Towards $N = 2$ Minimal Models

Tianshu Liu

School of Mathematics and Statistics,

The University of Melbourne

tianshul@student.unimelb.edu.au

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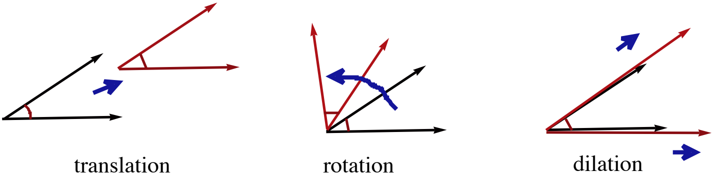
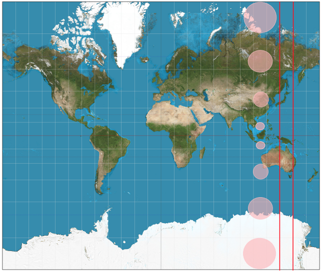
Conformal Transformations

conformal: angle-perserving



Mercator projection



Conformal Field Theory (CFT)

Applications: string theory, critical statistical model, pure mathematics...

Virasoro Algebra ($\mathfrak{Vir}$)

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

where $c \in \mathbb{C}$ is the 'central charge'.

Virasoro Minimal Models – $\mathcal{M}(p,q)$:

- $p, q \geq 2$, coprime
- a finite number of **irreducible** Virasoro representations
- **unitary** $|p-q| = 1$, e.g., Ising model $\mathcal{M}(3,4)$
- **non-unitary** $|p-q| \neq 1$, e.g., Yang-Lee singularity $\mathcal{M}(2,5)$

conformal dimension for $\mathcal{M}(p, q)$

$$h_{r,s} = \frac{(qr - ps)^2 - (p - q)^2}{4pq},$$

where $1 \leq r \leq p - 1$ and $1 \leq s \leq q - 1$

Kac table for $\mathcal{M}(4, 5)$

s \ r	1	2	3	4
1	0	1/10	3/5	3/2
2	7/16	3/80	3/80	7/16
3	3/2	3/5	1/10	0

$N = 2$ Super Conformal Field Theory

Motivation: Calabi-Yau manifold, Ashkin-Teller model, etc.

Operators: L_n, G_r^+, G_r^- and J_n

$N = 2$ super conformal algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

$$[L_m, J_n] = -nJ_{m+n}$$

$$[L_n, G_r^\pm] = (\frac{n}{2} - r)G_{n+r}^\pm$$

$$[J_n, G_r^\pm] = \pm G_{n+r}^\pm$$

$$\{G_r^+, G_s^-\} = 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}$$

$$[J_m, J_n] = \frac{mc}{3}\delta_{m+n,0}$$

$$\{G_r^+, G_s^+\} = \{G_r^-, G_s^-\} = 0$$

Representations

$$\textbf{fermionic fields: } G^{\pm}(ze^{2\pi i}) = \begin{cases} G^{\pm}(z) & \rightarrow \text{ NS } \rightarrow G^{\pm}_{r\in\mathbb{Z}+\frac{1}{2}} \\ -G^{\pm}(z) & \rightarrow \text{ R } \rightarrow G^{\pm}_{r\in\mathbb{Z}} \end{cases}$$

$N = 2$ minimal models $\mathcal{M}^{N=2}(k)$:

- **unitary**: $k \in \mathbb{Z}_{\geq 0}$

  J_0-E.V. $$j^m_\lambda = \frac{m}{k+2} + \frac{1}{4}\left(1 - (-1)^{\lambda+m}\right)$$

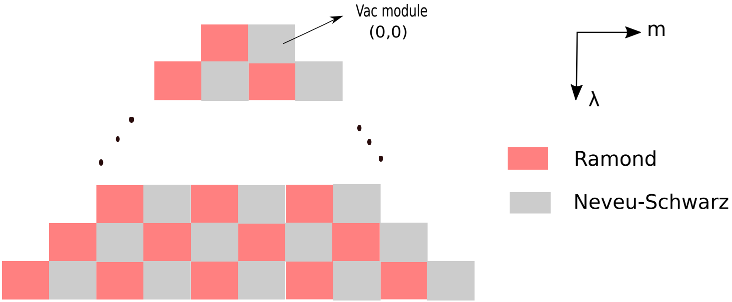
  L_0-E.V. $$h^m_\lambda = \frac{\lambda(\lambda+2) - m^2}{4(k+2)} + \frac{1}{16}\left(1 - (-1)^{\lambda+m}\right)$$

  where $\lambda = 0, 1, ..., k$ and $m = -\lambda - 1, -\lambda, ..., \lambda$
- **non-unitary**: $k + 2 = \frac{u}{v}$, where $u, v \geq 2$, coprime

$N = 2$ Unitary Minimal Model Results

For $\mathcal{M}^{N=2}(k)$, number of irreps: $(k+1)(k+2)$, in λ and m

$N = 2$ **Kac table**



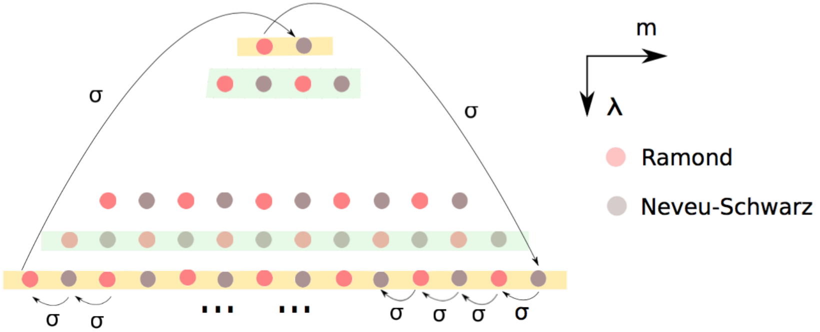
Automorphisms of the $N = 2$ Algebra

Automorphisms cut down the complexity through symmetries.

spectral flow (σ):

$$\sigma(L_n) = L_n - \tfrac{1}{2}J_n + \tfrac{c}{24}\delta_{n,0},$$

$$\sigma(J_n) = J_n - \tfrac{c}{6}\delta_{n,0}, \quad \sigma(G_s^{\pm}) = G^{\mp}_{s\mp\frac{1}{2}}$$



$(k+1)(k+2)$ times cuts down to $\sim$ k/2 times

Coset Construction

$$\mathcal{M}^{N=2}(k) = \frac{\hat{\mathfrak{sl}}(2)_k \otimes \hat{\mathfrak{bc}}}{\hat{\mathfrak{gl}}(1)}$$

Embeddings: $\partial\varphi(z) = H(z) + 2 : bc : (z)$,

$$J(z) = \cdots,\ G^{\pm}(z) = \cdots,\ T(z) = \cdots$$

Theorem [Creutzig-Kanade-Linshaw-Ridout]

An irreducible $\hat{\mathfrak{sl}}(2)_k \otimes \hat{\mathfrak{bc}}$ representation decomposes into $\hat{\mathfrak{gl}}(1) \otimes \mathcal{M}^{N=2}(k)$ representations as

$$\bigoplus_{\mu} \mathcal{F}_\mu \otimes C_\mu$$

where C_μ are the **irreducible** representations of $\mathcal{M}^{N=2}(k)$.

Unitary $N = 2$ minimal models

Results

- Complete irreducibility $\rightarrow$ branching rules
- Character and supercharacter of the $N = 2$
- Modular transformations and Verlinde Formula
- Fusion rules with parities

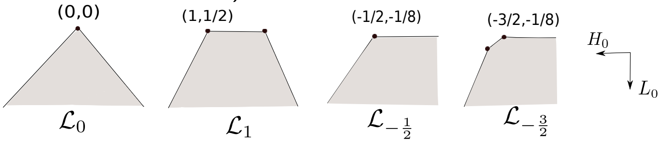
Non-unitary $N = 2$ minimal models

$\mathcal{M}^{N=2}(k)$, where $k + 2 = \frac{u}{v}$. Example, $k = -\frac{1}{2}$, $c = -1$

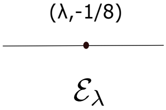
$$\mathcal{M}^{N=2}(-\frac{1}{2}) = \frac{\hat{\mathfrak{sl}}(2)_{-\frac{1}{2}} \otimes \hat{\mathfrak{bc}}}{\hat{\mathfrak{gl}}(1)}$$

$\hat{\mathfrak{sl}}(2)_{-\frac{1}{2}}$ **modules**:

$\mathcal{L}$-type (irred. h.w. modules)



$\mathcal{E}$-type (relaxed modules, $\lambda \in (-1, 1]$, irred. for $\lambda \neq \pm\frac{1}{2}$)



Non-unitary $N = 2$ minimal models

representation decomposition $\rightarrow$ infinite $N = 2$ representations

$$\hat{\mathfrak{sl}}(2)_{-\frac{1}{2}} \otimes \hat{\mathfrak{bc}} \qquad \mathcal{M}^{N=2}(-\tfrac{1}{2}) \otimes \hat{\mathfrak{gl}}(1)$$

$$\mathcal{L}_{\lambda}^{(-\frac{1}{2})} \longrightarrow C_{\lambda}^{m}, \text{ where } \lambda \in \{0, 1, -\tfrac{1}{2}, -\tfrac{3}{2}\} \text{ and } m \in \mathbb{Z}$$

$$\mathcal{E}_{\lambda}^{(-\frac{1}{2})} \longrightarrow E_{\lambda}^{n}, \text{ where } \lambda \in (-1, 1] \text{ and } n \in \mathbb{Z}$$

Both are irreducible highest-weight modules.

Non-unitary $N = 2$ Results

- **character formulae**:
  E-type: from the component characters of its coset
  C-type: from the residue method, in terms of the E-type
- **Branching rules**, e.g.,

$$\mathcal{L}_0^{(-\frac{1}{2})} \otimes NS = \bigoplus_{\mu \in 2\mathbb{Z}} \mathcal{F}_\mu \otimes C_0^\mu$$

- Verlinde Formula, **fusion rules**, e.g.,

$$C_1^0 \times C_1^0 = C_0^0,$$
$$C_1^0 \times E_\lambda = E_\lambda$$
$$E_\lambda \times E_\mu = E_{\lambda+\mu-\frac{3}{2}} \oplus E_{\lambda+\mu+\frac{3}{2}}$$

- Repeated with $\mathcal{M}^{N=2}(-\frac{4}{3})$

Future Direction

- Justify the $E_\lambda \times E_\mu$ fusion rules involving reducible rep. $E_{\pm\frac{1}{2}}$
- Mock modular forms of the characters
- Generalisation to any k, with $k+2=\frac{u}{v}$, fusion rules
- Generalisation to more complicated coset — Kazama-Suzuki models
- The string theory compactified on Calabi-Yau manifold