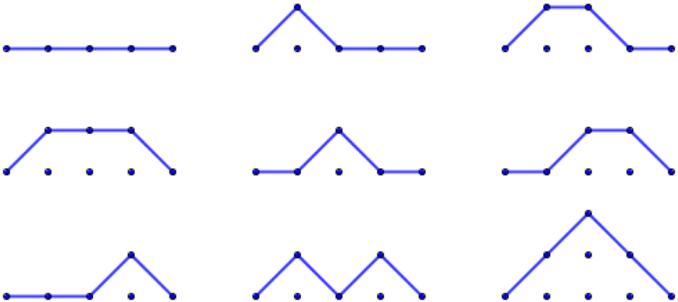

The nine Motzkin paths of length $n = 4$.

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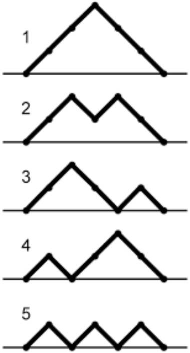


The five Dyck paths of length $2n = 6$.

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**Theorem (Flajolet 1980)**

- The Jacobi–Rogers polynomial $J_n(\boldsymbol{\beta}, \boldsymbol{\gamma})$ is the generating polynomial for Motzkin paths of length $n$, in which each rise gets weight 1, each level step at height $i$ gets weight $\gamma_i$, and each fall from height $i$ gets weight $\beta_i$.
- The Stieltjes–Rogers polynomial $S_n(\boldsymbol{\alpha})$ is the generating polynomial for Dyck paths of length $2n$, in which each rise gets weight 1 and each fall from height $i$ gets weight $\alpha_i$.

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**Theorem (A.S. 2014, based on Viennot 1983)**

The sequence $(S_n(\alpha))_{n \geq 0}$ of Stieltjes–Rogers polynomials is coefficientwise Hankel-totally positive in the polynomial ring $\mathbb{Z}[\alpha]$.

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Proof uses the Lindström–Gessel–Viennot lemma on families of nonintersecting paths.

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Can now specialize $\alpha$ to *nonnegative* elements in any partially ordered commutative ring, and get Hankel-TP.

I will show some examples . . .

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- It is not even totally positive in $\mathbb{R}$ for all $\boldsymbol{\beta},\boldsymbol{\gamma} \ge 0$.
- Rather, the total positivity of $H_\infty(\boldsymbol{J})$ holds only when $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ satisfy suitable *inequalities*.

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Form the infinite tridiagonal matrix

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Proof uses the method of *production matrices.*

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- But the method is much more general.
- Big computational problem: Given a Hankel-TP sequence $\boldsymbol{a} = (a_n)_{n \geq 0}$, find a TP production matrix that generates $\boldsymbol{a}$ as the zeroth column of its output matrix. Does one even necessarily exist?

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- Elementary "renewal" argument on Dyck paths implies

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- **Conclusion:** The sequence $(N_n(x))_{n\geq 0}$ of Narayana polynomials is coefficientwise Hankel-totally positive.

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- See Sokal–Zeng 2020 for extensions to even more variables.

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- There is no S-type continued fraction *in the ring of polynomials*: we have

$$\alpha_1, \alpha_2, \ldots \;=\; 1+x,\; \frac{2x}{1+x},\; \frac{1+x^2}{1+x},\; \frac{x+x^2}{1+x^2},\; \frac{1+x^3}{1+x^2},\; \frac{x+x^3}{1+x^3},\; \ldots$$

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- **Conclusion** (A.S. unpublished 2014, Wang–Zhu 2016):

  The sequence $(W_n(x))_{n\geq 0}$ of Narayana polynomials of type B is coefficientwise Hankel-totally positive.

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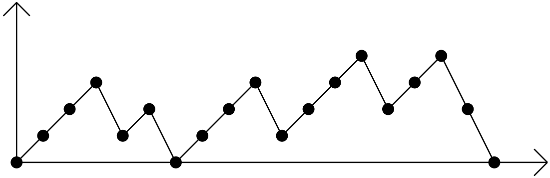

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Proof is essentially *identical* to the one for $m = 1$!

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But we hardly ever use these formulae.

We use (a) the graphical description, and/or (b) recurrences.

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- The $\alpha$ are indeed positive, but what the hell are they???

# Branched continued fractions: An example

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- See Pétréolle–A.S.–Zhu 2018 for details

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- Don't know (even conjecturally) any continued fraction or production matrix. (Or any good combinatorial interpretation.)

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*Dedicated to the memory of Philippe Flajolet (1948–2011)*