A tour of combinatorics and statistical mechanics: In memory of Richard Brak

6-7 February, 2022

Bordeaux, June 1994

Critical phenomena and polyominoes

R. Brak
30 juin.
1994

$$G(y,q) = \sum c_n(m)\, y^n q^m$$

($n \to$ perimeter, $m \to$ area)

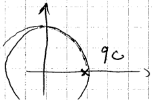
$$= \sum A_m(y)\, q^m = \sum P_n(q)\, y^n$$

Parall $G(y,q) = y\left(\frac{J_0(yq)}{J_0(y)} - 1\right)$

Valeurs exactes ou asymptotiques de $c_n(m)$, $P_n(q)$, $G(y,q)$?

G.F and asymptotics.

1) $A(q) = \sum c(m)\, q^m$



(souvent)

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Modèle: pol. vert. convexes.

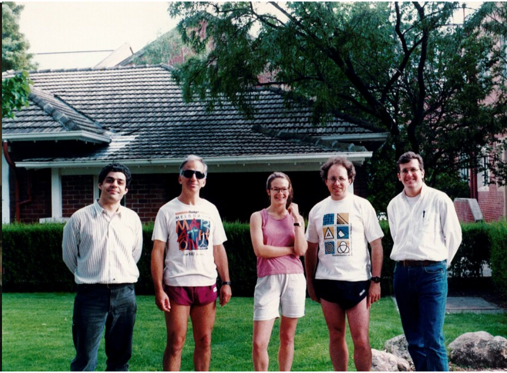
State: configuration

Energy: → J_1 for edge
→ J_2 per square } et/ou $\Rightarrow e^{-\frac{E}{T}} = \begin{cases} e^{-nJ_1/T} = y^n \\ e^{-mJ_2/T} = q^n \end{cases}$

$\Rightarrow$ Partition function $= \sum_{\varphi} y^{n(\varphi)}$ = Generating function.

Si l'énergie est $nJ_1 + mJ_2 \rightarrow G(y,q)$

1995: a Melbourne spring

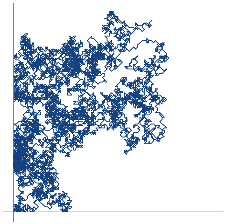


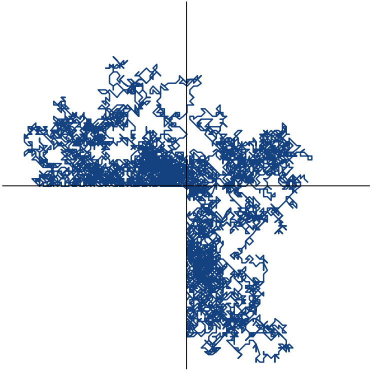
Richard's maths

- lattice walks, polyominoes and polygons
- exclusion processes
- osculating walks and alternating sign matrices
- percolation
- ...

Counting walks in a cone: a mini-survey

Mireille Bousquet-Mélou, CNRS, Bordeaux, France

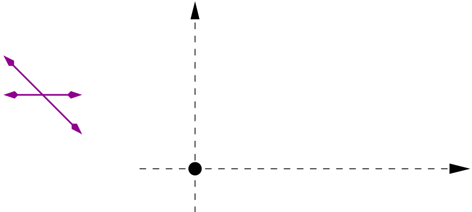




A typical question (in two dimensions)

Let $\mathcal{S}$ be a finite subset of $\mathbb{Z}^2$ (set of steps) and $p_0 \in \mathbb{Z}^2$ (starting point).

Example. $\mathcal{S} = \{\leftarrow, \nwarrow, \rightarrow, \searrow\}$, $p_0 = (0,0)$



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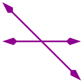

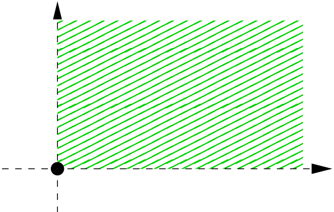

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Questions

- What is the number $a(n)$ of n-step walks starting at p_0 and contained in C?
- For $(i,j) \in C$, what is the number $a(i,j;n)$ of such walks that end at (i,j)?

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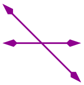

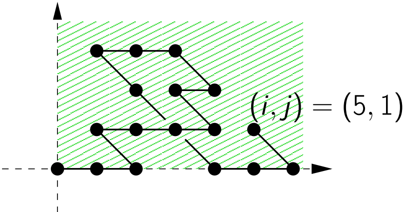

An attractive topic with a long history

- Many discrete objects can be encoded in that way:
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- Melbourne, FPSAC 2002...

- A Melbourne topic!

N. Beaton, R. Brak, A. Elvey Price, A. Owczarek, A. Rechnitzer, R. Xu...

Generating functions

- Our original question:

$$a(n) = ? \qquad a(i,j;n) = ?$$

- Generating functions:

$$A(t) = \sum_{n\geq 0} a(n)t^n, \qquad A(x,y;t) = \sum_{i,j,n} a(i,j;n)x^i y^j t^n$$

$$= \sum_{w \text{ walk}} x^{i(w)} y^{j(w)} t^{|w|}$$

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- $A(1,1;t) = A(t)$
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Can one express these series? What is their *nature*?

A hierarchy of formal power series

- Rational series

$$A(t) = \frac{1-t}{1-t-t^2}$$

- Algebraic series

$$1 - A(t) + tA(t)^2 = 0$$

- Differentially finite series (D-finite)

$$t(1-16t)A''(t) + (1-32t)A'(t) - 4A(t) = 0$$

- D-algebraic series

$$(2t + 5A(t) - 3tA'(t))A''(t) = 48t$$

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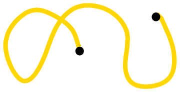

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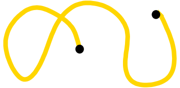
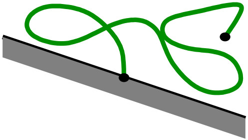


- Half-space: algebraic series



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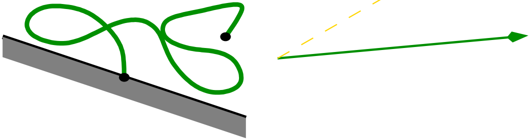
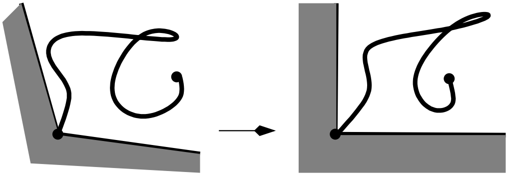


- Convex cone ≡ quadrant



- Non-convex cone ≡ three quadrants

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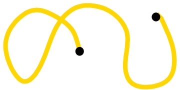

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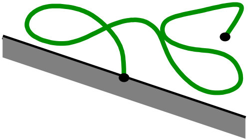

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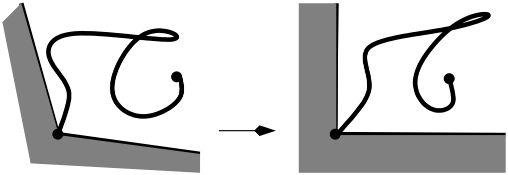

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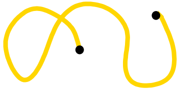

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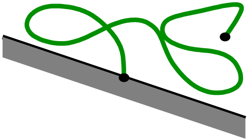

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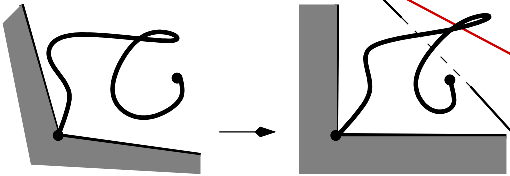
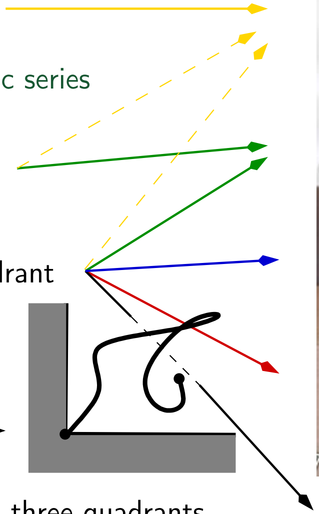
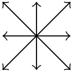




- Non-convex cone ≡ three quadrants

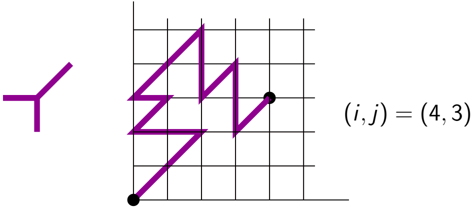
Focus: quadrant walks with small steps

- Small steps: $\mathcal{S} \subset \{-1,0,1\}^2 \setminus \{(0,0)\}$. Only 2^8 models.



- Some models are trivial, or equivalent to a half plane problem

$\Rightarrow$ 79 really interesting and distinct models [mbm-Mishna 10]



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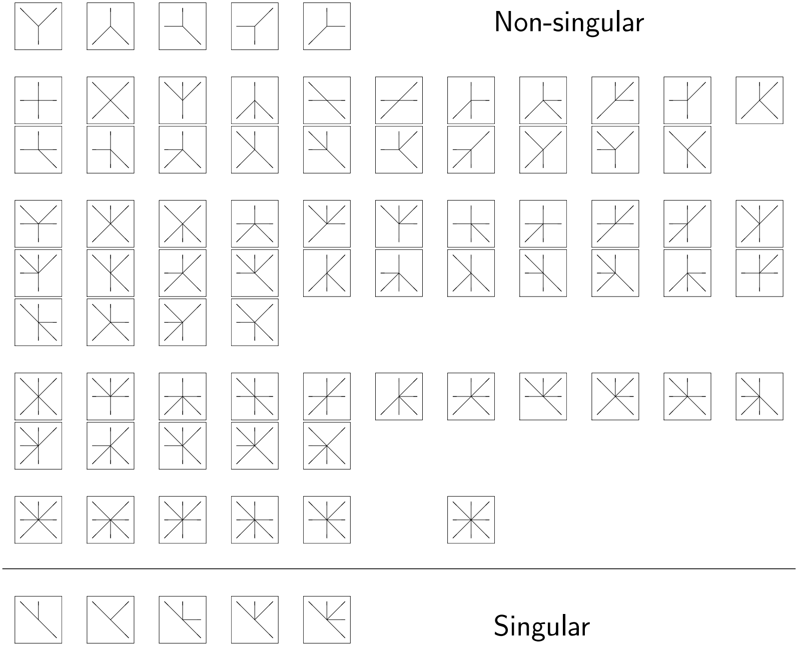

A systematic approach

- One can always write a recurrence relation for the numbers $q(i,j;n)$
- or equivalently, a linear functional equation for $Q(x,y;t) \equiv Q(x,y)$

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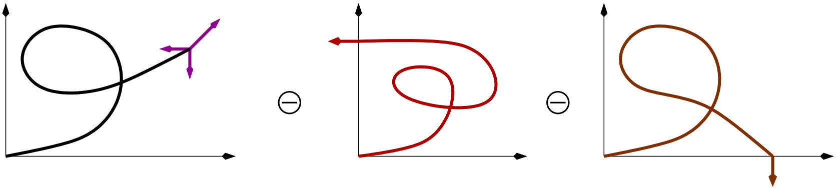
Example: $\mathcal{S} = \{\leftarrow, \downarrow, \nearrow\}$, with $\bar{x} := 1/x$ and $\bar{y} := 1/y$

$$Q(x, y) = 1 + t(\bar{x} + \bar{y} + xy)Q(x, y) - t\bar{x}Q(0, y) - t\bar{y}Q(x, 0).$$



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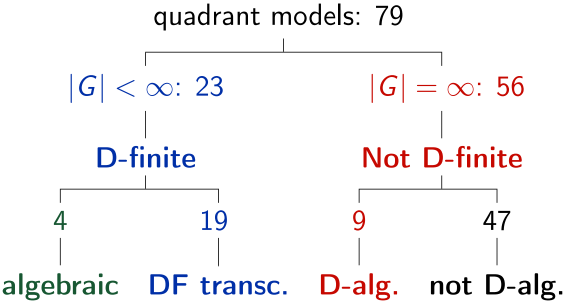
Equivalently,

$$\boxed{\big(1 - t(\bar{x} + \bar{y} + xy)\big)xyQ(x,y) = xy - tyQ(0,y) - txQ(x,0)}$$

The (Laurent) polynomial $S(x,y) := \bar{x} + \bar{y} + xy$ is the step polynomial of this model, and $K(x,y) := 1 - t(\bar{x} + \bar{y} + xy)$ is the kernel.

Twenty years later: classification of quadrant walks

(small steps)



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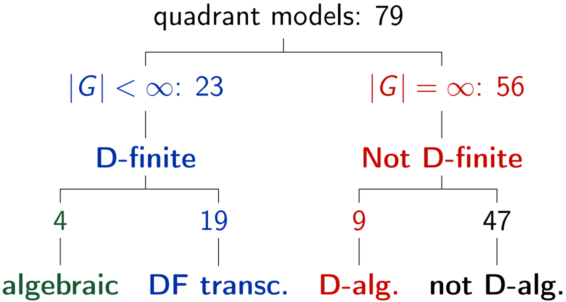


Formal power series algebra

Complex analysis

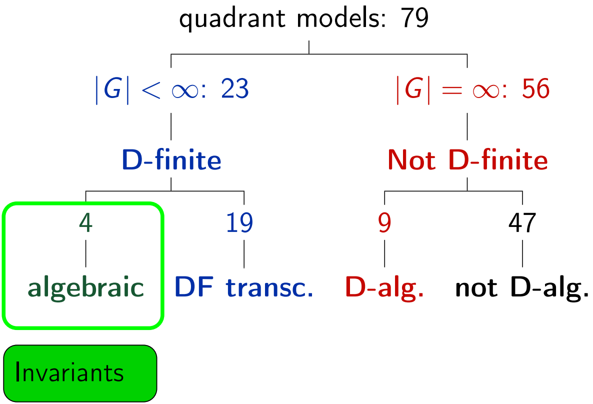
Differential Galois theory

Computer algebra

D-finite series
- G-functions
- arithmetic properties
- asymptotics

Random walks

Invariants



Invariants: a notion introduced by W. Tutte to count properly coloured planar triangulations (1973-1984)

What are invariants?

Fix a step set $\mathcal{S}$, with step polynomial $S(x, y)$.

• A pair of series $(I(x), J(y))$ in t with coefficients in $\mathbb{Q}(x)$ and $\mathbb{Q}(y)$ (respectively) is a pair of invariants if

$$I(x) - J(y) = (1 - tS(x, y))H(x, y)$$

where $H(x, y)$ has poles of bounded order at $x = 0$ and $y = 0$.

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By this, we mean that there exists i, j such that

$$x^i y^j H(x, y) = \sum_n \frac{p_n(x, y)}{d_n(x) d'_n(y)} t^n$$

where $d_n(0) \neq 0$ and $d'_n(0) \neq 0$.

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• We say that $I(x) - J(y)$ is divisible by $K(x, y) = 1 - tS(x, y)$.

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Example: Trivial invariants: take $I(x) = J(y) \in \mathbb{Q}((t))$

Some non-trivial invariants

- Rational invariants: let $I_0(x) = \bar{x}^2 - \bar{x}/t - x$ and $J_0(y) = I_0(y)$. Then

$$I_0(x) - J_0(y) = (1 - t(\bar{x} + \bar{y} + xy))\frac{x-y}{txy} = K(x,y)H_0(x,y),$$

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$$K(x,y)xyQ(x,y) = xy - txQ(x,0) - tyQ(0,y)$$

and the decoupling relation

$$xy = -\bar{x} - \bar{y} + \frac{1}{t} - \frac{1}{t}K(x,y),$$

one derives

$$K(x,y)\left(xyQ(x,y) - \frac{1}{t}\right) = \left(\frac{1}{2t} - \bar{x} - txQ(x,0)\right) + \left(\frac{1}{2t} - \bar{y} - tyQ(0,y)\right)$$

A relation between (I_1, J_1) and the rational invariants (I_0, J_0)

We have two pairs of invariants:

$$I_0(x) = \bar{x}^2 - \bar{x}/t - x, \qquad J_0(y) = I_0(y),$$
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New invariants from old ones

The componentwise sum (resp. product) of two pairs of invariants $(I_0(x), J_0(y))$, $(I_1(x), J_1(y))$ is another pair of invariants.

$$(I_0(x) + I_1(x), J_0(y) + J_1(y)), \qquad (I_0(x)I_1(x), J_0(y)J_1(y))$$

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A pair of invariants with no pole at zero is trivial.

Observation: $I_1(x)^2 - I_0(x)$ has no pole! Define

$$I(x) := I_1(x)^2 - I_0(x) \qquad J(y) := J_1(y)^2 - J_0(y).$$

Then $(I(x), J(y))$ is a pair of invariants with no pole at zero.
Hence $I(x) = J(y)$ is a series in t only.

A relation between (I_1, J_1) and the rational invariants (I_0, J_0)

The series

$$I(x) = I_1(x)^2 - I_0(x) = (txQ(x,0) - 1/(2t))^2 + 2tQ(x,0) + x$$

depends only on t. Thus, $I(x) = I(0)$, that is,

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Summary: starting from an equation between

$$Q(x,y), \qquad Q(x,0), \quad \text{and} \quad Q(0,y),$$

we have obtained an equation between

$$Q(x,0) \quad \text{and} \quad Q(0,0).$$

One less variable.

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GF of Kreweras' walks in the quadrant

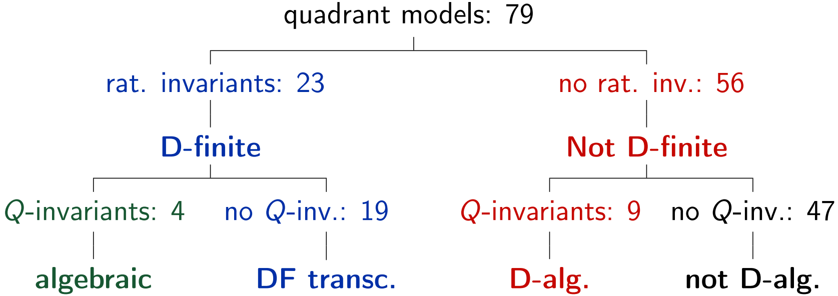
Let $Z \equiv Z(t)$ be the only series in t such that $Z = t(2 + Z^3)$. Then

$$Q(x,0) = \frac{1}{tx}\left(\frac{1}{2t} - \frac{1}{x} - \left(\frac{1}{Z} - \frac{1}{x}\right)\sqrt{1 - xZ^2}\right).$$

[Kreweras 65], [Gessel 86], [mbm 05]...

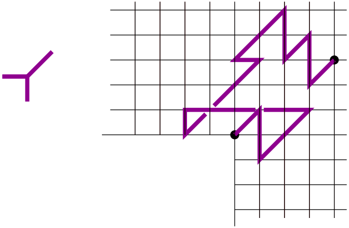
Revisiting the quadrant classification with invariants

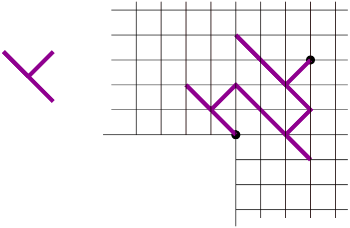
The whole picture



[Bernardi-mbm-Raschel, Dreyfus-Hardouin-Roques-Singer]

Three quadrant problems with small steps

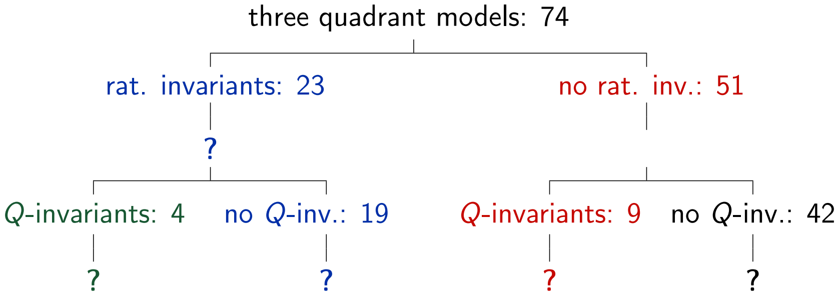




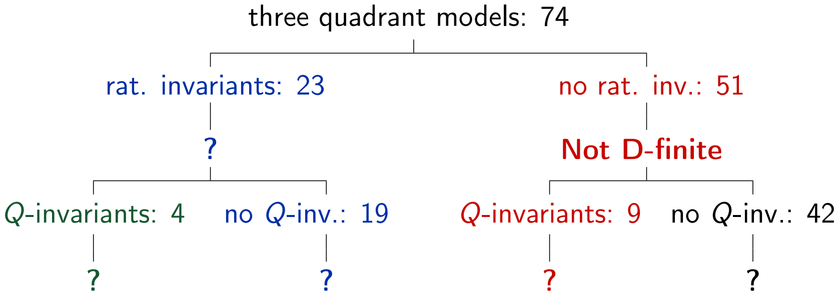
The 5 singular models become trivial (rational GF).

Functional equation for the series $C(x, y; t)$

Classification of three quadrant problems?



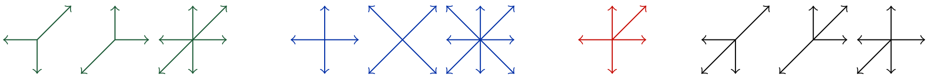
Classification of three quadrant problems?



[Mustapha 19]

Ten diagonally symmetric models

- 10 models
  - rat. invariants: 6
    - **D-finite**
      - Q-invariants: 3
        - **algebraic**
      - no Q-inv.: 3
        - **DF transc.**
  - no rat. inv.: 4
    - **Not D-finite**
      - Q-invariants: 1
        - **D-alg.**
      - no Q-inv.: 3
        - **not D-alg.**



[mbm, Wallner, Raschel, Trotignon, Mustapha, Dreyfus]

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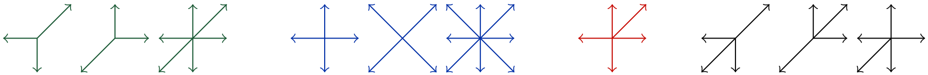
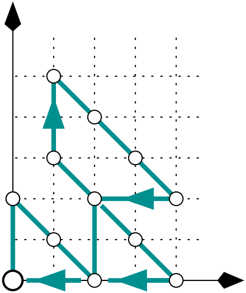


[mbm, Wallner, Raschel, Trotignon, Mustapha, Dreyfus]

Elvey Price: same nature as the quadrant series, at least in x and y (in preparation)

Beyond small steps in 2D

• Arbitrary steps in the quadrant $\Rightarrow$ equivalence between D-finiteness and finite "group"?



Beyond small steps in 2D

• Arbitrary steps in the quadrant ⇒ equivalence between D-finiteness and finite “group”?

• Walks with small steps in $\mathbb{N}^3$: some non-D-finite models with a finite group?

Example. The model $\{111, \bar{1}00, 0\bar{1}0, 00\bar{1}\}$ has a finite group of order 24. Is it D-finite? Most likely not...