The powered Catalan numbers

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A Tour of Combinatorics and Statistical Mechanics
in Memory of Richard Brak

The sequence

OEIS sequence A113227:

$$(u_n)_{n\geq 0} = (1, 1, 2, 6, 23, 105, 549, 3207, 20577, 143239, 1071704, 8555388, 72442465, \dots)$$

First (?) observed by David Callan circa 2005 as the number of permutations of length n which avoid the generalised pattern 1-23-4.

$\rightarrow \pi = (\pi_1, \pi_2, \dots, \pi_n) \in \mathcal{S}_n$ such that there are no $i, j, k \in \{1, \dots, n\}$ with

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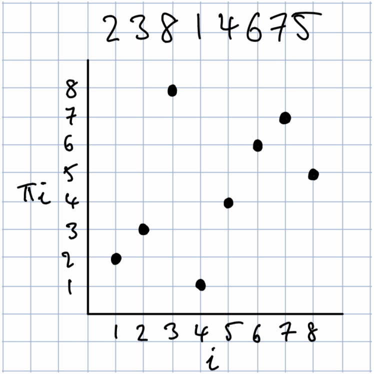

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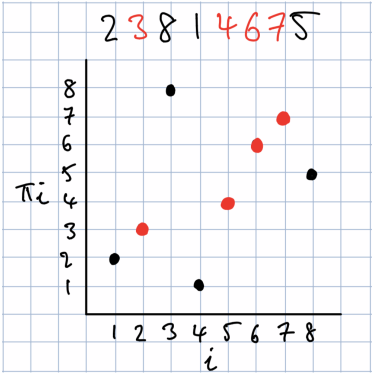

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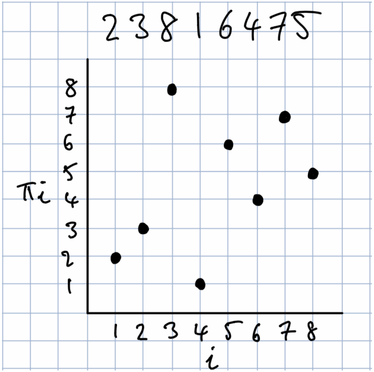


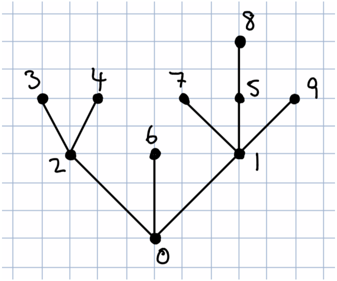
A recurrence for trees

In 2010 Callan found a two-index recurrence for u_n:

$$u_n = \sum_{k=1}^{n} u_{n,k}, \quad \text{where} \quad u_{n,k} = u_{n-1,k-1} + k \sum_{j=k}^{n-1} u_{n-1,j} \quad \text{for} \quad 1 \le k \le n$$

with initial conditions $u_{0,0} = 1$ and $u_{n,0} = 0$ for $n \ge 1$.

In the process he used a (very complicated) bijection with increasing ordered trees with increasing leaves:



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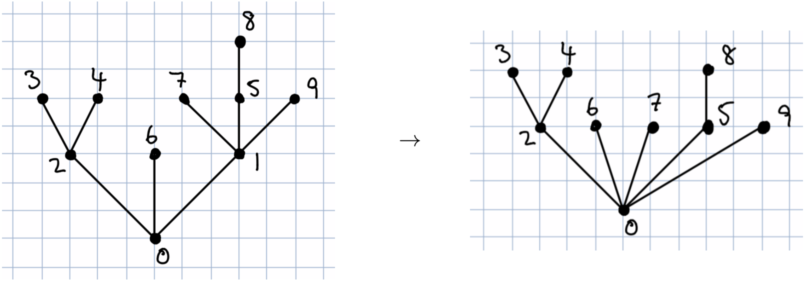

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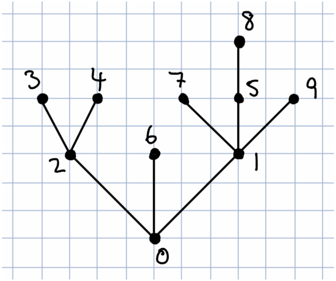

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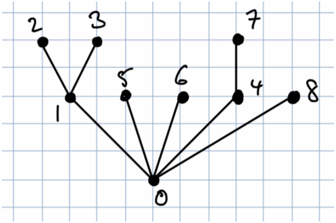

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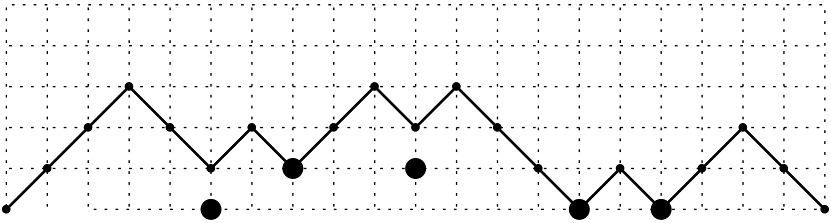
Valley-marked Dyck paths

Callan also noticed that u_n counts valley-marked Dyck paths (VMDPs) of length $2n$.



Each valley at height h gets a mark at one of $0, 1, \ldots, h$.

$u_{n,k}$ is the number of VMDPs of length $2n$ with final descent k.

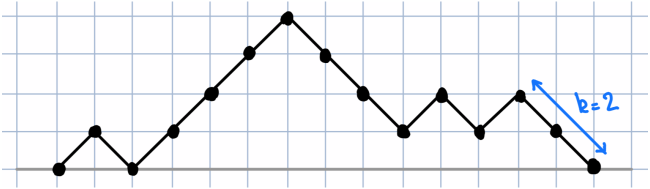
Generating trees

Why the name powered Catalan numbers?

A generating tree or succession rule is method for illustrating the recursive growth of combinatorial objects. For example, one recurrence for the Catalan numbers is

$$C_n = \sum_{k=1}^{n} C_{n,k}, \quad \text{where} \quad C_{n,k} = \sum_{j=k-1}^{n-1} C_{n-1,j} \quad \text{for} \quad 1 \leq k \leq n$$

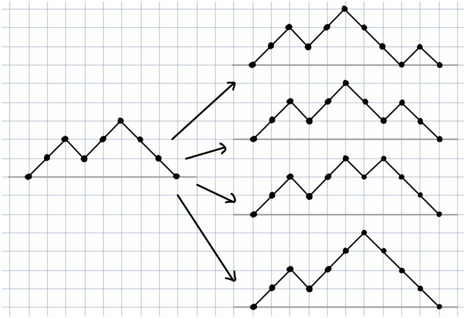
with initial conditions $C_{0,0} = 1$ and $C_{n,0} = 0$ for $n \geq 1$. ($C_{n,k}$ is the number of Dyck paths of length $2n$ with final descent k.)



Generating trees cont'd

Label each Catalan object (eg. Dyck path) with (k). Then the initial object has label (0), and something with label (k) can be grown into something with label $(1), (2), \ldots, (k+1)$. Write this as

$$\Omega_{\mathrm{Cat}} = \begin{cases} (0) \\ (k) \rightsquigarrow (1), (2), \ldots, (k+1) \end{cases}$$



Then Callan's recurrence for $u_{n,k}$ is

$$\Omega_{\mathrm{pCat}} = \begin{cases} (0) \\ (k) \rightsquigarrow (1), (2), (2), (3), (3), (3), \ldots, (k)^k, (k+1) \end{cases}$$

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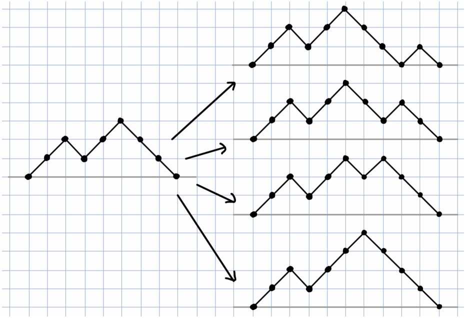

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Inversion sequences

We found the sequence in another place: inversion sequences.

An inversion sequence (IS) of length n is a sequence $(e_1, \ldots, e_n)$ such that $0 \leq e_i < i$.

The number of inversion sequences $|\mathcal{I}_n|$ of length n is $n!$, and they are in bijection with permutations. First map a permutation to its (left) inversion table:

$$\pi = (\pi_1, \ldots, \pi_n) \mapsto (t_1, \ldots, t_n), \quad \text{where} \quad t_i = |\{j : j > i \text{ and } \pi_j < \pi_i\}|$$

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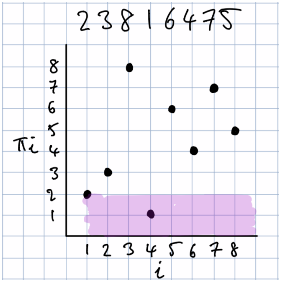

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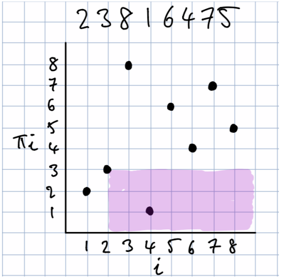

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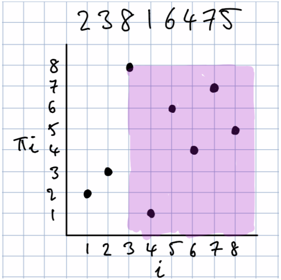

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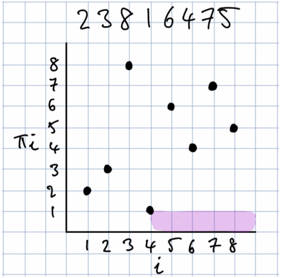

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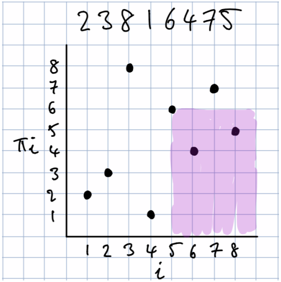

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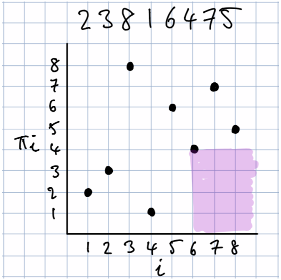

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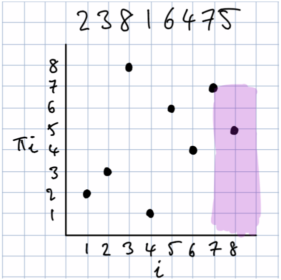

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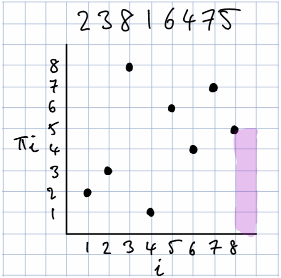

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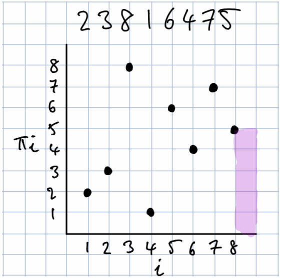

$$t = (1,1,5,0,2,0,1,0) \qquad \mapsto \qquad e = (0,1,0,2,0,5,1,1)$$

Then reverse t to get an inversion sequence.

Pattern-avoiding inversion sequences

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For example, the ISs avoiding the patterns $\{201, 210, 110, 101, 100, 000\}$ are counted by the Catalan numbers. These are ISs $(e_1, e_2, \ldots, e_n)$ for which there is no triple $i < j < k$ such that $e_i \geq e_j$ and $e_i \geq e_k$.

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These grow according to Ω_{Cat} as follows:

- For $1 \leq i \leq n+1$, let $e \odot i$ be the operation which inserts $i-1$ at position i. Note that if e is a Catalan IS then $e \odot (n+1)$ and $e \odot n$ are also Catalan; in general if $e \odot i$ is Catalan then we say i is an active position of e.
- Label each Catalan IS with (k), where the number of active positions is $k+1$.
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There is also another growth rule for Catalan ISs, which describes how to add an element to the end:

$$\Omega_{\text{Cat}_2} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (0,k+1)^h, (h+1,k), (h+2,k-1), \ldots, (h+k,1) \end{cases}$$

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Not (very) hard to show that powered Catalan ISs grow according to Ω_{pCat}. The label (k) counts the number of 0 entries. The growth rule is essentially

- **i.** increase all non-0 entries by 1;
- **ii.** increase some of the 0 entries by 1, in such a way as to maintain 110-pattern-avoidance; and
- **iii.** prepend a new 0 entry.

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But along the way we found another succession rule which also generates u_n:

$$\Omega_{\mathrm{steady}} = \begin{cases} (0,1) \\ (h,k) \rightsquigarrow (h+k-1,2),(h+k-2,3),\ldots,(h+1,k), \\ \qquad\qquad (0,k+1),\ldots,(0,h+k+1) \end{cases}$$

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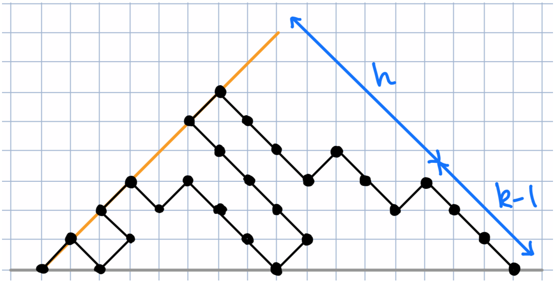
And in fact it is must simpler to show that (1-23-4)-PAPs grow according to Ω_{steady} than Callan's proof that they grow according to Ω_{pCat}.

Steady paths

Why steady? A steady path of size n is a generalised Dyck path which may also step "back" $(-1, 1)$ (but remains self-avoiding), with the constraint that any consecutive pair of steps UU or BU creates a diagonal "wall" that the walk must stay below. (The wall is initially the line $y = x$.)

n is the number of U steps.



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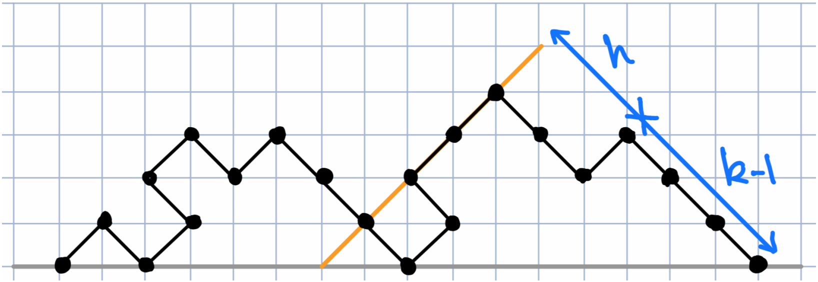

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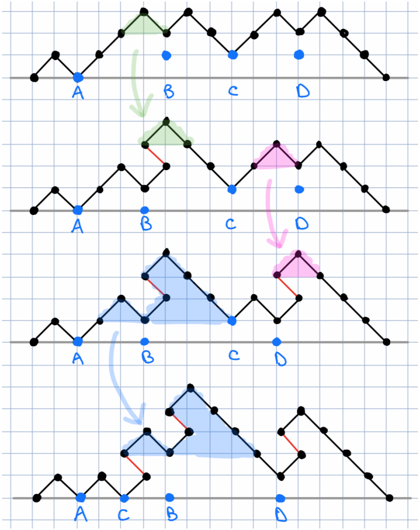
A bijection between Ω_{pCat} and Ω_{steady}

All of the classes counted by u_n (there are others) seem to naturally grow according to either Ω_{pCat} or Ω_{steady}. Is there a simple way to connect the two?

Yes! Just mess around with the paths...



A bijection between Ω_{pCat} and Ω_{steady} cont'd

Multiple statistics are conserved:

- number of U steps along the first diagonal $y = x$
- total height of marks $\mapsto$ total number of B steps
- number of valleys with highest possible mark $\mapsto$ number of valleys at height 0

(1-34-2)-PAPs

(But wait, there's more...)

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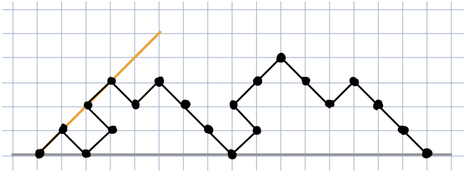
A steady path is completely characterised by the locations of its U steps.



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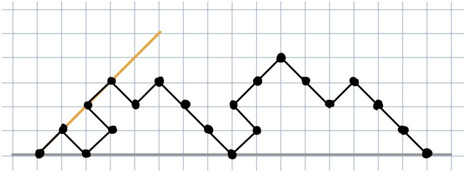

Writing down the distance from each U step to the main diagonal, reading from right to left, gives a sequence $(t_1, \ldots, t_n)$. eg. the above has sequence $(5, 3, 3, 4, 1, 0, 1, 0)$.

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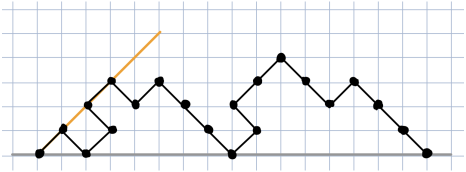

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Since $0 \leq t_k \leq n - k$, these are inversion tables of permutations, eg. the above corresponds to the permutation 64582173.

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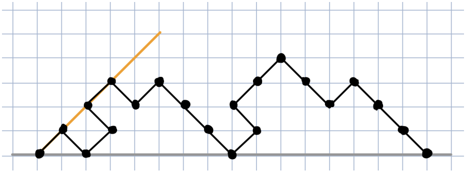

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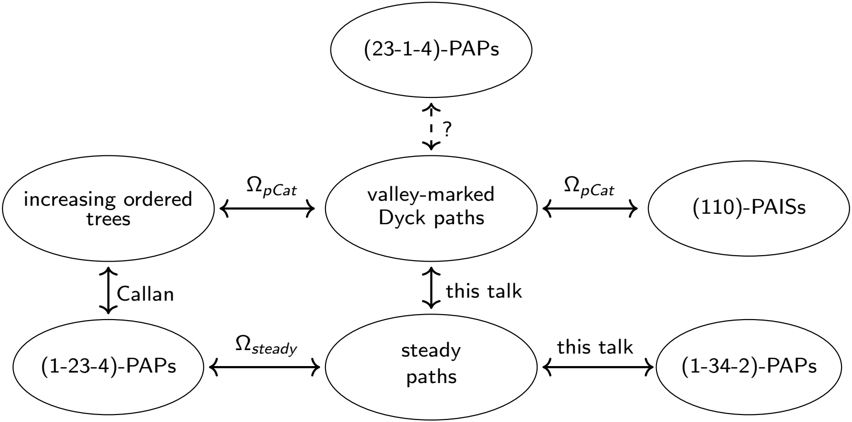
In fact (proof omitted) these turn out to be exactly (1-34-2)-PAPs.

The family



(23-1-4)-PAPs
?
increasing ordered trees
Ω_{pCat}
valley-marked Dyck paths
Ω_{pCat}
(110)-PAISs
Callan
this talk
(1-23-4)-PAPs
Ω_{steady}
steady paths
this talk
(1-34-2)-PAPs

A hierarchy of PAISs

Let $\rho_1, \rho_2, \rho_3 \in \{<, >, \leq, \geq, =, \neq, \bullet\}$, where $x \bullet y$ is true for all integers x, y.

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Then write $\mathbf{I}_n(\rho_1, \rho_2, \rho_3)$ as the set of ISs $(e_1, \ldots, e_n)$ for which there is no triple $i < j < k$ such that

$$e_i \rho_1 e_j, \quad e_j \rho_2 e_k, \quad e_i \rho_3 e_k,$$

and $\mathbf{I}(\rho_1, \rho_2, \rho_3) = \bigcup_n \mathbf{I}_n(\rho_1, \rho_2, \rho_3)$.

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We have

$\mathbf{I}(\geq, \bullet, \geq)$	→	$\mathbf{I}(\geq, \geq, \geq)$	→	$\mathbf{I}(\geq, \geq, >)$	→	$\mathbf{I}(\geq, >, \bullet)$	→	$\mathbf{I}(=, >, >)$
avoiding 201, 210, 110, 101, 100, 000		avoiding 210, 110, 100, 000		avoiding 210, 110, 100		avoiding 210, 110		avoiding 110
seq. A000108 (Catalan)		seq. A108307		seq. A001181 (Baxter)		seq. A117106 (semi-Baxter)		seq. A113227 (powered Catalan)

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