Staircase polygons revisited: an open question by Richard Brak

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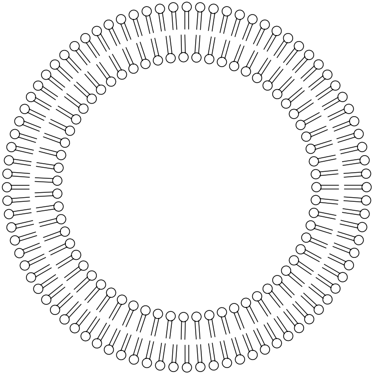
Richard Brak Memorial Symposium

Melbourne, February 2022

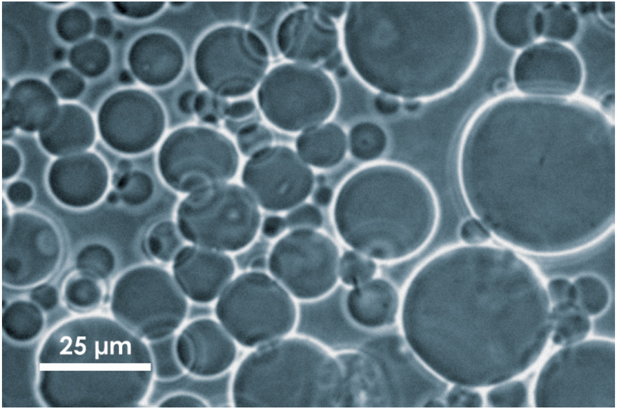
Biological Vesicles

- Vesicles: closed membranes formed of lipid bi-layers



Biological Vesicles

- Vesicles commercially produced by electro-swelling



Modelling Vesicles

We model vesicles as two-dimensional self-avoiding polygons. This is known as the Fisher-Guttmann-Whittington vesicle model.

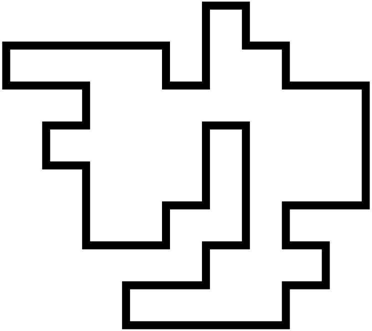


Figure: A self-avoiding polygon of perimeter $2n = 52$ and area $a = 37$.

The Vesicle Generating Function

$$G(t,q)=\sum_{n,a} c_{n,a}\, t^n\, q^a$$

where $c_{n,a}$ is the number of SAP with perimeter $2n$ and area a.

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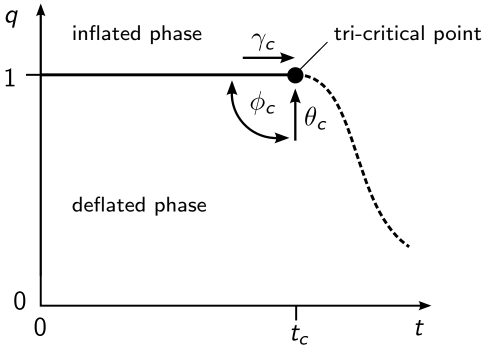

Relevant Papers

Understanding the scaling behaviour

- **R. Brak**, A. L. Owczarek, and TP, “A Scaling Theory of the Collapse Transition in Geometric Cluster Models of Polymers and Vesicles,” J. Phys. A: Math. Gen. **26** (1993) 4565-4579

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- TP and **R. Brak**, "Critical Exponents from Non-Linear Functional Equations for Partially Directed Cluster Models," J. Stat. Phys. **78** (1995) 701-730

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and finally computing a scaling function

- TP, "Uniform q-Series Asymptotics for Staircase Polygons," J. Phys. A: Math. Gen. **28** (1995) 1289-1304

Richard's most cited paper (on Scopus)

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Article • *Open access*
Critical exponents from nonlinear functional equations for partially directed cluster models
Prellberg, T., Brak, R.
Journal of Statistical Physics, 1995, 78(3-4), pp. 701–730

85
Citations

Also relevant

- M Bousquet-Mélou and X G Viennot, "Empilements de segments et *q*-énumération de polyominioes convex dirigés," J. Comb. Th. A **60** (1992) 196-224

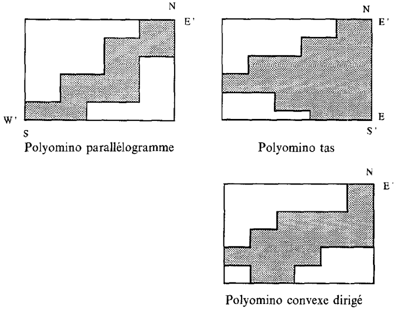
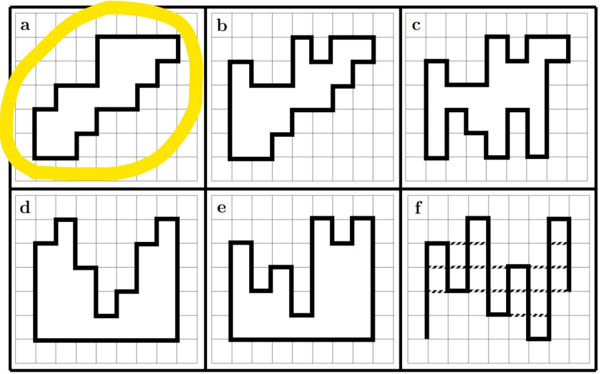


FIG. 0.4. Différentes familles de polyominos convexes.

(first cited by us in 1999!)

Staircase polygons



Panel (a) shows staircase polygons, a.k.a. skew-Ferrers diagrams or parallelogram polyominos

Staircase polygon generating function I

Differentiate horizontal/vertical steps with variables x/y, respectively, so $G(t,q)$ becomes $S(x,y,q)$ with

$$S(x,y,q) = bt \left(\frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-{\color{red}q}x)^n}{\prod_{k=1}^{n}(1-q^k)\prod_{k=1}^{n}(1-yq^k)}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-x)^n}{\prod_{k=1}^{n}(1-q^k)\prod_{k=1}^{n}(1-yq^k)}} - 1 \right)$$

- $G(t,q) = S(t,t,q)$
- notice q-Bessel functions (I will avoid q-series notation)
- $S(x,y,q)$ not explicitly symmetric in x and y!

Methods for finding an exact solution

- bijections to other objects and using known results
- heaps of pieces ("recognisable" from the given expression)
- combinatorial decomposition, recurrences/functional equations

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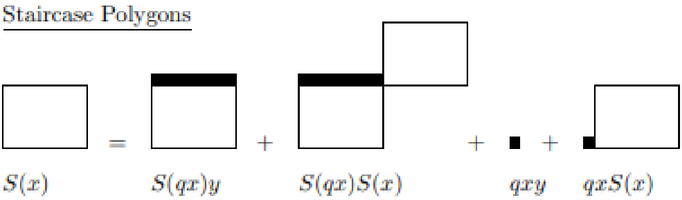
With $S(x) \equiv S(x, y, q)$



Solving the functional equation

- Linearise $S(x) = S(qx)y + S(qx)S(x) + qxy + qxS(x)$ with

$$S(x) = y\left(\frac{T(qx)}{T(x)} - 1\right)$$

to get

$$T(x) - (1+y)T(qx) + yT(q^2x) = -qxT(qx)$$

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$$(1-q^n)(1-yq^n)t_n + q^n t_{n-1} = 0$$

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- Finish off by iterating the recurrence

$$t_n = \frac{q^{n(n+1)/2}(-1)^n}{\prod_{k=1}^{n}(1-q^k)\prod_{k=1}^{n}(1-yq^k)}$$

The symmetric decomposition

- Back in 1992/3, Richard was bothered that our approach did not use

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- From the original functional equation one can easily deduce that

$$S(x,y,q) = qxy + qxS(x,y,q) + qyS(x,y,q) + S(qx,qy,q)S(a,b,q)$$

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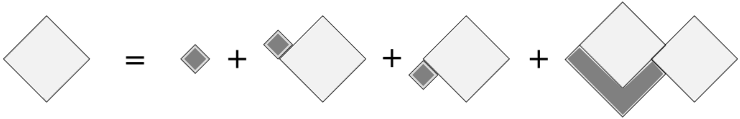

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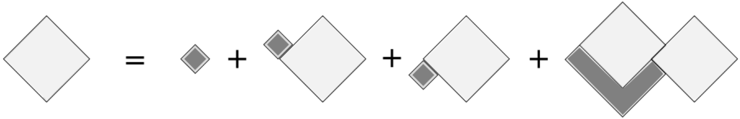

But we did not solve it back then!

Solving the symmetric functional equation

- Change notation: with $S(t) \equiv S(tx, ty, q)$ we need to consider

$$S(t) = qxyt^2 + (qxt + qyt)S(t) + S(qt)S(t)$$

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$$S(t) = qxyt^2 \frac{U(qt)}{U(t)}$$

to get

$$U(qt) = U(t) + q(x+y)U(qt) + q^3 t^2 U(q^2 t)$$

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- Insert $U(t) = \sum_{n=0}^{\infty} q^{n(n+1)/2} u_n t^n$ to get recurrence

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$$u_n + (x+y)u_{n-1} + xyu_{n-2} = q^n u_n$$

- This two-term recurrence is somewhat harder to solve

Solving the two-term recurrence (1 of 2)

- The left hand side of

$$u_n + (x+y)u_{n-1} + xyu_{n-2} = q^n u_n$$

is a constant term recurrence with characteristic polynomial

$$(\lambda + x)(\lambda + y)$$

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- The right hand side suggests an expansion in powers of q^n, so we let

$$u_n = \lambda^n \sum_{m=0}^{\infty} q^{mn} v_m q^{m^2+m}$$

and get a one-term recurrence

$$(\lambda q^m + x)(\lambda q^m + y)v_m = \lambda^2 v_{m-1} \quad \text{with} \quad v_0(\lambda + x)(\lambda + y) = 0$$

which we can iterate

Solving the two-term recurrence (2 of 2)

- We thus have two linearly independent solutions of $u_n + (x+y)u_{n-1} + xyu_{n-2} = q^n u_n$:

$$u_n(\lambda) = \sum_{m=0}^{\infty} \frac{q^{mn+m^2+m}\lambda^{2m+n}}{\prod_{k=1}^{m}(x+\lambda q^k)\prod_{k=1}^{m}(y+\lambda q^k)}$$

for $\lambda = -x$ and $\lambda = -y$

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- To satisfy the initial condition, the required solution is now

$$u_n = Au_n(-x) + Bu_n(-y) \quad \text{with} \quad u_{-1} = 0$$

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- And now put it all together ...

Staircase polygon generating function II

$$S(x,y,q) = xyq\frac{U(qx,qy,q)}{U(x,y,q)}$$

where

$$U(x,y,q) =$$
$$x\sum_{m=0}^{\infty}\frac{y^m q^{m^2}}{\prod_{k=1}^{m}(x-yq^k)\prod_{k=1}^{m}(1-q^k)}\sum_{n=0}^{\infty}q^{n(n+1)/2}(-x)^n\sum_{m=0}^{\infty}\frac{x^m q^{m^2+mn+m}}{\prod_{k=1}^{m}(y-xq^k)\prod_{k=1}^{m}(1-q^k)}$$
$$-y\sum_{m=0}^{\infty}\frac{x^m q^{m^2}}{\prod_{k=1}^{m}(y-xq^k)\prod_{k=1}^{m}(1-q^k)}\sum_{n=0}^{\infty}q^{n(n+1)/2}(-y)^n\sum_{m=0}^{\infty}\frac{y^m q^{m^2+mn+m}}{\prod_{k=1}^{m}(x-yq^k)\prod_{k=1}^{m}(1-q^k)}$$

Mireille's work

Compare

$$U(qt) = U(t) + q(x+y)U(qt) + q^3t^2U(q^2t)$$

with Lemma 4.5 in Mireille and Xavier's "Empilements de segments et q-énumération de polyominioes convex dirigés" (1992)

LEMME 4.5. *L'application L définie dans la proposition* 4.3 *par*

$$L(x, y) = \sum_{n \geqslant 0,\, m \geqslant 0} \frac{(-1)^{n+m} x^n y^m q^{\binom{n+m+1}{2}}}{(q)_n (q)_m}$$

est l'unique série formelle solution de la q-équation en H,

$$H(x, y) = (1 - q(x+y))\, H(xq, yq) - xyq^3\, H(xq^2, yq^2),$$

vérifiant $H(0, 0) = 1$.

In that paper, the functional equation is actually found from the exact expression for U (or L), which was obtained by other means

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It pays off to be able to read French language papers!

Staircase polygon generating function III

The solution to

$$U(qt) = U(t) + q(x+y)U(qt) + q^3t^2U(q^2t)$$

is given by

$$U(x,y,q) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{q^{(n+m)(n+m+1)/2}(-x)^m(-y)^n}{\prod_{k=1}^{m}(1-q^k)\prod_{k=1}^{n}(1-q^k)}$$

$$= x\sum_{m=0}^{\infty}\frac{y^m q^{m^2}}{\prod_{k=1}^{m}(x-yq^k)\prod_{k=1}^{m}(1-q^k)}\sum_{n=0}^{\infty}q^{n(n+1)/2}(-x)^n\sum_{m=0}^{\infty}\frac{x^m q^{m^2+mn+m}}{\prod_{k=1}^{m}(y-xq^k)\prod_{k=1}^{m}(1-q^k)}$$

$$-y\sum_{m=0}^{\infty}\frac{x^m q^{m^2}}{\prod_{k=1}^{m}(y-xq^k)\prod_{k=1}^{m}(1-q^k)}\sum_{n=0}^{\infty}q^{n(n+1)/2}(-y)^n\sum_{m=0}^{\infty}\frac{y^m q^{m^2+mn+m}}{\prod_{k=1}^{m}(x-yq^k)\prod_{k=1}^{m}(1-q^k)}$$

Summary

To me, this talk (and the small calculation presented here) has been more reminiscing than forward looking.

All the techniques used here I learned while a postdoc in Melbourne 1991-1994.

I still dabble in things I encountered while working in Melbourne; working with Richard has clearly influenced my career significantly.