Odds and Ends About Osculating Walkers

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Osculating Walkers

Consider p directed walkers on the square lattice rotated through 45°.

Walks take steps in the $(1,1)$ or $(1,-1)$ directions.

The walkers are labelled $k = 1, 2, \cdots, p$.

y_t^k is the ordinate of the k'th walker after t steps.

Walkers never cross but they may share vertices so $y_t^k \leq y_t^{k+1}$.

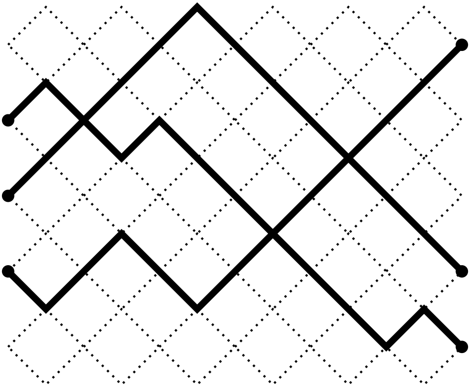
However, they are not allowed to share an edge so must separate after meeting fleetingly.

Easy to enumerate. Polynomial time algorithm of order n^{p-1} (keep track of gaps between walkers).

Generating functions solutions of Fuchsian ODEs so D-finite series.

Star Configurations

Walkers start on neighbouring sites, but can finish anywhere.



Known results

The dominant (physical) singularity is at $x = x_c = 1/2^p$.

The number of vicious stars of length n with p walkers is, from Guttmann, Owczarek and Viennot, J Phys A **31** 8123 (1998)

$$\prod_{1 \le i \le j \le n} \frac{p+i+j-1}{i+j-1}$$

For osculating 3-stars (vicious boundaries) Mireille Bousquet-Mélou proved [J Phys Conf **42** 35 (2006)]

$$\mathcal{O}_3(x) = \frac{3 - 15x - 4x^2 - 3(1-x)\sqrt{1-8x}}{8x^2(1+x)}.$$

Star Configurations

p	Init	Order	Degree	Singularities	Exponents
4	Vic	5	14	$1/16$	$0, 1, 2, 2, 3$
				4	$0, 1, 2, 3, -1/2$
				$-1/2$	$0, 1, 2, -1, 1/2$
4	Osc	5	9	$1/16$	$0, 1, 2, 2, 3$
				4	$0, 1, 2, 3, -1/2$
				$-1/2$	$0, 1, 2, -1, 1/2$
5	Vic	6	29	$1/32$	$0, 1, 2, 3, 4, 4$
				2	$0, 1, 2, 3, -1, 1/2$
				$-1/4$	$0, 1, 2, 3, -1/2, 1/2$
5	Osc	6	12	1/32	$0, 1, 2, 3, 4, 4$
				2	$0, 1, 2, 3, 4, 1/2$
				$-1/4$	$0, 1, 2, 3, -1/2, 1/2$

4 walkers – Osculating Initial Configuration

Differential operator is a direct sum of operators of order 1 and 4:

$$L_5 = L_1 \oplus L_4,$$

where L_4 is a product of two order 1 operators and an order 2 operator

$$L_4 = L_2 \cdot M_1 \cdot N_1.$$

The solutions of these operators are:

$$L_1 : \quad \frac{3 + 11x}{x(1 + 2x)}$$

$$N_1 : \quad \frac{(1 - x)}{\sqrt{x(4 - x)}\,(1 + 2x)}$$

$$M_1 : \quad \frac{\left(2 - 6x - 6x^2 + x^3\right)}{x^2\,(4 - x)\,(1 - x)\,(1 + 2x)^{\frac{3}{2}}}$$

4 walkers – Osculating Initial Configuration

$$L_2: \quad \frac{(10-135x+24x^2+20x^3)\cdot \text{hypergeom}([3/2,3/2],[3],16x)}{2x^6-19x^5+26x^4+70x^3+10x^2-8x}$$

$$+ \quad \frac{6x(1-16x)(10-3x-2x^2)\cdot \text{hypergeom}([5/2,5/2],[4],16x)}{8x-10x^2-70x^3-26x^4+19x^5-2x^6}$$

$$L_2: \quad \frac{(10-135x+24x^2+20x^3)\cdot \text{hypergeom}([3/2,3/2],[1],1-16x)}{2x^6-19x^5+26x^4+70x^3+10x^2-8x}$$

$$- \quad \frac{18x(1-16x)(10-3x-2x^2)\cdot \text{hypergeom}([5/2,5/2],[2],1-16x)}{8x-10x^2-70x^3-26x^4+19x^5-2x^6}$$

5 walkers – Osculating Initial Configuration

Differential operator is a direct sum of two operators of order 1 and and operator of order 4:

$$L_6 = L_1^{a} \oplus L_1^{b} \oplus L_4,$$

where L_4 is a product of two order 1 operators and an order 2 operator

$$L_4 = L_2 \cdot M_1 \cdot N_1.$$

The solutions of these operators are:

$$L_1^{a}: \quad 86 - 81/x + 42/x^2$$

$$L_1^{b}: \quad \frac{(656386943 + 25794996300x)}{x^2\sqrt{1+4x}}$$

$$N_1: \quad \frac{(2-x)}{x^2\sqrt{1+4x}}$$

$$M_1: \quad \frac{\left(1 + 2x - 32x^2 + 12x^3\right)}{x^2\,(1+4x)\,\sqrt{x(2-x)}}$$

5 walkers – Osculating Initial Configuration

$$L_2 : \quad \frac{(1-9x+47x^2)\cdot \text{hypergeom}([-1/2,-1/2],[1],32x)}{x^3+6x^4-24x^5-116x^6+48x^7}$$

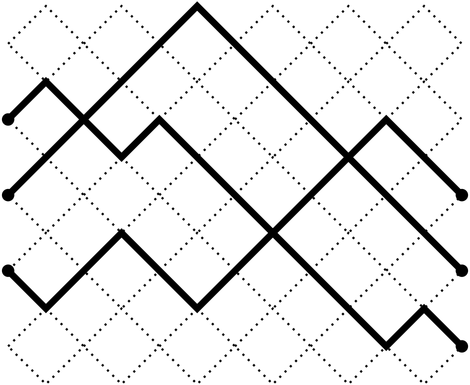
$$- \quad \frac{2x(1-32x)(3-8x)\cdot \text{hypergeom}([1/2,1/2],[2],32x)}{x^3+6x^4-24x^5-116x^6+48x^7}$$

$$L_2 : \quad \frac{(1-25x-1553x^2+45120x^3-82944x^4)\cdot \text{hypergeom}([3/2,3/2],[3],1-32x)}{x^3+6x^4-24x^5-116x^6+48x^7}$$

$$- \quad \frac{6x(1-32x)^3(3-8x)\cdot \text{hypergeom}([5/2,5/2],[4],1-32x)}{x^3+6x^4-24x^5-116x^6+48x^7}$$

Watermelon Configurations

Walkers start and finish on neighbouring sites.



Known results

The dominant (physical) singularity is at $x = x_c = 1/2^p$ with critical exponent $\alpha_p = (p^2 - 3)/2$, and when integer valued the critical behaviour is of the form $(1 - x/x_c)^{\alpha_p} \log(1 - x/x_c)$.

John Essam and Tony Guttmann proved that the generating function $\mathcal{V}_3(x)$ for vicious 3-watermelons can be expressed in terms of a Heun function

$$\begin{aligned}
\mathcal{V}_3(x) &= \frac{1}{3x^3}\left[-1 + x - 3x^2 + \mathrm{HeunG}\left(-\frac{1}{8}, -\frac{1}{4}; -1, -2, 2, -2; -x\right)\right] \\
&= \frac{1}{3x^3}\left[-1 + x - 3x^2 + \mathrm{HeunG}\left(-8, 2; -1, -2, 2, -2; 8x\right)\right]
\end{aligned}$$

For osculating 3-watermelons (vicious boundaries)

$$\mathcal{O}_3(x) = \frac{-1 + x + (1 - x)^2\mathcal{V}_3(x)}{x(1 + x)}.$$

Watermelon Configurations

p	Init	Order	Degree	Singularities
4	Vic	7	38	
4	Osc	7	21	$1/16, 4, -1/4, -1/2, -1$
5	Vic	10	93	
5	Osc	10	50	$1/32, 1, 2, -1/4, -1, 1+11x-x^2$

Simple solutions: There are 3 (4) rational solutions for $p = 4$ (5).

For $p = 4$ osculating boundary conditions:

$$L_7 = L_1 \oplus L_6, \quad L_6 = L_3 \cdot L_1^{\mathrm{c}} \cdot L_1^{\mathrm{b}} \cdot L_1^{\mathrm{a}}$$

Solution of L_1^{a} : $(1-3x^2+2x^3)\sqrt{4-x}/x^{5/2}$.

Rational solution of L_1^{b}.