Spiral walks on the triangular lattice.

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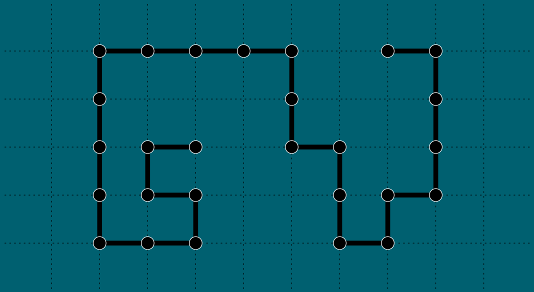
Self-avoiding walks

A typical square-lattice SAW



How many are there?

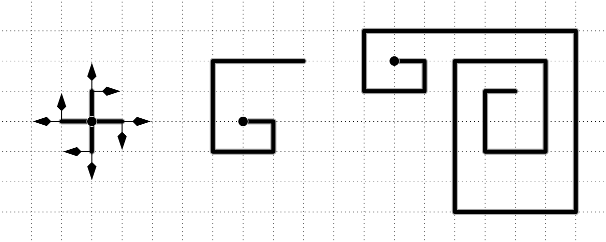
- Let c_n be the number of n-step SAW equivalent up to a translation.
- By concatenation arguments, $c_{n+m} \leq c_n c_m$, hence $c_n \geq k^n$, with $k > 1$.
- This implies that there exists $\mu > 1$ such that $\lim_{n\to\infty}(c_n)^{1/n} = \mu$.
- It is universally accepted that $c_n \sim const \times \mu^n n^g$.
- However all that has been proved is $c_n \sim const \times \mu^{n+O(\sqrt{n})}$.

What is believed?

- It has not even been proved that the sub-dominant term n^g exists for SAW in dimension $d = 2$ or $d = 3$.
- But it is widely believed that $g = \frac{11}{32}$ when $d = 2$.
- This form, combining dominant exponential growth with a sub-dominant power law term is widely observed in lattice statistics.
- Examples include percolation clusters, Ising and Potts model graphs.
- It reflects the expected power-law generating function $\sum c_n z^n \sim const. \times (1 - \mu z)^{-(1+g)}$.

Spiral SAW

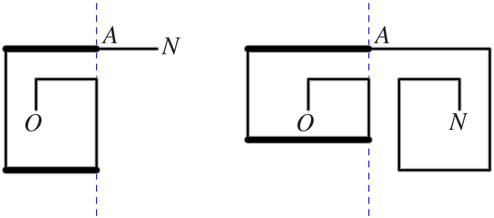
A solvable version arises if left turns are forbidden. Proposed by Privman (1983). Solved by Blotë and Hilhorst (1984) and Guttmann and Wormald (1984), with full asymptotic expansion by Joyce (1984).



Note that such walks can only grow by spiraling outwards—once their projection can "see" the walk, they are doomed!

SPIRAL SAW

Solution of Wormald and AJG used results of Szekeres (1951) for $q(n,k)$, the number of partitions of n into $k = \mathrm{O}(\sqrt{n})$ parts.



We considered single spirals, and then worked out how to put them together.

- The number of single spirals s_n^* has ogf

$$S^*(x) = \sum s_n^* x^n = \frac{x}{1-x} \prod_{n=1}^{\infty} (1-x^n)^{-1},$$

- hence

$$s_n^* \sim \frac{1}{4\sqrt{3}n} \exp(\pi\sqrt{2n/3}).$$

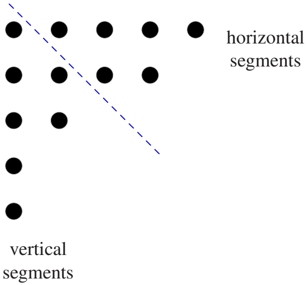
- The number of spiral SAWs is then

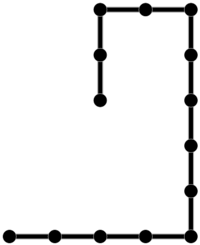
$$s_n \sim \frac{\pi}{4 \cdot 3^{5/4} \cdot n^{7/4}} \exp(2\pi\sqrt{n/3}).$$

- Joyce subsequently obtained the full asymptotic expansion.

SPIRAL SAW

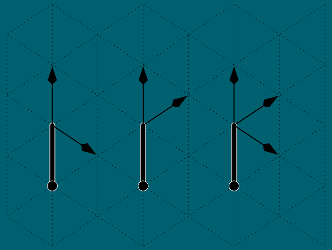
Szekeres and Richmond recognised the result for single spirals—just the formula for partitions of the integers. Hirschhorn found a bijection to prove this:





Triangular spiral SAW—three cases

There are three ways the "no left turn" rule can be implemented on the triangular lattice



We call these Cases I, II, III

Model 1

- This case was first considered by Richard Brak and Geoff Joyce.
- It involves a partition problem similar to that encountered in the square lattice case.
- The number of single spirals s_n^* has ogf

$$S^*(x) \sum s_n^* x^n = \frac{x}{1-x} \prod_{n=1}^{\infty} (1 + x^n),$$

-

$$s_n^* \sim \frac{3^{1/4}}{2\pi n^{1/4}} \exp(\pi\sqrt{n}/\sqrt{3}).$$

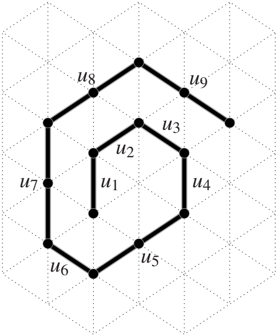
-

$$s_n \sim \frac{\pi \cdot 6^{1/4}}{9 \cdot n^{5/4}} \exp(2\pi\sqrt{n/6}) + \mathrm{O}(1/\sqrt{n}).$$

Cases II and III

- I started to work on Cases II and III in 1984, and quickly realised that more subtle partition problems arose.
- Checking references I kept encountering papers by Szekeres, by Erdös, by Turán and by Szalay.
- I visited UNSW to discuss the problem with George, and we immediately began a collaboration.

Formulation of model II



We label successive segments $u_1 \cdots u_k$. Notice there is no "obvious" partition problem.

Note that $u_i > 0$, for $i = 1 \ldots k$. We define $u_0 = 0$.
A single spiral is defined by the constraint

$$u_{i-1} + u_i < u_{i+2} + u_{i+3}, \quad 1 \le i \le k-3.$$

Next, set $t_i = u_i + u_{i-1}$, then the above conditions can be simplified to

$$0 < t_i < t_{i+3}, \quad 1 \le i \le k-3.$$

Then

$$u_i = \sum_{k=0}^{i-1} (-1)^k t_{i-k} > 0.$$

The problem is then to count the number of partitions of
$n = \sum u_i = \sum_{j \ge 0} t_{k-2j}$ with t_i satisfying the above two conditions.

The next step is to define yet another variable,

$$d_i = t_i - t_{i-3}, \quad i = 1 \ldots k.$$

Then

$$n = \sum_{j \geq 0} d_{k-3j} + \sum_{j \geq 0} d_{k-3j-2} + \sum_{j \geq 0} d_{k-3j-4} + \ldots$$

while the constraint equation becomes

$$\begin{aligned} u_i &= (d_i - d_{i-1} + d_{i-2}) + (d_{i-6} - d_{i-7} + d_{i-8}) + \\ &+ (d_{i-12} - d_{i-13} + d_{i-14}) + \ldots > 0. \end{aligned}$$

George's idea was to focus on the partition problem first, then worry about the constraint later.

This required two new results. The first can be deduced from his 1951 paper (used in earlier work on square spirals):

Theorem 1. Let $q_r(m)$ denote the number of partitions of m into r unequal positive integer parts. Then for $\lambda = r - (2/\pi)\sqrt{3m}\log 2 = \mathrm{O}(m^{1/3})$ we have asymptotically, for m large

$$q_r(m) \sim \frac{1}{4m\sqrt{(6\gamma)}} \exp[\pi\sqrt{(m/3)} - \pi\lambda^2/2\gamma\sqrt{(3m)}]$$

where $\gamma = 1 - 12(\log 2/\pi)^2 = 0.4158391...$
This comes from considering where the (sharp) maximum of $q_r(m)$ for fixed m occurs when the number of summands is in the neighbourhood of $(2/\pi)\sqrt{3m}\log 2$. The point is that the distribution is Gaussian about r_0 with variance $\gamma\sqrt{(3m)}/\pi$.

The next theorem is new, and the proof was subsequently given in the Bulletin of the AustMS in 1987 by Szekeres.

Theorem 2. Define $\sigma = \sqrt{n}\exp(\frac{-k\pi}{2\sqrt{3n}})$. Let $Q_k(n)$ denote the number of the above partitions of n into unequal parts of which k is the largest summand. Then for large n and $1/\sigma = \mathrm{O}(n^{1/6})$,

$$Q_k(n) \sim Q(n)\frac{\sigma}{\sqrt{n}}\exp[-2\sqrt{3}\sigma)/\pi]$$

where $Q(n)$ is the total number of partitions of n into unequal parts. This extends a theorem of Erdös and Lehmer. A corollary is that for

almost all partitions of n into unequal parts the largest summand is

$$k = \frac{\sqrt{(3n)}}{\pi}\log n + \mathrm{O}(\sqrt{n}W(n))$$

where $W(n)$ diverges arbitrarily slowly.

MASTER THEOREM.

In our application, we need to consider the distribution of unequal partitions of n in which both the number of summands r, and the largest summand k, varies. This is given by

Theorem 3 Let $q(n;r,k)$ denote the number of partitions of n into r unequal positive integer parts, with maximal summand k. Then for large n and for $\lambda = r - (2/\pi)\sqrt{3n}\log 2 = \mathrm{O}(n^{1/3})$, $\sigma = \sqrt{n}\exp(\frac{-k\pi}{2\sqrt{3n}})$, and $1/\sigma = \mathrm{O}(n^{1/6})$, we have

$$q(n;r,k) \sim \frac{\sigma}{4n^{3/2}\sqrt{(6\gamma)}}\exp[\pi\sqrt{(n/3)} - 2\sigma\sqrt{3}/\pi - \pi\lambda^2/2\gamma\sqrt{(3n)}].$$

This contains theorems 1 and 2 as special cases.

Next, we rewrite n as the sum of six terms:

$$n = \sum_{i=0}^{5} n_i$$

where

$$n_0 = d_k + 2d_{k-6} + 3d_{k-12} + 4d_{k-18} + \ldots, \quad \text{etc.}$$

and all n_i are of the form

$$m_1 + 2m_2 + 3m_3 + 4m_4 + \ldots.$$

All terms except d_{k-1} appear in this reformulation.
The number $q_r(m)$ of decompositions of m in the form

$$m = \sum_{k=1}^{r} km_k, \quad m_k > 0$$

is just the number of partitions of m into r distinct summands.

Approximate sum

Then disregarding the condition $u_i > 0$ for the moment, and the fact that d_{k-1} doesn't appear, we have for the number of single spirals

$$\hat{s}_n^* = \sum_{k>0} \sum_{n=\sum n_i} \prod_{i=0}^{5} q_{r_i}(n_i)$$

where $r_i = [\frac{k+5-i}{6}]$.
We use Theorem 1, noting that $q_r(n) \sim q_{r+1}(n) \sim \max_r(q_r n)$ as n and r get large, to get a messy expression which has a sharp maximum when all the n_i are approximately equal to each other and hence to $n/6$. The sum over r is performed by integration, as is the final sum over n, which is a five-dimensional integral, yielding

$$\hat{s}_n^* \sim \frac{\exp(\pi\sqrt{2n})}{64\sqrt{3}\gamma^{5/2}n^2}.$$

Imposing constraints

Now we fix this up, by considering the effect of the constraints we have neglected. It turns out that d_{k-1} is only constrained by the requirements that it be positive and less than the maximum summand of n_0 into unequal partitions. Thus

$$d_{k-1} \leq \frac{\sqrt{n}}{\pi\sqrt{2}} \log(\frac{n}{6}) + \mathrm{O}(\sqrt{n}\mathrm{W}(n)).$$

Thus $\hat{s}_n^*$ must be multiplied by this factor to account for the freedom of d_{k-1}.

The final factor arises from the constraint $u_i > 0$. A little asymptotic analysis, shows that the effect of this constraint is to introduce a multiplicative constant factor, given by

$$\phi = \sum 2^{-(d_1+d_2+d_3+\ldots d_r)},$$

where the sum is over all d_i satisfying the constraint.
We were only able to determine this sum numerically, by constructing a sequence of inequalities, giving a monotone decreasing sequence of estimates, which we could extrapolate to give $\phi \approx 0.009$.
Thus we obtain

$$s_n^* = \frac{\phi \log(n/6) \exp(\pi\sqrt{2n})}{64\sqrt{6}\pi\gamma^{5/2}n^{3/2}}.$$

FULL SPIRALS

Finally, we need to consider how we can concatenate two spirals, which is quite complicated, and we can only obtain an asymptotic estimate, invoking theorem 2. This, and replacing sums by integrals, finally yields

$$s_n = \frac{\phi^2 \log n/12 \exp(2\pi\sqrt{n})}{768\gamma^5 n^{13/4}}.$$

Model III

This behaves in exactly the same way as model II, the only change being in the value of the pre-multiplying constant ϕ, which increases to $\phi \approx 0.16$.

See *Spiral self-avoiding walks on the triangular lattice* by G. Szekeres and A. J. Guttmann, J. Phys. A:Math. Gen **20** (1987) 481-493.