The Izergin–Korepin model at roots of unity

A Garbali

School of Maths&Stats UoM ACEMS

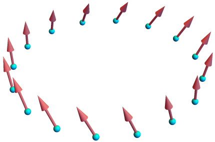
ANZAMP 2015

Contents

- One of the fundamental models in physics is the Heisenberg spin chain

$$H = \sum_{i=1}^{N} \left\{ J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z \right\}.$$

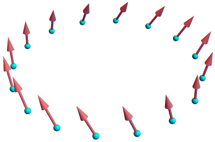
This model describes interacting spins on a circle (or a line)



- One of the fundamental models in physics is the Heisenberg spin chain

$$H = \sum_{i=1}^{N} \left\{ J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z \right\}.$$

This model describes interacting spins on a circle (or a line)



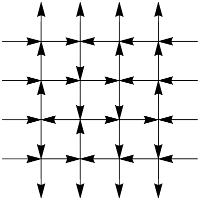
- Reducing anisotropy in the XY plane we get the XXZ spin chain

$$H_{XXZ} = J \sum_{i=1}^{N} \left\{ \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta(\sigma_i^z \sigma_{i+1}^z - 1) \right\}.$$

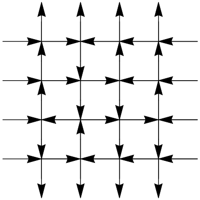
This model is the prototype for many integrable models.

- The XXZ model can be reformulated in the six vertex language

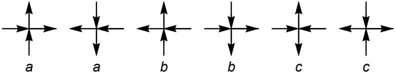
- The XXZ model can be reformulated in the six vertex language
- The six vertex model is defined on a square lattice. Here is a typical configuration



- The XXZ model can be reformulated in the six vertex language
- The six vertex model is defined on a square lattice. Here is a typical configuration



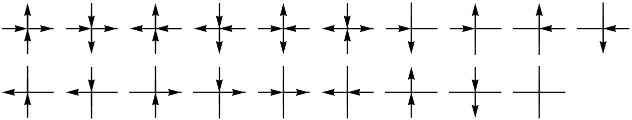
- It is build by choosing at each vertex of the square lattice one of the configurations

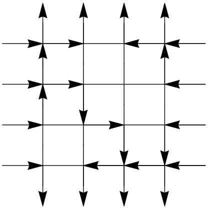


In particular, the six vertex model provides a very powerful technique for diagonalizing H_{XXZ} which is called the Algebraic Bethe Ansatz

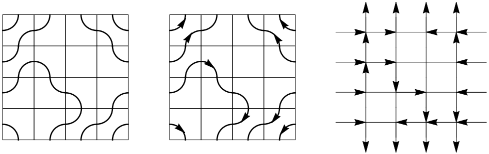
- The XXZ model is associated with the algebra $U_q(\hat{sl}_2)$ ($U_q(A_1^{(1)})$). We will consider a "generalisation" of XXZ by requiring the algebra to be $U_q(A_2^{(2)})$. This model is called the Izergin–Korepin model.

- The XXZ model is associated with the algebra $U_q(\hat{sl}_2)$ ($U_q(A_1^{(1)})$). We will consider a "generalisation" of XXZ by requiring the algebra to be $U_q(A_2^{(2)})$. This model is called the Izergin–Korepin model.
- The IK model corresponds to a 19-vertex (19v) model

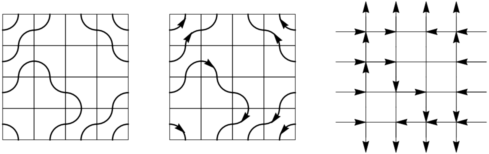




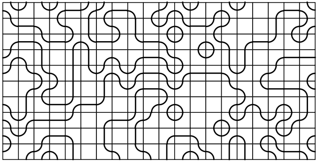
- The IK 19v model is related to the dilute O(n) Temperley–Lieb (dTL) loop models via the mapping



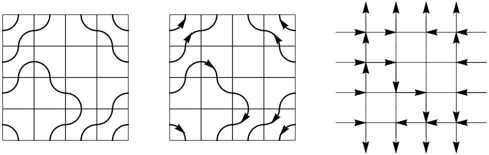
- The IK 19v model is related to the dilute O(n) Temperley–Lieb (dTL) loop models via the mapping



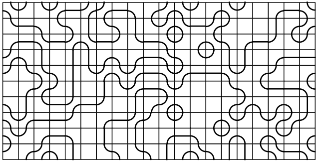
- A typical configuration of this model is



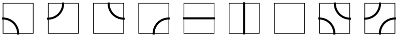
- The IK 19v model is related to the dilute O(n) Temperley–Lieb (dTL) loop models via the mapping



- A typical configuration of this model is



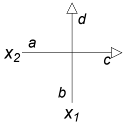
- It is built in the bulk using the nine plaquettes



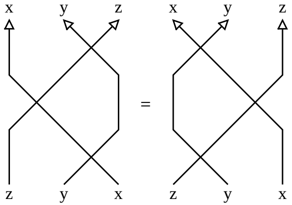
The IK 19v model is is defined through its R-matrix, $R \in End(V \otimes V)$

$$R(x_2/x_1) = \sum_{a,b,c,d} r_{a,b}^{c,d}(x_2/x_1) e_{a,c} \otimes e_{b,d}.$$

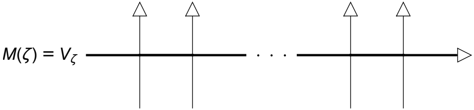
$r_{a,b}^{c,d}(x_2/x_1)$ =



We define R_i by the action of R on the $i, i+1$ part of $\mathcal{H}_L = \mathbb{C}^3 \otimes .. \otimes \mathbb{C}^3$. The basis in $\mathbb{C}^3$ is represented by $v_1 = \uparrow,\ v_2 = \circ,\ v_3 = \downarrow$
The R-matrix satisfies the Yang–Baxter equation



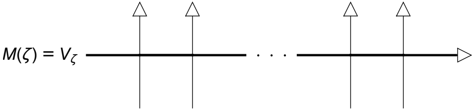
- Using this R-matrix we can construct the transfer matrix of the IK model



$$T(\zeta) = \mathrm{Tr}_V M(\zeta)\tau = \mathrm{Tr}_V \left[R_1(\zeta/z_1) R_2(\zeta/z_2) \dots R_L(\zeta/z_L) \times \tau \right],$$

where we assumed periodic boundary condition with the twist $\tau = e^{\phi \sigma_L^z}$.

- Using this R-matrix we can construct the transfer matrix of the IK model



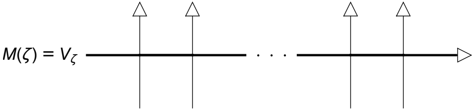
$$T(\zeta) = \mathrm{Tr}_V M(\zeta)\tau = \mathrm{Tr}_V \left[R_1(\zeta/z_1)R_2(\zeta/z_2)\dots R_L(\zeta/z_L) \times \tau\right],$$

where we assumed periodic boundary condition with the twist $\tau = e^{\phi\sigma_L^z}$.

- The IK Hamiltonian H_{IK} is related to the homogeneous transfer matrix $T(\zeta)$

$$H_{IK} = T^{-1}(\zeta)\frac{dT(\zeta)}{d\zeta}|_{\zeta=1}.$$

- Using this R-matrix we can construct the transfer matrix of the IK model



$$T(\zeta) = \mathrm{Tr}_V M(\zeta)\tau = \mathrm{Tr}_V \left[R_1(\zeta/z_1)R_2(\zeta/z_2)\ldots R_L(\zeta/z_L) \times \tau\right],$$

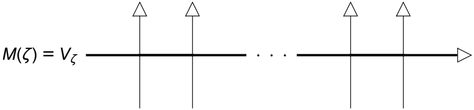
where we assumed periodic boundary condition with the twist $\tau = e^{\phi\sigma_L^z}$.

- The IK Hamiltonian H_{IK} is related to the homogeneous transfer matrix $T(\zeta)$

$$H_{IK} = T^{-1}(\zeta)\frac{dT(\zeta)}{d\zeta}|_{\zeta=1}.$$

- Therefore H_{IK} shares the eigenvectors with $T(\zeta)$

- Using this R-matrix we can construct the transfer matrix of the IK model



$$T(\zeta) = \mathrm{Tr}_V M(\zeta)\tau = \mathrm{Tr}_V \left[R_1(\zeta/z_1)R_2(\zeta/z_2)\dots R_L(\zeta/z_L) \times \tau\right],$$

where we assumed periodic boundary condition with the twist $\tau = e^{\phi\sigma^z_L}$.

- The IK Hamiltonian H_{IK} is related to the homogeneous transfer matrix $T(\zeta)$

$$H_{IK} = T^{-1}(\zeta)\frac{dT(\zeta)}{d\zeta}|_{\zeta=1}.$$

- Therefore H_{IK} shares the eigenvectors with $T(\zeta)$
- Starting from the loop model R-matrix one can as well define the transfer matrix of the dTL O(n) loop model.

- We can study this model by means of the algebraic and coordinate Bethe Ansatz.

- We can study this model by means of the algebraic and coordinate Bethe Ansatz.
- The monodromy matrix in the auxiliary space can be written

$$M(\zeta) = \begin{bmatrix} A_1(\zeta) & B_1(\zeta) & B_2(\zeta) \\ C_1(\zeta) & A_2(\zeta) & B_3(\zeta) \\ C_2(\zeta) & C_3(\zeta) & A_3(\zeta) \end{bmatrix}.$$

$A_i(\zeta), B_i(\zeta), C_i(\zeta)$ form the Yang–Baxter algebra for the IK model. The defining relations of the YB algebra are given by

$$R(x/y)M(x) \otimes M(y) = M(y) \otimes M(x)R(x/y).$$

- We can study this model by means of the algebraic and coordinate Bethe Ansatz.
- The monodromy matrix in the auxiliary space can be written

$$M(\zeta) = \begin{bmatrix} A_1(\zeta) & B_1(\zeta) & B_2(\zeta) \\ C_1(\zeta) & A_2(\zeta) & B_3(\zeta) \\ C_2(\zeta) & C_3(\zeta) & A_3(\zeta) \end{bmatrix}.$$

$A_i(\zeta), B_i(\zeta), C_i(\zeta)$ form the Yang–Baxter algebra for the IK model. The defining relations of the YB algebra are given by

$$R(x/y)M(x) \otimes M(y) = M(y) \otimes M(x)R(x/y).$$

- The (untwisted) transfer matrix T can be written in terms of YB algebra

$$T(\zeta) = A_1(\zeta) + A_2(\zeta) + A_3(\zeta).$$

The eigenvectors of T are given as polynomials in the YB algebra elements which can be shown using the RMM equation and assuming additional conditions (Bethe equations)

- According to Tarasov's BA, the N-magnon eigenstate $|\Psi\rangle$ of T equals to a polynomial Φ_N acting on the reference state: $|0\rangle = \uparrow .. \uparrow$ s.t.

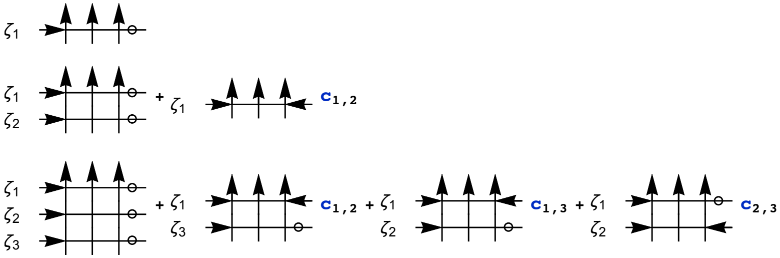
$$\Phi_N(\zeta_1, .., \zeta_N) = B_1(\zeta_1)\Phi_{N-1}(\zeta_2, .., \zeta_N) +$$
$$B_2(\zeta_1) \sum_{i>1} c_{1,i}(\zeta_1, .., \zeta_N)\Phi_{N-2}(\zeta_2, .., \hat{\zeta}_i, .., \zeta_N).$$

- According to Tarasov's BA, the N-magnon eigenstate $|\Psi\rangle$ of T equals to a polynomial Φ_N acting on the reference state: $|0\rangle = \uparrow .. \uparrow$ s.t.

$$\Phi_N(\zeta_1, .., \zeta_N) = B_1(\zeta_1)\Phi_{N-1}(\zeta_2, .., \zeta_N) +$$

$$B_2(\zeta_1) \sum_{i>1} c_{1,i}(\zeta_1, .., \zeta_N)\Phi_{N-2}(\zeta_2, .., \hat{\zeta}_i, .., \zeta_N).$$

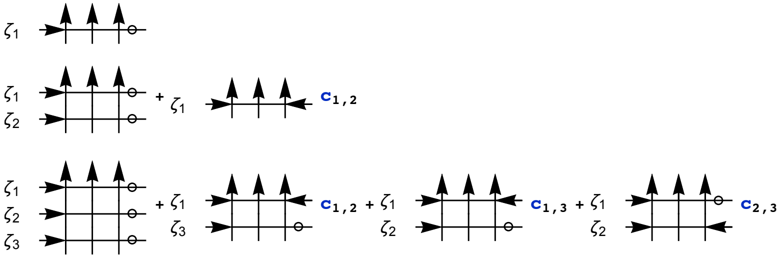
- Here are $1, 2$ and 3 particle states of the system of length $L = 3$



- According to Tarasov's BA, the N-magnon eigenstate $|\Psi\rangle$ of T equals to a polynomial Φ_N acting on the reference state: $|0\rangle = \uparrow .. \uparrow$ s.t.

$$\Phi_N(\zeta_1, .., \zeta_N) = B_1(\zeta_1)\Phi_{N-1}(\zeta_2, .., \zeta_N) +$$
$$B_2(\zeta_1)\sum_{i>1} c_{1,i}(\zeta_1, .., \zeta_N)\Phi_{N-2}(\zeta_2, .., \hat{\zeta}_i, .., \zeta_N).$$

- Here are $1, 2$ and 3 particle states of the system of length $L = 3$



- The ABA for the six vertex model gives simply

$$\Phi_N^{6v}(\zeta_1, .., \zeta_L) = \prod_{i=1}^{L} B(\zeta_i)$$

- The parameters $\zeta_1, .., \zeta_N$ of N-particle eigenstate satisfies Bethe equations

- The parameters $\zeta_1, .., \zeta_N$ of N-particle eigenstate satisfies Bethe equations
- Let $F(\zeta) = \prod_{i=1}^{L} \left(\zeta^2 - z_i^2\right)$, $Q(\zeta) = \prod_{i=1}^{N} \left(\zeta^2 - \zeta_i^2\right)$;
  then $\zeta_1, .., \zeta_N$ must satisfy the Bethe equations

$$qF\left(q\zeta_j\right) Q\left(q^{-1}\zeta_j\right) Q\left(iq^{1/2}\zeta_j\right) + F\left(\zeta_j\right) Q\left(iq^{-1/2}\zeta_j\right) Q\left(q\zeta_j\right) = 0,$$

- The parameters $\zeta_1, .., \zeta_N$ of N-particle eigenstate satisfies Bethe equations
- Let $F(\zeta) = \prod_{i=1}^{L} \left(\zeta^2 - z_i^2\right)$, $Q(\zeta) = \prod_{i=1}^{N} \left(\zeta^2 - \zeta_i^2\right)$; then $\zeta_1, .., \zeta_N$ must satisfy the Bethe equations

$$qF\left(q\zeta_j\right) Q\left(q^{-1}\zeta_j\right) Q\left(iq^{1/2}\zeta_j\right) + F\left(\zeta_j\right) Q\left(iq^{-1/2}\zeta_j\right) Q\left(q\zeta_j\right) = 0,$$

- and the eigenvalues $\mathcal{T}(\zeta)$ of $T(\zeta)$ are given by

$$q^{-L}\mathcal{T}(\zeta) = \frac{F(\zeta)F(iq^{1/2}\zeta)}{qF(q\zeta)F(iq^{3/2}\zeta)} \frac{Q(\zeta)Q(iq^{3/2}\zeta)}{Q(\zeta)Q(iq^{1/2}\zeta)} + \frac{F(\zeta)}{F(q\zeta)} \frac{Q(iq^{-1/2}\zeta)Q(q\zeta)}{Q(\zeta)Q(iq^{1/2}\zeta)} + q\frac{Q(q^{-1}\zeta)Q(iq^{1/2}\zeta)}{Q(\zeta)Q(iq^{1/2}\zeta)}.$$

- The parameters $\zeta_1, .., \zeta_N$ of N-particle eigenstate satisfies Bethe equations
- Let $F(\zeta) = \prod_{i=1}^{L} \left(\zeta^2 - z_i^2\right)$, $Q(\zeta) = \prod_{i=1}^{N} \left(\zeta^2 - \zeta_i^2\right)$;
then $\zeta_1, .., \zeta_N$ must satisfy the Bethe equations

$$qF\left(q\zeta_j\right) Q\left(q^{-1}\zeta_j\right) Q\left(iq^{1/2}\zeta_j\right) + F\left(\zeta_j\right) Q\left(iq^{-1/2}\zeta_j\right) Q\left(q\zeta_j\right) = 0,$$

- and the eigenvalues $\mathcal{T}(\zeta)$ of $T(\zeta)$ are given by

$$q^{-L}\mathcal{T}(\zeta) = \frac{F(\zeta)F(iq^{1/2}\zeta)}{qF(q\zeta)F(iq^{3/2}\zeta)} \frac{Q(\zeta)Q(iq^{3/2}\zeta)}{Q(\zeta)Q(iq^{1/2}\zeta)} + \frac{F(\zeta)}{F(q\zeta)} \frac{Q(iq^{-1/2}\zeta)Q(q\zeta)}{Q(\zeta)Q(iq^{1/2}\zeta)} + q\frac{Q(q^{-1}\zeta)Q(iq^{1/2}\zeta)}{Q(\zeta)Q(iq^{1/2}\zeta)}.$$

- where $q = e^{i\theta}$ is the interaction parameter

- The parameters $\zeta_1,..,\zeta_N$ of N-particle eigenstate satisfies Bethe equations
- Let $F(\zeta)=\prod_{i=1}^{L}\left(\zeta^2-z_i^2\right)$, $Q(\zeta)=\prod_{i=1}^{N}\left(\zeta^2-\zeta_i^2\right)$; then $\zeta_1,..,\zeta_N$ must satisfy the Bethe equations

$$qF\left(q\zeta_j\right)Q\left(q^{-1}\zeta_j\right)Q\left(iq^{1/2}\zeta_j\right)+F\left(\zeta_j\right)Q\left(iq^{-1/2}\zeta_j\right)Q\left(q\zeta_j\right)=0,$$

- and the eigenvalues $\mathcal{T}(\zeta)$ of $T(\zeta)$ are given by

$$\begin{aligned}q^{-L}\mathcal{T}(\zeta)=&\frac{F(\zeta)F(iq^{1/2}\zeta)}{qF(q\zeta)F(iq^{3/2}\zeta)}\frac{Q(\zeta)Q(iq^{3/2}\zeta)}{Q(\zeta)Q(iq^{1/2}\zeta)}\\&+\frac{F(\zeta)}{F(q\zeta)}\frac{Q(iq^{-1/2}\zeta)Q(q\zeta)}{Q(\zeta)Q(iq^{1/2}\zeta)}+q\frac{Q(q^{-1}\zeta)Q(iq^{1/2}\zeta)}{Q(\zeta)Q(iq^{1/2}\zeta)}.\end{aligned}$$

- where $q=e^{i\theta}$ is the interaction parameter
- We need to solve the Bethe equations and find a good representation for correlation functions

- For calculations of correlation functions a crucial quantity is the scalar product

$$S_N(\mu_1, .., \mu_N; \zeta_1, .., \zeta_N) = \langle \bar{\Psi}(\mu_1, .., \mu_N) | \Psi(\zeta_1, .., \zeta_N) \rangle.$$

- For calculations of correlation functions a crucial quantity is the scalar product

$$S_N(\mu_1, .., \mu_N; \zeta_1, .., \zeta_N) = \langle \bar{\Psi}(\mu_1, .., \mu_N) | \Psi(\zeta_1, .., \zeta_N) \rangle.$$

- As a fist step we propose a closed formula for the eigenstates Ψ.

- For calculations of correlation functions a crucial quantity is the scalar product

$$S_N(\mu_1, .., \mu_N; \zeta_1, .., \zeta_N) = \langle \bar{\Psi}(\mu_1, .., \mu_N) | \Psi(\zeta_1, .., \zeta_N) \rangle.$$

- As a fist step we propose a closed formula for the eigenstates Ψ.
- Introducing certain normal ordering we find a representation

$$|\Psi_N\rangle = \oint \frac{\mathrm{d}x}{x^{N+1}} : \prod_{i=1}^{N} e^{x\mathcal{B}(\zeta_i;x)} : |0\rangle,$$

$$\mathcal{B}(\zeta_i; x) = xB_2(\zeta_i) \sum_{i<j\leq N} c_{i,j} + B_1(\zeta_i).$$

- For calculations of correlation functions a crucial quantity is the scalar product

$$S_N(\mu_1, .., \mu_N; \zeta_1, .., \zeta_N) = \langle \bar{\Psi}(\mu_1, .., \mu_N) | \Psi(\zeta_1, .., \zeta_N) \rangle.$$

- As a fist step we propose a closed formula for the eigenstates Ψ.
- Introducing certain normal ordering we find a representation

$$|\Psi_N\rangle = \oint \frac{\mathrm{d}x}{x^{N+1}} : \prod_{i=1}^{N} e^{x\mathcal{B}(\zeta_i;x)} : |0\rangle,$$

$$\mathcal{B}(\zeta_i; x) = xB_2(\zeta_i) \sum_{i<j\le N} c_{i,j} + B_1(\zeta_i).$$

- We can write a similar representation for the dual state. S_N becomes

$$S_N = \oint \frac{\mathrm{d}x\mathrm{d}y}{x^{N+1}y^{N+1}} \langle 0| : \prod_{i=1}^{N} e^{x\mathcal{C}(\mu_i;x)} : : \prod_{i=1}^{N} e^{y\mathcal{B}(\zeta_i;y)} : |0\rangle.$$

- For calculations of correlation functions a crucial quantity is the scalar product

$$S_N(\mu_1, .., \mu_N; \zeta_1, .., \zeta_N) = \langle \bar{\Psi}(\mu_1, .., \mu_N) | \Psi(\zeta_1, .., \zeta_N) \rangle.$$

- As a fist step we propose a closed formula for the eigenstates Ψ.
- Introducing certain normal ordering we find a representation

$$|\Psi_N\rangle = \oint \frac{\mathrm{d}x}{x^{N+1}} : \prod_{i=1}^{N} e^{x\mathcal{B}(\zeta_i;x)} : |0\rangle,$$

$$\mathcal{B}(\zeta_i; x) = xB_2(\zeta_i) \sum_{i<j\leq N} c_{i,j} + B_1(\zeta_i).$$

- We can write a similar representation for the dual state. S_N becomes

$$S_N = \oint \frac{\mathrm{d}x\mathrm{d}y}{x^{N+1}y^{N+1}} \langle 0| : \prod_{i=1}^{N} e^{x\mathcal{C}(\mu_i;x)} : \ : \prod_{i=1}^{N} e^{y\mathcal{B}(\zeta_i;y)} : |0\rangle.$$

- Remark: This also holds for $sl(2)$ spin-1 (FZ) and $osp(2|1)$ models

Now we would like to recall some coordinate Bethe Ansatz results

- The free energy f_L of the $L \times \infty$ cylinder for large L is

$$f_L(v) \sim f_\infty - c\frac{\pi}{6L^2}\cosh 2\rho_\theta v,$$

where c is the conformal anomaly

Now we would like to recall some coordinate Bethe Ansatz results

- The free energy f_L of the $L \times \infty$ cylinder for large L is

$$f_L(v) \sim f_\infty - c\frac{\pi}{6L^2}\cosh 2\rho_\theta v,$$

  where c is the conformal anomaly

- There are three regimes (ϕ is the twist parameter)

$$c = 1 - \frac{3\phi^2}{\pi\theta}, \qquad \text{for } 0 < \theta < \pi$$

$$c = \frac{3}{2} - \frac{3\phi^2}{\pi(\pi+\theta)}, \qquad \text{for } -\pi < \theta < \pi/3$$

$$c = 2 + \frac{3\phi^2}{\pi\theta}, \qquad \text{for } -\pi/3 < \theta < 0 \quad \text{and } \phi \leq -\theta$$

$$c = -1 + \frac{3(\phi-\pi)^2}{\pi(\pi+\theta)}, \qquad \text{for } -\pi/3 < \theta < 0 \quad \text{and } \phi \geq -\theta$$

(Nienhuis, Blöte, Warnaar, Batchelor, ..)

The eigenstates of the IK Hamiltonian are split into N-particle sectors. At $\theta = \pi/3$ it is possible to diagonalize the IK Hamiltonian in a large portion (2^{L-1} states) of the 0-particle sector.

- We can write corresponding eigenvalues exactly

$$\mathcal{T}_\epsilon(t) = \frac{q^2 \left(F_\epsilon\left(-tq\right) F_\epsilon\left(-tq^2\right) + F_\epsilon\left(tq\right) F_\epsilon\left(tq^2\right)\right)}{F_\epsilon\left(-tq\right) F_\epsilon\left(tq\right)},$$

where $\epsilon = \{\epsilon_1, .., \epsilon_N\}$, $\epsilon_i = \pm 1$ and $F_\epsilon(x) = \prod_{i=1}^{N} \left(x - \epsilon_i z_i\right)$ and $q = e^{i\theta}$

The eigenstates of the IK Hamiltonian are split into N-particle sectors. At $\theta = \pi/3$ it is possible to diagonalize the IK Hamiltonian in a large portion (2^{L-1} states) of the 0-particle sector.

- We can write corresponding eigenvalues exactly

$$\mathcal{T}_\epsilon(t) = \frac{q^2\left(F_\epsilon(-tq)\,F_\epsilon\left(-tq^2\right) + F_\epsilon(tq)\,F_\epsilon\left(tq^2\right)\right)}{F_\epsilon(-tq)\,F_\epsilon(tq)},$$

  where $\epsilon = \{\epsilon_1, .., \epsilon_N\}$, $\epsilon_i = \pm 1$ and $F_\epsilon(x) = \prod_{i=1}^{N}(x - \epsilon_i z_i)$ and $q = e^{i\theta}$

- This implies that the corresponding eigenvectors have polynomial in z_i entries

The eigenstates of the IK Hamiltonian are split into N-particle sectors. At $\theta = \pi/3$ it is possible to diagonalize the IK Hamiltonian in a large portion (2^{L-1} states) of the 0-particle sector.

- We can write corresponding eigenvalues exactly

$$\mathcal{T}_\epsilon(t) = \frac{q^2\left(F_\epsilon\left(-tq\right)F_\epsilon\left(-tq^2\right) + F_\epsilon\left(tq\right)F_\epsilon\left(tq^2\right)\right)}{F_\epsilon\left(-tq\right)F_\epsilon\left(tq\right)},$$

  where $\epsilon = \{\epsilon_1, .., \epsilon_N\}$, $\epsilon_i = \pm 1$ and $F_\epsilon(x) = \prod_{i=1}^{N}\left(x - \epsilon_i z_i\right)$ and $q = e^{i\theta}$

- This implies that the corresponding eigenvectors have polynomial in z_i entries
- Conjecturally, the ground state corresponds to $\epsilon_i = 1$ for all i

The eigenstates of the IK Hamiltonian are split into N-particle sectors. At $\theta = \pi/3$ it is possible to diagonalize the IK Hamiltonian in a large portion (2^{L-1} states) of the 0-particle sector.

- We can write corresponding eigenvalues exactly

$$\mathcal{T}_\epsilon(t) = \frac{q^2 \left(F_\epsilon\left(-tq\right) F_\epsilon\left(-tq^2\right) + F_\epsilon\left(tq\right) F_\epsilon\left(tq^2\right)\right)}{F_\epsilon\left(-tq\right) F_\epsilon\left(tq\right)},$$

  where $\epsilon = \{\epsilon_1, .., \epsilon_N\}$, $\epsilon_i = \pm 1$ and $F_\epsilon(x) = \prod_{i=1}^{N}\left(x - \epsilon_i z_i\right)$ and $q = e^{i\theta}$
- This implies that the corresponding eigenvectors have polynomial in z_i entries
- Conjecturally, the ground state corresponds to $\epsilon_i = 1$ for all i
- This simplification may allow one to study correlation functions in this sector

The eigenstates of the IK Hamiltonian are split into N-particle sectors. At $\theta = \pi/3$ it is possible to diagonalize the IK Hamiltonian in a large portion (2^{L-1} states) of the 0-particle sector.

- We can write corresponding eigenvalues exactly

$$\mathcal{T}_\epsilon(t) = \frac{q^2 \left(F_\epsilon\left(-tq\right) F_\epsilon\left(-tq^2\right) + F_\epsilon\left(tq\right) F_\epsilon\left(tq^2\right)\right)}{F_\epsilon\left(-tq\right) F_\epsilon\left(tq\right)},$$

where $\epsilon = \{\epsilon_1, .., \epsilon_N\}$, $\epsilon_i = \pm 1$ and $F_\epsilon(x) = \prod_{i=1}^{N} (x - \epsilon_i z_i)$ and $q = e^{i\theta}$

- This implies that the corresponding eigenvectors have polynomial in z_i entries
- Conjecturally, the ground state corresponds to $\epsilon_i = 1$ for all i
- This simplification may allow one to study correlation functions in this sector
- In particular, the domain wall partition function is a determinant

The eigenstates of the IK Hamiltonian are split into N-particle sectors. At $\theta = \pi/3$ it is possible to diagonalize the IK Hamiltonian in a large portion (2^{L-1} states) of the 0-particle sector.

- We can write corresponding eigenvalues exactly

$$\mathcal{T}_\epsilon(t) = \frac{q^2\left(F_\epsilon\left(-tq\right)F_\epsilon\left(-tq^2\right) + F_\epsilon\left(tq\right)F_\epsilon\left(tq^2\right)\right)}{F_\epsilon\left(-tq\right)F_\epsilon\left(tq\right)},$$

  where $\epsilon = \{\epsilon_1, .., \epsilon_N\}$, $\epsilon_i = \pm 1$ and $F_\epsilon(x) = \prod_{i=1}^{N}\left(x - \epsilon_i z_i\right)$ and $q = e^{i\theta}$
- This implies that the corresponding eigenvectors have polynomial in z_i entries
- Conjecturally, the ground state corresponds to $\epsilon_i = 1$ for all i
- This simplification may allow one to study correlation functions in this sector
- In particular, the domain wall partition function is a determinant
- On the O(n=1) loop model side we can compute the norm of the ground state and simple correlation functions
  (Fehér, Nienhuis, A G)

Let us focus now on the third regime

- IK model in the continuum limit corresponds to a nonimpact CFT with one compact and one nonimpact bosons

Let us focus now on the third regime

- IK model in the continuum limit corresponds to a nonimpact CFT with one compact and one nonimpact bosons
- The IK model in this regime is dual to Witten Euclidean black hole CFT [coset $SL(2,\mathbb{R})/U(1)$]

Let us focus now on the third regime

- IK model in the continuum limit corresponds to a nonimpact CFT with one compact and one nonimpact bosons
- The IK model in this regime is dual to Witten Euclidean black hole CFT [coset $SL(2,\mathbb{R})/U(1)$]
- When $n \to 0$ the dilute O(n) model describes 'the' (Θ-transition) collapse of two dimensional polymers. This is the transition between the dilute phase and the dense phase of polymers with short range attraction
(Duplantier, Saleur; Nienhuis, Blöte; Vernier, Jacobsen, Saleur, ...)

We consider the regime $\theta = -\pi/4$. This translates to the condition $n = 0$ in the language of the dilute O(n) loop model.

- As in the previous situation, the conjectural ground state of the IK model has polynomial (in z_i) entries

We consider the regime $\theta = -\pi/4$. This translates to the condition $n = 0$ in the language of the dilute O(n) loop model.

- As in the previous situation, the conjectural ground state of the IK model has polynomial (in z_i) entries
- Using the free field realization of vertex operators $\Phi_j(z_k)$ of $U_q(A_2^{(2)})$ algebra we can write integral formulae for the ground state components

$$\psi_{i_1,..,i_L}(z_1,..,z_L) = \langle \Phi_{i_1}(z_1)..\Phi_{i_L}(z_L) \rangle,$$

where $i_j = +, -, 0$.

We consider the regime $\theta = -\pi/4$. This translates to the condition $n = 0$ in the language of the dilute O(n) loop model.

- As in the previous situation, the conjectural ground state of the IK model has polynomial (in z_i) entries
- Using the free field realization of vertex operators $\Phi_j(z_k)$ of $U_q(A_2^{(2)})$ algebra we can write integral formulae for the ground state components

$$\psi_{i_1,..,i_L}(z_1,..,z_L) = \langle \Phi_{i_1}(z_1)..\Phi_{i_L}(z_L) \rangle,$$

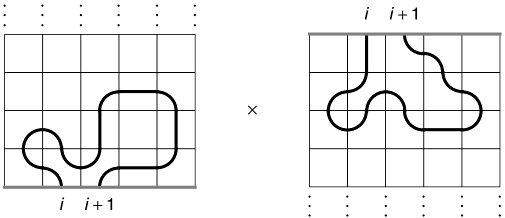
  where $i_j = +, -, 0$.
- This allows us, using the map between the IK model and the loop model, to obtain some components ψ° of the ground state eigenvector of the loop model. This loop model is defined on half infinite $L \times \infty$ region of the square lattice with inhomogeneities $z_1, ..z_L$.

  The components ψ°_π are labeled by the connectivity π of the loops on the rim of the cylinder (for periodic b.c.)

As an application consider the partition function of a single closed loop

$$Z_1(z_1, .., z_L) = \sum_{\alpha,\beta} \psi^\circ_\alpha(z_1, .., z_L)\bar{\psi}^\circ_\beta(z_1, .., z_L),$$

where $\bar{\psi}^\circ_\beta$ is a component of the dual ground state and the sum runs over all matching connectivities which form a single closed loop, for example



$= \psi^\circ_{i,i+1}\overline{\psi}^\circ_{i,i+1}$

The partition function $Z_1^{(L)}$ is a polynomial in z_i. It is possible to derive a recurrence relation in size L for $Z_1^{(L)}$. Pick a horizontal position i and set two inhomogeneities proportional $z_i = e^{\theta} z, \; z_{i+1} = e^{-\theta} z$

$$
\begin{aligned}
Z_1^{(L)}(.., e^{\theta} z, e^{-\theta} z, ..) &= \mathrm{const}(\theta) \prod (z_i^2 + z^2) Z_1^{(L-1)}(.., z, ..) \\
&+ \psi^{\circ}_{i,i+1}(.., e^{\theta} z, e^{-\theta} z, ..) \bar{\psi}^{\circ}_{i,i+1}(.., e^{\theta} z, e^{-\theta} z, ..)
\end{aligned}
$$

Where the $\psi^{\circ}_{i,i+1}$ is the component corresponding to the link pattern with the points i and $i+1$ connected

The above equation determines Z_1 together with a simple additional equation which occurs at $z_i = 0$.

The component $\psi^{\circ}_{i,i+1}$ can be written in terms of two components of the IK model

$$
\psi^{\circ}_{i,i+1} = (z_1/z_2)^{1/2} \psi_{+-} - e^{2\theta} (z_1/z_2)^{-1/2} \psi_{-+}
$$

Using the vertex operators we get

$$
\psi^{\circ}_{i,i+1} \propto \oint .. \oint \prod_{i>2} \frac{dw_i}{(z_i + q w_i)(w_i + q z_i)} \prod_{i<j} \frac{(w_i - w_j)(w_i - q^2 w_j)}{(z_i + q w_j)(w_i + q z_j)(w_i - q w_j)} \Big|_{w_1 = w_2 = a}
$$

where $a = -e^{2\theta} z_1^2$. (Fehér, Nienhuis, A G)

Conclusion

- The expression for the partition function with the domain wall boundary condition at generic θ remains unknown. This expression is important in particular for the calculation of S_N

Conclusion

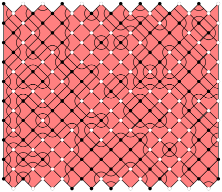
- The expression for the partition function with the domain wall boundary condition at generic θ remains unknown. This expression is important in particular for the calculation of S_N
- We found explicit eigenvalues for $\theta = \pi/3$ in 0-particle sector. It would be interesting to understand which part of the spectrum they correspond to

Conclusion

- The expression for the partition function with the domain wall boundary condition at generic θ remains unknown. This expression is important in particular for the calculation of S_N
- We found explicit eigenvalues for $\theta = \pi/3$ in 0-particle sector. It would be interesting to understand which part of the spectrum they correspond to
- We need to find a good representation for the scalar product. This will lead to form factors and thus will allow one to study analytically the polymer partition function and correlation functions for critical site percolation

Thank you!

Let's look at the simplest case $\theta = \pi/3$ or $n = 1$ in the loop model description. In this case the ground state of dTL O(1) model describes critical site percolation



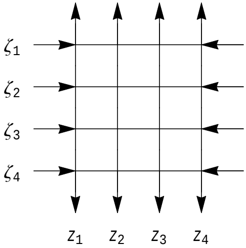
The ground state vector satisfies the reduced qKZ equations. It is possible to describe the solution and write the normalisation in a determinant form (Di Francesco, B. Nienhuis; A G).

$$Z_L^p = \det_{1\le i,j\le L} E_{3j-2i}(z_1, ..),$$

$$Z_L^o = \det_{1\le i,j\le L} (E_{3j-2i}(z_1, .., z_1^{-1}, ..) - E_{3j-2i+4L}(z_1, .., z_1^{-1}, ..))$$

Recently this allowed to calculate a simple correlation function in the open dTL(n=1) model (G. Fehér & B. Nienhuis).

In the IK model at this point it is possible to calculate the domain wall partition function



$$Z_N = \sum_{\varepsilon \in \text{states}} \prod_{1 \le i,j \le N} w_{i,j}^{(\varepsilon)}.$$

The solution is written in a determinant form

$$Z_N(\zeta_1, .., \zeta_N, z_1, .., z_N) = \det_{1 \le i,j \le N-1} \Delta_{3j-i,N}(\zeta_1, .., \zeta_N, z_1, .., z_N),$$

where $\Delta_{i,N}$ is written in terms of elementary symmetric polynomials

$$\Delta_{2N-i,N} = 2 \sum_{\substack{1 \le n_1, n_2 \le N \\ n_1 + n_2 = i}}^{N} \cos(\theta(n_2 - n_1)) E_{N-n_1}(\zeta_1, .., \zeta_N) E_{N-n_2}(z_1, .., z_N),$$

We start with the algebra $\mathcal{U} = U_q(A_2^{(2)})_0$ and its representations. $\mathcal{U}$ is defined by the Drinfeld generators $x_r^{\pm}$, $h_{\pm m}$ and K, K^{-1} $(r \in \mathbb{Z},\ m \in \mathbb{Z}_+)$ and relations

$$
\begin{aligned}
&KK^{-1} = K^{-1}K = 1, \quad Kh_m = h_m K, \quad h_m h_l = h_l h_m,\\
&Kx_r^{\pm}K^{-1} = q^{\pm 1}x_r^{\pm},\\
&[x_r^+, x_s^-] = \frac{\psi_{r+s}^+ - \psi_{r+s}^-}{q - q^{-1}},\\
&[h_r, x_s^{\pm}] = \pm\frac{[r]}{r}(q^r + q^{-r} + (-1)^{r+1})x_{r+s}^{\pm},\\
&x_{r+2}^{\pm}x_s^{\pm} + (q^{\mp 1} - q^{\pm 2})x_{r+1}^{\pm}x_{s+1}^{\pm} - q^{\pm 1}x_r^{\pm}x_{s+2}^{\pm}\\
&= q^{\pm 1}x_s^{\pm}x_{r+2}^{\pm} + (q^{\pm 2} - q^{\mp 1})x_{s+1}^{\pm}x_{r+1}^{\pm} - q^{\pm 1}x_{s+2}^{\pm}x_r^{\pm},\\
&\text{[some more cubic relations]},
\end{aligned}
$$

where h_m and $\psi_m^{\pm}$ are related by

$$
\Psi^{\pm}(u) = \sum_{k=0}^{\infty}\psi_{\pm k}^{\pm}u^{\pm k} = K^{\pm 1}\exp\left(\pm(q - q^{-1})\sum_{l=1}^{\infty}h_{\pm l}u^l\right).
$$

- Any irreducible finite dimensional $\mathcal{U}$-module V is presented as $\mathcal{U}v = V$ with v

$$x_r^+ v = 0, \qquad \Psi^{\pm}(z)v = \Phi(z)v, \qquad \Phi(z) = q^{\deg P} \frac{P(q^{-1}z)}{P(qz)},$$

where $P(z) \in \mathbb{C}[z],\ P(0) = 1$ is a Drinfeld polynomial. The correspondence between V and $P(z)$ is bijective (Chary & Pressley).

- Any irreducible finite dimensional $\mathcal{U}$-module V is presented as $\mathcal{U}v = V$ with v

$$x_r^+ v = 0, \qquad \Psi^{\pm}(z)v = \Phi(z)v, \qquad \Phi(z) = q^{\deg P}\frac{P(q^{-1}z)}{P(qz)},$$

  where $P(z) \in \mathbb{C}[z],\ P(0) = 1$ is a Drinfeld polynomial. The correspondence between V and $P(z)$ is bijective (Chary & Pressley).

- We study the Kirillov–Reshetikhin (KR) modules which are defined by

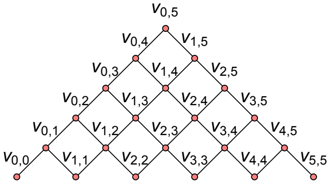
$$P(u) = (1-u)(1-q^2u)..(1-q^{2k-2}u).$$

  The corresponding vector space $V^{(k)}$ is of dimension $(k+1)(k+2)/2$

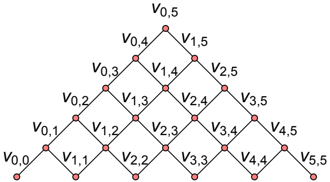
$$V^{(k)} = \bigoplus_{0\le n_1\le n_2\le k} v_{n_1,n_2}.$$

  We need to find the action of the elements of the algebra $\mathcal{U}$ on the space $V^{(k)}$.

- After solving certain commutation relations of the elements of $\mathcal{U}$ in $V^{(k)}$



- After solving certain commutation relations of the elements of $\mathcal{U}$ in $V^{(k)}$



- From here we see that there are two modes for each $x_r^{\pm}$, we can write

$$
\begin{aligned}
K &= q^k {\kappa_1}^{-1} {\kappa_2}^{-1}, \\
x_r^+ &= (-1)^r q^{2r} a_1 \kappa_1^{-2r} + q^r a_2 \kappa_2^{-2r}, \\
x_r^- &= (-1)^r a_1^\dagger \kappa_1^{-2r} + q^{-r} a_2^\dagger \kappa_2^{-2r},
\end{aligned}
$$

where $\{a_1, a_2, a_1^\dagger, a_2^\dagger, \kappa_1, \kappa_2\} \in \mathcal{A}$ is a new algebra which acts on $V^{(k)}$ as

$$
\kappa_1 v_{i,j} = q^i v_{i,j}, \quad \kappa_2 v_{i,j} = q^j v_{i,j}, \quad a_1 v_{i,j} = v_{i-1,j}, \quad a_2^\dagger v_{i,j} = v_{i,j+1}
$$

$$
a_2 v_{i,j} = \frac{q^{3+k+i-j}\left(q^{2j+1}+1\right)\left(q^{2j}-q^{2i}\right)\left(q^{-2}-q^{2j-2k}\right)}{(q-1)^2(q+1)\left(q^{2i+1}+q^{2j}\right)\left(q^{2i}+q^{2j+1}\right)} v_{i,j-1}
$$

$$
a_1^\dagger v_{i,j} = \frac{q^{-i+j+k+2}\left(q^{2(i+1)}-1\right)\left(q^{2j}-q^{2i}\right)\left(q^{2i-2k+1}+q^{-2}\right)}{(q-1)^2(q+1)\left(q^{2i+1}+q^{2j}\right)\left(q^{2i}+q^{2j+1}\right)} v_{i+1,j}
$$

- We will use KR modules to construct important integrability objects: $R^{(k)}$-matrices

$$R^{(k)} = (\rho_{V^{(1)}} \otimes \rho_{V^{(k)}})\,\mathcal{R},$$

where $\mathcal{R} \in \mathcal{U}\hat{\otimes}\mathcal{U}$ is the universal $\mathcal{R}$-matrix. $\mathcal{R}$ is defined by

$$\Delta'(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1}, \quad \forall x \in \mathcal{U},$$
$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{1,3}\mathcal{R}_{2,3}, \quad (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{1,3}\mathcal{R}_{1,2},$$

where Δ is the coproduct and Δ' is the opposite coproduct of $\mathcal{U}$.

- We will use KR modules to construct important integrability objects: $R^{(k)}$-matrices

$$R^{(k)} = (\rho_{V^{(1)}} \otimes \rho_{V^{(k)}})\,\mathcal{R},$$

where $\mathcal{R} \in \mathcal{U}\hat{\otimes}\mathcal{U}$ is the universal $\mathcal{R}$-matrix. $\mathcal{R}$ is defined by

$$\Delta'(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1}, \quad \forall x \in \mathcal{U},$$
$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{1,3}\mathcal{R}_{2,3}, \quad (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{1,3}\mathcal{R}_{1,2},$$

where Δ is the coproduct and Δ' is the opposite coproduct of $\mathcal{U}$.

- An explicit form of $\mathcal{R}$ is given by the Khoroshkin–Tolstoy (KT) formula

$$\mathcal{R} = \prod_{\alpha \in D^+} \exp_{q_\alpha}\left((q - q^{-1})e_\alpha \otimes f_\alpha\right)\mathcal{K},$$

where D^+ is an ordered set of positive roots of $\mathcal{U}$, $\mathcal{K}$ is some simple element of the tensor product of two Cartan subalgebras, e_α and f_α are the Drinfeld–Jimbo root vectors (linear or quadratic functions of $x_r^\pm$) and

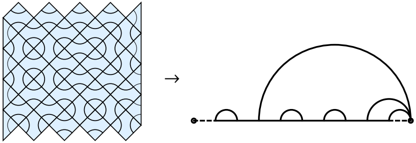
$$\exp_q(x) = 1 + x + \frac{x^2}{(2)_q!} + \frac{x^3}{(3)_q!} + .., \qquad (n)_q = \frac{q^n - 1}{q - 1}.$$

Using the KT formula and KR modules we find $R^{(k)}$ which we write as a matrix in the first tensor component $V^{(1)} = \mathbb{C}^3$

$$R^{(k)}(\zeta) = \begin{pmatrix} \mathcal{A}_1 & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C}_1 & \mathcal{A}_2 & \mathcal{B}_3 \\ \mathcal{C}_2 & \mathcal{C}_3 & \mathcal{A}_3 \end{pmatrix},$$

$$\begin{aligned}
\mathcal{A}_1 &= \lambda_1^{(1)}, \\
\mathcal{A}_2 &= \lambda_2^{(1)} + \lambda_2^{(2)} a_1^\dagger a_2 + \lambda_2^{(3)} a_2^\dagger a_1, \\
\mathcal{A}_3 &= \lambda_3^{(1)} + \lambda_3^{(2)} a_1^\dagger a_2 + \lambda_3^{(3)} a_2^\dagger a_1 + \lambda_3^{(4)} a_1^{\dagger\,2} a_2^2 + \lambda_3^{(5)} a_2^{\dagger\,2} a_1^2, \\
\mathcal{B}_1 &= \nu_1^{(1)} a_1^\dagger + \nu_1^{(2)} a_2^\dagger, \\
\mathcal{B}_2 &= \nu_2^{(1)} a_1^{\dagger\,2} + \nu_2^{(2)} a_2^{\dagger\,2} + \nu_2^{(3)} a_1^\dagger a_2^\dagger, \\
\mathcal{B}_3 &= \nu_3^{(1)} a_1^\dagger + \nu_3^{(2)} a_2^\dagger + \nu_3^{(3)} a_1^{\dagger\,2} a_2 + \nu_3^{(4)} a_2^{\dagger\,2} a_1, \\
\mathcal{C}_1 &= \mu_1^{(1)} a_1 + \mu_1^{(2)} a_2, \\
\mathcal{C}_2 &= \mu_2^{(1)} a_1^2 + \mu_2^{(2)} a_2^2 + \mu_2^{(3)} a_1 a_2, \\
\mathcal{C}_3 &= \mu_3^{(1)} a_1 + \mu_3^{(2)} a_2 + \mu_3^{(3)} a_2^\dagger a_1^2 + \mu_3^{(4)} a_1^\dagger a_2^2.
\end{aligned}$$

Link pattern basis for open boundary conditions. Erase all closed loops and the paths connecting two boundary points



States for $L = 3$ are

The IK R-matrix

$$\begin{pmatrix} x_1(\zeta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2(\zeta) & 0 & x_5(\zeta) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3(\zeta) & 0 & x_6(\zeta) & 0 & x_7(\zeta) & 0 & 0 \\ 0 & y_5(\zeta) & 0 & x_2(\zeta) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_6(\zeta) & 0 & x_4(\zeta) & 0 & x_6(\zeta) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_2(\zeta) & 0 & x_5(\zeta) & 0 \\ 0 & 0 & y_7(\zeta) & 0 & y_6(\zeta) & 0 & x_3(\zeta) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_5(\zeta) & 0 & x_2(\zeta) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1(\zeta) \end{pmatrix}$$

$$x_1(\zeta) = \left(\zeta q^2 - 1\right)\left(\zeta q^3 + 1\right), \quad x_2(\zeta) = (\zeta - 1)\, q\left(\zeta q^3 + 1\right),$$

$$x_3(\zeta) = (\zeta - 1)\, q^2\,(\zeta q + 1)\,,$$

$$x_4(\zeta) = -\zeta + \zeta q^5 + \zeta\,(\zeta - 1)\, q^4 - \zeta q^3 + \zeta q^2 + (\zeta - 1)\, q$$

$$x_5(\zeta) = \sqrt{\zeta}\left(q^2 - 1\right)\left(\zeta q^3 + 1\right), \quad x_6(\zeta) = \sqrt{\zeta}\,(\zeta - 1)\,(-\sqrt{q})\left(q^2 - 1\right),$$

$$x_7(\zeta) = \zeta\left(q^2 - 1\right)\left(\zeta q^3 + (\zeta - 1)\, q + 1\right), \quad y_5(\zeta) = \sqrt{\zeta}\left(q^2 - 1\right)\left(\zeta q^3 + 1\right),$$

$$y_6(\zeta) = \sqrt{\zeta}\,(\zeta - 1)\, q^{5/2}\left(q^2 - 1\right), \quad y_7(\zeta) = \left(q^2 - 1\right)\left(\zeta q^3 - (\zeta - 1)\, q^2 + 1\right).$$

Hamiltonian of the IK model in terms of Gell-Mann matrices reads

$$H = \sum_{j=1}^{N} H_{j,j+1}.$$

(write this in a nicer way)

$$\begin{aligned}
H_{j,k} &= (q^{1/2} + q^{-1/2})(q^2 + q^{-2})(\lambda_1 \otimes \lambda_1 + \lambda_2 \otimes \lambda_2) \\
&+ i(q^{1/2} + q^{-1/2})(q^2 - q^{-2})(-\lambda_1 \otimes \lambda_2 + \lambda_2 \otimes \lambda_1) \\
&+ 2(q^{1/2} + q^{-1/2})\lambda_3 \otimes \lambda_3 \\
&+ (q^{3/2} + q^{-3/2})(q + q^{-1})(\lambda_4 \otimes \lambda_4 + \lambda_5 \otimes \lambda_5 + \lambda_6 \otimes \lambda_6 + \lambda_7 \otimes \lambda_7) \\
&+ i(q^{3/2} + q^{-3/2})(q - q^{-1})(\lambda_4 \otimes \lambda_5 - \lambda_5 \otimes \lambda_4 + \lambda_6 \otimes \lambda_7 - \lambda_7 \otimes \lambda_6) \\
&+ (q - q^{-1})^2(\lambda_4 \otimes \lambda_6 + \lambda_6 \otimes \lambda_4 - \lambda_5 \otimes \lambda_7 - \lambda_7 \otimes \lambda_5) \\
&+ i(q^2 - q^{-2})(-\lambda_4 \otimes \lambda_7 + \lambda_7 \otimes \lambda_4 - \lambda_5 \otimes \lambda_6 + \lambda_6 \otimes \lambda_5) \\
&+ \frac{2}{3}(-(q^{1/2} + q^{-1/2}) + 2(q^{3/2} + q^{-5/2}) + 2(q^{5/2} + q^{-5/2}))\lambda_8 \otimes \lambda_8 \\
&+ 3^{-3/2}(-(q^{1/2} + q^{-1/2}) + 2(q^{3/2} + q^{-3/2}) - (q^{5/2} + q^{-5/2}))(\lambda_8 \otimes \mathrm{id} + \mathrm{id} \otimes \lambda_8).
\end{aligned}$$

- For calculations of correlation functions a crucial quantity is the scalar product

$$S_N(\mu_1, .., \mu_N; \zeta_1, .., \zeta_N) = \langle \bar{\Psi}(\mu_1, .., \mu_N) | \Psi(\zeta_1, .., \zeta_N) \rangle.$$

- For calculations of correlation functions a crucial quantity is the scalar product

$$S_N(\mu_1, .., \mu_N; \zeta_1, .., \zeta_N) = \langle \bar{\Psi}(\mu_1, .., \mu_N) | \Psi(\zeta_1, .., \zeta_N) \rangle.$$

- As a fist step we propose a closed formula for the eigenstates Ψ. Introduce an algebra $\mathcal{F}$ with elements $\{f_1, .., f_N\}$ which satisfy

$$[f_i, f_j] = 0, \quad f_i^2 = 0, \quad {}_N\langle \tilde{0} | \prod_{i=1}^{n} f_i | \tilde{0} \rangle_N = \delta_{n,N}$$

- For calculations of correlation functions a crucial quantity is the scalar product

$$S_N(\mu_1, .., \mu_N; \zeta_1, .., \zeta_N) = \langle \bar{\Psi}(\mu_1, .., \mu_N) | \Psi(\zeta_1, .., \zeta_N) \rangle.$$

- As a fist step we propose a closed formula for the eigenstates Ψ. Introduce an algebra $\mathcal{F}$ with elements $\{f_1, .., f_N\}$ which satisfy

$$[f_i, f_j] = 0, \quad f_i^2 = 0, \quad {}_N\langle \tilde{0} | \prod_{i=1}^{n} f_i | \tilde{0} \rangle_N = \delta_{n,N}$$

- For example, one can choose the representation space $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ for

$$f_i = I_1 \otimes \cdots \otimes \sigma_i^+ \otimes \cdots \otimes I_N, \qquad |\tilde{0}\rangle_N = \otimes_{j=1}^N v_+, \qquad {}_N\langle \tilde{0}| = \otimes_{j=1}^N \, v_-,$$

where I is the identity in $\mathbb{C}^2$ and $v_+ = (1, 0), \; v_- = (0, 1)$.

- Using the algebra $\mathcal{F}$ we define

$$\beta(\zeta_i|\zeta_{i+1},..,\zeta_N) = \mathbb{I} + B_1(\zeta_i)f_i + B_2(\zeta_i) \times \sum_{j>i} c_{i,j} f_j f_i,$$

then the Tarasov's recurrence is solved by

$$|\Psi_N(\zeta_1,..,\zeta_N)\rangle =_N \langle\tilde{0}| \prod_{i=1}^{N} \beta(\zeta_i|\zeta_{i+1},..,\zeta_N)|\tilde{0}\rangle_N \otimes |0\rangle.$$

- Using the algebra $\mathcal{F}$ we define

$$\beta(\zeta_i|\zeta_{i+1},..,\zeta_N) = \mathbb{I} + B_1(\zeta_i)f_i + B_2(\zeta_i) \times \sum_{j>i} c_{i,j} f_j f_i,$$

then the Tarasov's recurrence is solved by

$$|\Psi_N(\zeta_1,..,\zeta_N)\rangle =_N \langle\tilde{0}| \prod_{i=1}^{N} \beta(\zeta_i|\zeta_{i+1},..,\zeta_N)|\tilde{0}\rangle_N \otimes |0\rangle.$$

- Introducing certain normal ordering we find an equivalent representation

$$|\Psi_N\rangle = \oint \frac{\mathrm{d}x}{x^{N+1}} : \prod_{i=1}^{N} e^{x\mathcal{B}(\zeta_i;x)} : |0\rangle,$$

$$\mathcal{B}(\zeta_i;x) = xB_2(\zeta_i) \sum_{i<j\leq N} c_{i,j} + B_1(\zeta_i).$$

- Using the algebra $\mathcal{F}$ we define

$$\beta(\zeta_i|\zeta_{i+1},..,\zeta_N) = \mathbb{I} + B_1(\zeta_i)f_i + B_2(\zeta_i) \times \sum_{j>i} c_{i,j} f_j f_i,$$

then the Tarasov's recurrence is solved by

$$|\Psi_N(\zeta_1,..,\zeta_N)\rangle =_N \langle\tilde{0}| \prod_{i=1}^{N} \beta(\zeta_i|\zeta_{i+1},..,\zeta_N)|\tilde{0}\rangle_N \otimes |0\rangle.$$

- Introducing certain normal ordering we find an equivalent representation

$$|\Psi_N\rangle = \oint \frac{dx}{x^{N+1}} : \prod_{i=1}^{N} e^{x\mathcal{B}(\zeta_i;x)} : |0\rangle,$$

$$\mathcal{B}(\zeta_i;x) = xB_2(\zeta_i) \sum_{i<j\leq N} c_{i,j} + B_1(\zeta_i).$$

- We can write a similar representation for the dual state. S_N becomes

$$S_N = \oint \frac{dx dy}{x^{N+1}y^{N+1}} \langle 0| : \prod_{i=1}^{N} e^{x\mathcal{C}(\mu_i;x)} : : \prod_{i=1}^{N} e^{y\mathcal{B}(\zeta_i;y)} : |0\rangle.$$