Pulling Spiders (and a Watermelon)

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Jointly with Stu Whittington
– February –

In honour of Richard Brak

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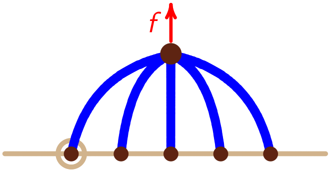
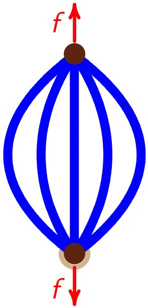
14.07.2005 02:48

Pulled spiders and a watermelon graph



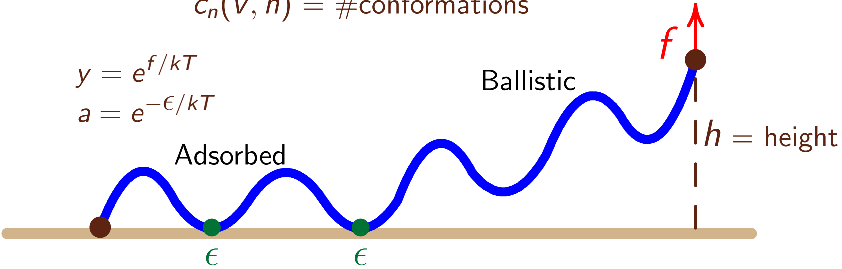


Pulled adsorbing self-avoiding walks

$c_n(v, h) = \#\text{conformations}$

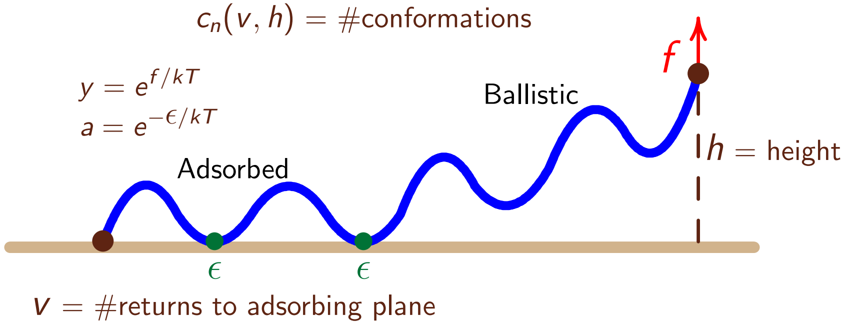
$y = e^{f/kT}$

$a = e^{-\epsilon/kT}$



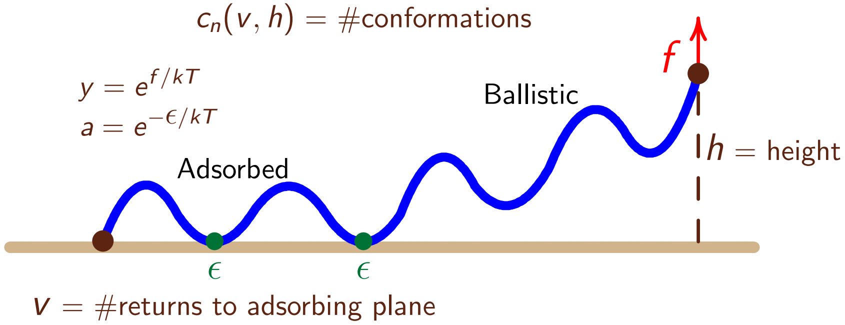
$v = \#\text{returns to adsorbing plane}$

Pulled adsorbing self-avoiding walks



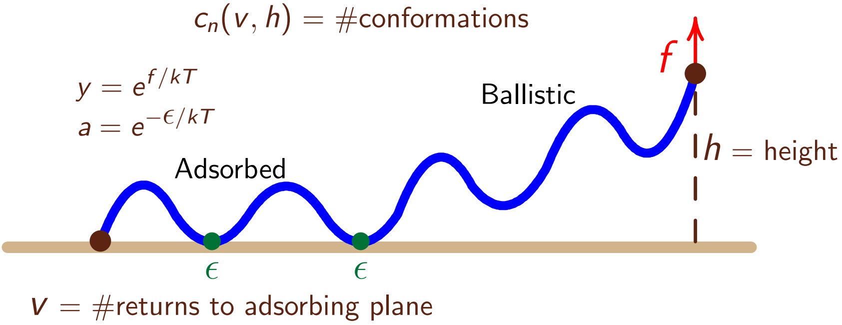
- Partition function $A_n(a, y) = \sum_{v,h} c_n(v, h)\, a^v y^h$

Pulled adsorbing self-avoiding walks



- Partition function $A_n(a,y) = \sum_{v,h} c_n(v,h)\, a^v y^h$
- Adsorbing walks free energy $\kappa(a) = \lim_{n\to\infty} \frac{1}{n} \log A_n(a,1)$
- Pulled walks free energy $\lambda(y) = \lim_{n\to\infty} \frac{1}{n} \log A_n(1,y)$

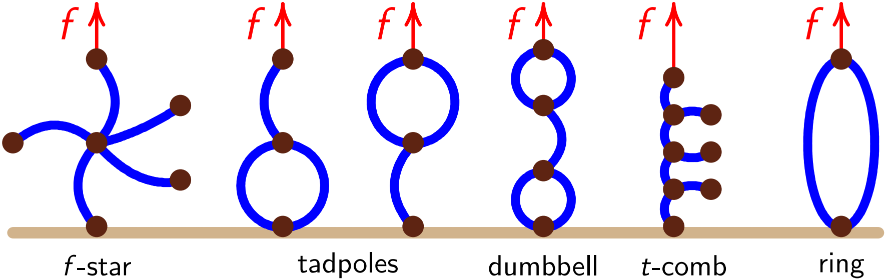
Pulled adsorbing self-avoiding walks



- Partition function $A_n(a,y) = \sum_{v,h} c_n(v,h)\, a^v y^h$
- Adsorbing walks free energy $\kappa(a) = \lim_{n\to\infty} \frac{1}{n} \log A_n(a,1)$
- Pulled walks free energy $\lambda(y) = \lim_{n\to\infty} \frac{1}{n} \log A_n(1,y)$
- Limiting free energy $\psi(a,y) = \max\{\kappa(a), \lambda(y)\}$
- Hammersley, Torrie and Whittington 1982

Models of uniform pulled self-avoiding graphs

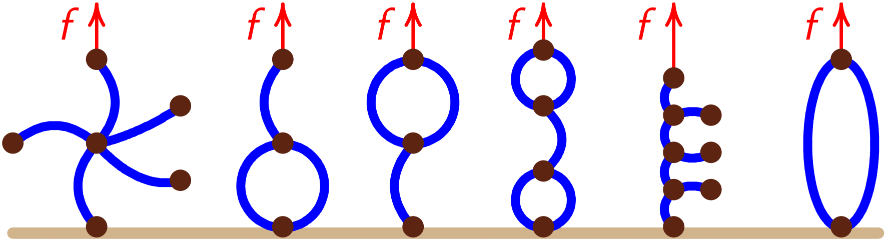
- Models of polymers pulled from a surface



- Phase diagram – free, adsorbed, and ballistic phases
- Partition function: $Z(a,y) = \sum_{v,h} s_n(v,h)\, a^v\, y^h$
- Free energy: $\psi_M(a,y) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(a,y)$

Models of uniform pulled self-avoiding graphs

- Models of polymers pulled from a surface



$\frac{1}{f}(2\lambda(y) + (f-2)\log \mu_3)$ $\frac{1}{2}(\lambda(y) + \lambda(y^{1/2}))$ $\frac{1}{3}(\lambda(y) + 2\lambda(y^{1/2}))$ $\frac{t+1}{2t+1}\lambda(y) + \frac{t}{2t+1}\log \mu_3$ $\lambda(y^{1/2})$

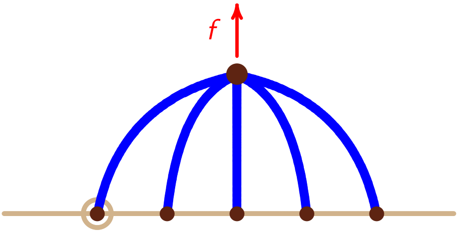
- Free energies when $a = 1$

- f-Star: $(2\lambda(y) + (f-2)\log \mu_3)/f$

- Polygon: $\lambda(y^{1/2})$

Pulled f-spiders

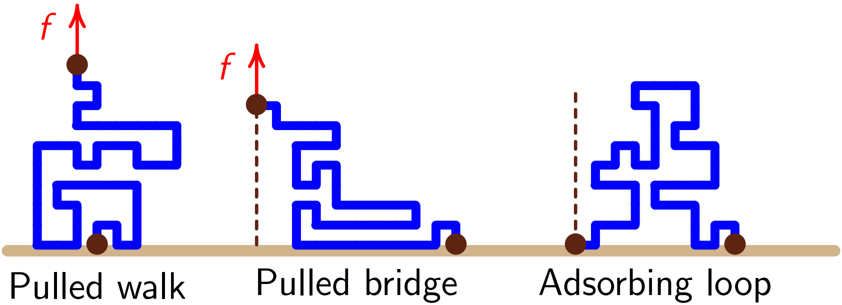
- Uniform f-star with end-vertices in the adsorbing plane
- Pulled in its middle vertex



- The free energy is $\lambda(y^{1/f})$
- Outline of the proof
- Ingredients (pioneering work of Chris Soteros)
  - Strategy bounds
  - Unfolded walks
  - Walks in wedges
  - Most popular arguments

Ingredients

- Pulled walks, Unfolded walks (bridges), Loops, all in Wedges



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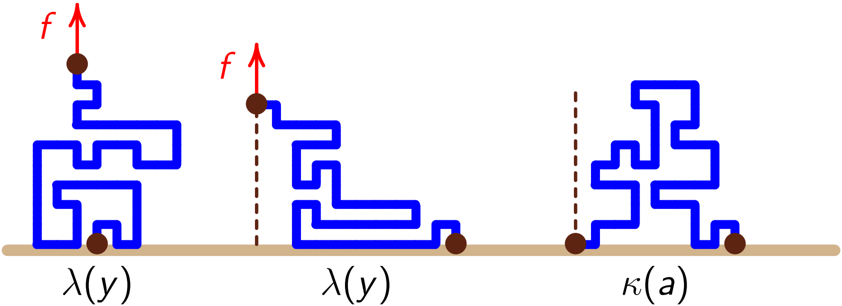
Ingredients

- Pulled and adsorbing walks, unfolded walks (bridges), and loops



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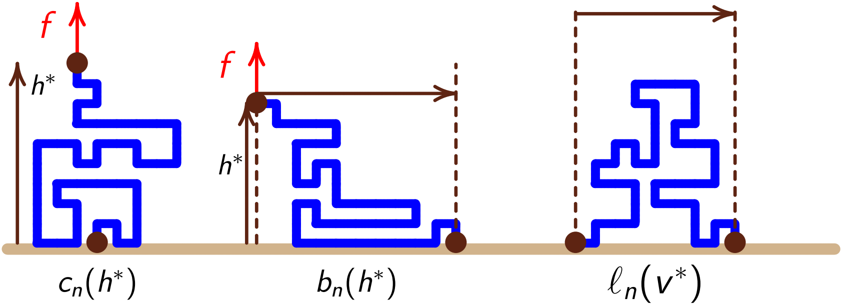
Most popular heights and widths

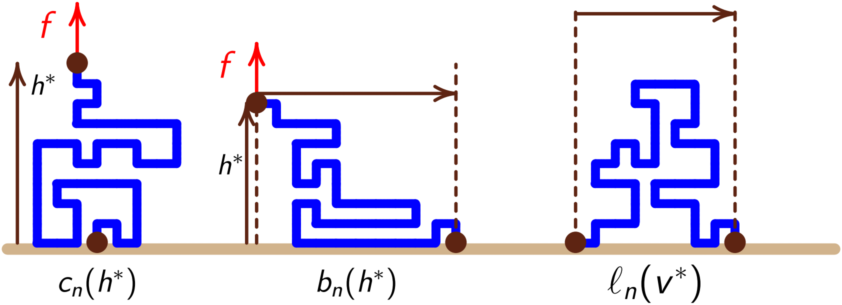
- Pulled and adsorbing walks, unfolded walks (bridges), and loops



- Most popular heights h^* and visits v^*

Most popular heights and widths

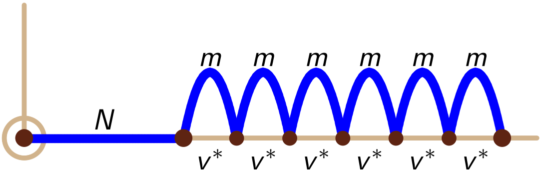
- Pulled and adsorbing walks, unfolded walks (bridges), and loops



- Most popular heights h^* and visits v^*
- $\lambda(y) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{h=0}^{n} b_n(h) y^h = \lim_{n\to\infty} \frac{1}{n} \log(b_n(h^*) y^{h^*})$
- $\kappa(a) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{v=0}^{n} \ell_n(v) a^v = \lim_{n\to\infty} \frac{1}{n} \log(\ell_n(v^*) a^{v^*})$

Adsorbing loops in a wedge

- $\ell_n^{(\alpha)}(v)$ – number of unfolded loops in an α-wedge
- Adsorbing loops in an α-wedge: $A_n^{(\alpha)}(a) = \sum_{n=0}^{\infty} \ell_n^{(\alpha)}(v)\, a^v$



- Most popular class adsorbing loops: $\kappa(a) = \lim_{n\to\infty} \frac{1}{n} \log(\ell_n(v^*)a^{v^*})$
- In the α-wedge: $\left(\ell_m(v^*)a^{v^*}\right)^M \leq A_{mM+N}^{(\alpha)}(a) \leq A_{mM+N}(a)$

Adsorbing loops in a wedge

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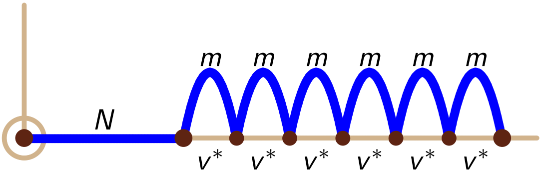
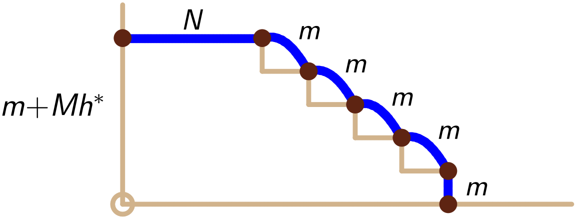


- Most popular class adsorbing loops: $\kappa(a) = \lim_{n\to\infty} \frac{1}{n} \log(\ell_n(v^*) a^{v^*})$
- In the α-wedge: $\left(\ell_m(v^*) a^{v^*}\right)^M \leq A_{mM+N}^{(\alpha)}(a) \leq A_{mM+N}(a)$
- $\frac{1}{m} \log(\ell_m(v^*) a^{v^*}) \leq \frac{1}{mM} \log A_{mM+N}^{(\alpha)}(a) \leq \frac{1}{mM} \log A_{mM+N}(a)$
- Take $M \to \infty$, $m \to \infty$, then $\kappa(a) = \lim_{n\to\infty} \frac{1}{n} \log A_n^{(\alpha)}(a)$

Pulled bridges in a wedge

- $b_n^{(\alpha)}(h)$ – number of bridges in an α-wedge
- Pulled bridges in an α-wedge: $B_n^{(\alpha)}(y) = \sum_{h=0}^{n} b_n^{(\alpha)}(h)\, y^h$



- Most popular class: $\lambda(y) = \lim_{n\to\infty} \frac{1}{n} \log(b_n(h^*) y^{h^*})$
- In the α-wedge: $\left(b_n^{(\alpha)}(h^*) y^{h^*}\right)^M \leq B_{mM+N}^{(\alpha)}(y) \leq B_{mM+N}(y)$

Pulled bridges in a wedge

- $b_n^{(\alpha)}(h)$ – number of bridges in an α-wedge
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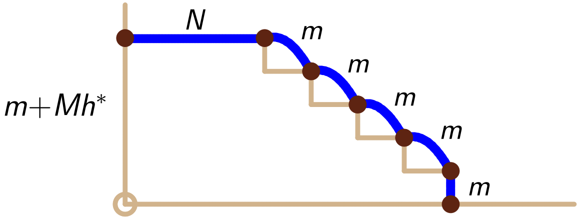


- Most popular class: $\lambda(y) = \lim_{n\to\infty} \frac{1}{n} \log(b_n(h^*) y^{h^*})$
- In the α-wedge: $\left(b_n^{(\alpha)}(h^*) y^{h^*}\right)^M \leq B_{mM+N}^{(\alpha)}(y) \leq B_{mM+N}(y)$
- $\frac{1}{m} \log B_m(h^*; y) \leq \frac{1}{mM} \log B_{mM+N}^{(\alpha)}(y) \leq \frac{1}{mM} \log B_{mM+N}(y)$
- Take $M \to \infty$, $m \to \infty$, then $\lambda(y) = \lim_{n\to\infty} \frac{1}{n} \log B_n^{(\alpha)}(y)$

Pulled adsorbing f-spiders

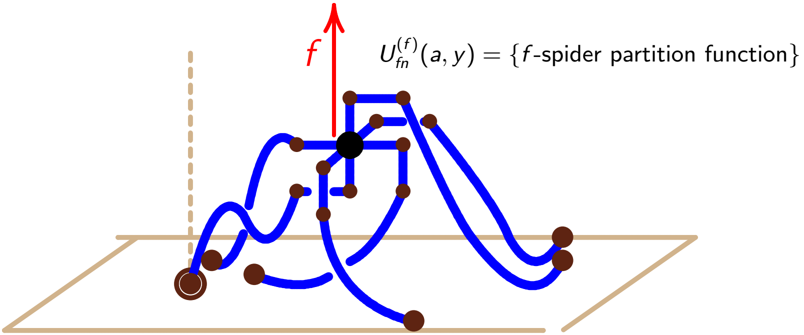
- Create adsorbing and pulled bridges in wedges:
$$B_n^{(\alpha)}(a,y) = \sum_{v,h} b_n^{(\alpha)}(v,h)\, a^v y^h$$
- $\lim_{n\to\infty} \frac{1}{n} \log B_n^{(\alpha)}(a,y) = \psi(a,y) = \max\{\kappa(a), \lambda(y)\}$

Pulled adsorbing f-spiders

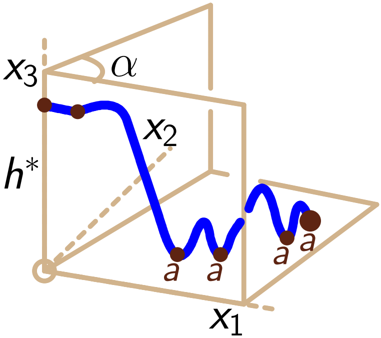
- Create adsorbing and pulled bridges in wedges:

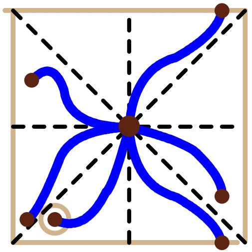
$$B_n^{(\alpha)}(a,y) = \sum_{v,h} b_n^{(\alpha)}(v,h)\, a^v y^h$$

- $\lim_{n\to\infty} \frac{1}{n} \log B_n^{(\alpha)}(a,y) = \psi(a,y) = \max\{\kappa(a), \lambda(y)\}$
- Put these together to create pulled and adsorbing spiders



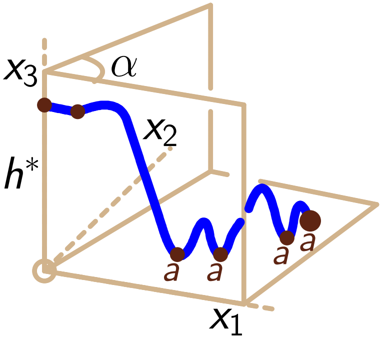
Lower bound

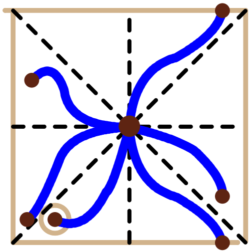




- $U_{fn}^{(f)}(a,y) \geq \left[\sum_v b_{n-k}^{(\alpha)}(v,h^*)\, a^v \right]^f y^{h^*} = \left[\sum_v b_{n-k}^{(\alpha)}(v,h^*)\, a^v y^{h^*/f} \right]^f$

Lower bound

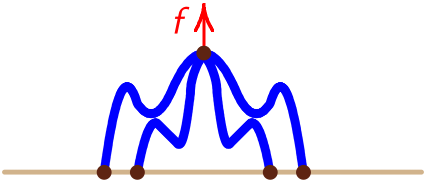




- $U_{fn}^{(f)}(a,y) \geq \left[\sum_v b_{n-k}^{(\alpha)}(v,h^*)\,a^v\right]^f y^{h^*} = \left[\sum_v b_{n-k}^{(\alpha)}(v,h^*)\,a^v y^{h^*/f}\right]^f$
- $\Rightarrow \liminf_{n\to\infty} \frac{1}{fn} \log U_{fn}^{(f)}(a,y) \geq \max\{\kappa(a), \lambda(y^{1/f})\} = \psi(a, y^{1/f})$

Upper bound

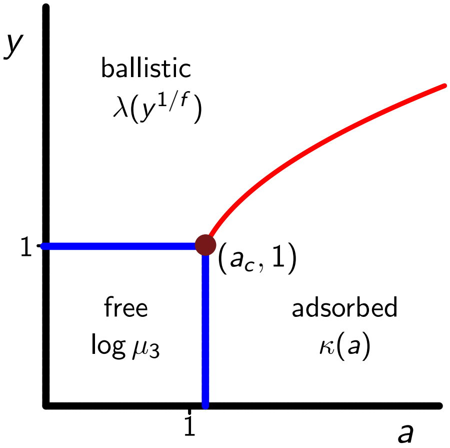
- Ignore intersections between the legs
- f adsorbing and pulled self-avoiding walks of most popular height h^*:



- $$\left[(n+1)\sum_{v=0}^{n} c_n(v,h^*)\, a^v y^{h^*/f}\right]^f \geq U_{nf}^{(f)}(a,y)$$
- Take logs, divide by nf and then $n \to \infty$:

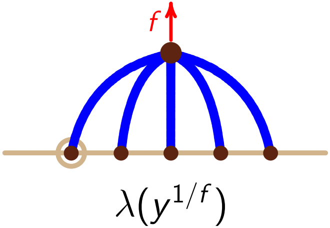
$$\psi(a, y^{1/f}) \geq \limsup_{n\to\infty} \frac{1}{nf} \log U_{nf}^{(f)}(a,y).$$

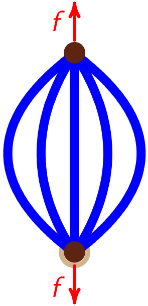
Phase diagram of pulled f-spiders



Pulled f-spiders and an f-watermelon graph

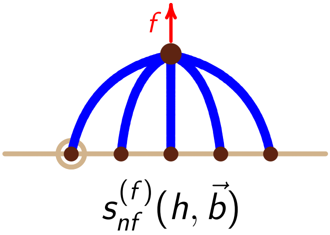
- $s_{nf}^{(f)}(h) = \#\{\text{spiders of length } nf \text{ and height } h\}$
- $w_{2nf}(h) = \#\{\text{watermelons of length } 2nf \text{ and extent } h\} \leq s_{2nf}^{(f)}(h)$

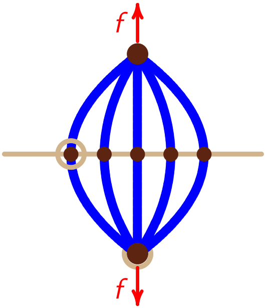




- $s_{nf}^{(f)}(h; \vec{b}^*) =$ Most popular leg endpoints $\{\vec{b}^*\}$
- $\lambda(y^{1/f}) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{h=0}^{n} s_{nf}^{(f)}(h; \vec{b}^*)\, y^h = \lim_{n\to\infty} \frac{1}{n} \log (s_{nf}^{(f)}(h^*; \vec{b}^*)\, y^{h^*})$

Pulled f-spiders and an f-watermelon graph





- $\left(s_{nf}^{(f)}(h^*; \vec{b}^*) y^{h^*}\right)^2 \leq \sum_{h=0}^{2n} w_{2nf}(h) y^h \leq \sum_{h=0}^{2n} s_{2nf}^{(f)}(h) y^h$
- Take logarithms, divide by $2nf$ and then $n \to \infty$

$$\lim_{n\to\infty} \frac{1}{2nf} \log \sum_{h=0}^{2n} w_{2nf}(h) y^h = \lambda(y^{1/f})$$

Conclusions

- Free energies are functions of connectivity
- The phase diagrams include first order and continuous transitions
- Multicritical points
- Copolymer spiders have mixed phases (both adsorbed and ballistic)

- Tony: thanks for the invitation to contribute to this meeting