The inhomogeneous asymmetric exclusion proces

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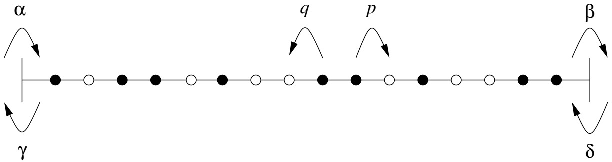
Outline

- Revision of ASEP
- Inhomogeneous version
- Matrix product Koornwinder polynomials

Partially Asymmetric Exclusion Process (PASEP)



Paradigm of classical non-equilibrium statistical mechanics: boundary driven hard-core diffusion on a line.

- One dimensional diffusion
- Non-equilibrium system: nonzero current
- Boundary driven phase transitions in one dimension

Configurations:

$$|\boldsymbol{\tau}\rangle = |\tau_1, \ldots, \tau_L\rangle = \bigotimes_{i=1}^{L} |\tau_i\rangle \qquad \tau_i = 0, 1.$$

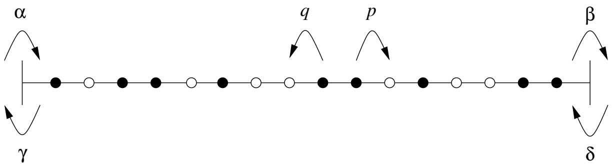
State at time t:

$$|P(t)\rangle = \sum_{\boldsymbol{\tau}} P_{\boldsymbol{\tau}}(t)|\boldsymbol{\tau}\rangle,$$

Time evolution is governed by the master equation:

$$\frac{d}{dt}|P(t)\rangle = M|P(t)\rangle.$$

Pair interactions



The PASEP transition matrix M consists of two-body interactions only:

$$M = \sum_{k=1}^{L-1} m_k + b_1 + + b_L$$

$$m_k = \mathbb{I}^{k-1} \otimes m \otimes \mathbb{I}^{L-k-1},$$

$$m = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & p & 0 \\ 0 & q & -p & 0 \\ 0 & 0 & 0 & 0. \end{pmatrix},$$

Matrix Product

We are interested in the stationary state

$$M|\Psi\rangle = 0$$

Assume

$$|\Psi\rangle = \sum_{\tau} \Psi_\tau |\tau\rangle = \langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes L} |V\rangle.$$

$$\Psi_{1001011} = \langle W|DEEDEDD|V\rangle$$

with matrices D and E, and row vector $\langle W|$ and column vector $|V\rangle$.

Matrix Product

$$M = \sum_{k=1}^{L-1} m_k + b_1 + + b_L$$

Stationarity of $\langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes L} |V\rangle$ is implied by

$$m \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} = \begin{pmatrix} E \\ D \end{pmatrix} \otimes I - I \otimes \begin{pmatrix} E \\ D \end{pmatrix},$$

Leads to q-oscillator algebra

$$pDE - qED = D + E.$$

Inhomogeneous ASEP

We will write $p = 1$, $q = t^2$ and

$$m = t\,e = t\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & -t & t^{-1} & 0\\ 0 & t & -t^{-1} & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

Define R-matrix:

$$\check{R}_i(x_i, x_{i+1}) = 1 + \frac{x_i - x_{i+1}}{tx_i - t^{-1}x_{i+1}} e_i.$$

This matrix satisfies the Yang-Baxter equation:

$$\check{R}_i(x_i, x_{i+1})\check{R}_{i+1}(x_{i+1}, x_{i+2})\check{R}_i(x_i, x_{i+1}) = \check{R}_{i+1}(x_{i+1}, x_{i+2})\check{R}_i(x_i, x_{i+1})\check{R}_{i+1}(x_{i+1}, x_{i+2})$$

With our definition of the R-matrix we can now define the transfer matrix as,

$$T(w|x_1,\ldots,x_L) = \mathrm{tr}_a R_{a,1}(w,x_1)\ldots R_{a,L}(w,x_L),$$

where $R_{a,i} = \mathbb{P}_{a,i}\check{R}_{i,a}$ and $\mathbb{P}_{a,i}$ is a permutation operator.

Stationary state is now given by

$$T(w|x_1,\ldots,x_L)|\Psi(x_1,\ldots,x_L)\rangle = \Lambda|\Psi(x_1,\ldots,x_L)\rangle,$$

with Λ symmetric.

Yang-Baxter equation implies

$$\check{R}_i(x_i,x_{i+1})T(\ldots,x_i,x_{i+1},\ldots) = T(\ldots,x_{i+1},x_i,\ldots)\check{R}_i(x_i,x_{i+1}).$$

and qKZ equation for Ψ:

$$\check{R}_i(x_i,x_{i+1})|\Psi(x_1,\ldots,x_L)\rangle = s_i \circ |\Psi(x_1,\ldots,x_L)\rangle.$$

Example

$$\check{R}_i(x_i, x_{i+1})|\Psi(x_1, \ldots, x_L)\rangle = s_i \circ |\Psi(x_1, \ldots, x_L)\rangle.$$

is written in components as

$$\Psi_{\ldots\circ\circ\ldots} = t^{-1} T_i \circ \Psi_{\ldots\circ\circ\ldots}$$
$$\Psi_{\ldots\bullet\bullet\ldots} = t^{-1} T_i \circ \Psi_{\ldots\bullet\bullet\ldots}$$
$$\Psi_{\ldots\bullet\circ\ldots} = t T_i \circ \Psi_{\ldots\circ\bullet\ldots}.$$

with

$$T_i = t - (t x_i - t^{-1} x_{i+1}) \cdot \partial_i$$

and divided difference operator

$$\partial_i = \frac{1 - s_i}{x_i - x_{i+1}}.$$

Hecke algebra

These operators satisfy the Hecke algebra:

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$$
$$(T_i - t)(T_i + t^{-1}) = 0,$$
$$[T_i, T_j] = 0 \text{ if } |i-j| > 1.$$

Solution for periodic boundaries

$$|\Psi(x_1, x_2)\rangle = \begin{pmatrix} \Psi_{\bullet\bullet}(x_1, x_2) \\ \Psi_{\circ\bullet}(x_1, x_2) \\ \Psi_{\bullet\circ}(x_1, x_2) \\ \Psi_{\circ\circ}(x_1, x_2) \end{pmatrix} = \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1 x_2 \end{pmatrix}$$

Any state with n particles can be represented by the following polynomial,

$$\Psi_{j_1,\ldots,j_n} = \prod_{k=1}^{n} x_{j_k},$$

where $1 \leq j_1 < \ldots < j_n \leq L$ denote the sites that are occupied by particles.

The normalisations of the periodic IASEP are elementary symmetric polynomials.

Open boundaries

We define the boundary scattering matrices as follows,

$$K_0(x|a,b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{x^2-1}{(x-a)(x-b)} \begin{pmatrix} -t_0^2 & 1 \\ t_0^2 & -1 \end{pmatrix},$$

$$\check{K}_L(x|c,d) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1-x^2}{(1-cx)(1-dx)} \begin{pmatrix} -1 & t_L^2 \\ 1 & -t_L^2 \end{pmatrix},$$

where $t_0 = -ab$ and $t_L = -cd$ and a, b, c, d are functions of $\alpha, \beta, \gamma, \delta$.

The qKZ equations now read

$$\begin{aligned} \check{R}_i(x_i, x_{i+1})|\Psi\rangle &= s_i \circ |\Psi\rangle, \\ K_0(x_1)|\Psi\rangle &= s_0 \circ |\Psi\rangle, \\ \check{K}_L(x_L)|\Psi\rangle &= s_L \circ |\Psi\rangle. \end{aligned}$$

$$s_0 : \; f(x_1, \ldots) \mapsto f(x_1^{-1}, \ldots), \qquad s_L : \; f(\ldots, x_L) \mapsto f(\ldots, x_L^{-1}).$$

The qKZ equations can now be written as

$$
\begin{aligned}
t\Psi_{\dots\circ\circ\dots} &= T_i \circ \Psi_{\dots\circ\circ\dots} \\
t\Psi_{\dots\bullet\bullet\dots} &= T_i \circ \Psi_{\dots\bullet\bullet\dots} \\
t^{-1}\Psi_{\dots\bullet\circ\dots} &= T_i \circ \Psi_{\dots\circ\bullet\dots}
\end{aligned}
$$

$$
\begin{aligned}
t_0^{-1}\Psi_{\circ\dots} &= T_0 \circ \Psi_{\bullet\dots}, \\
t_L^{-1}\Psi_{\dots\bullet} &= T_L \circ \Psi_{\dots\circ}.
\end{aligned}
$$

with

$$
\begin{aligned}
T_0 &:= t_0 + t_0^{-1}\frac{(x_1-a)(x_1-b)}{x_1^2-1}(s_0-1), \\
T_L &:= t_L + t_L^{-1}\frac{(1-cx_L)(1-dx_L)}{1-x_L^2}(s_L-1).
\end{aligned}
$$

$$t^{-1}\Psi_{\dots\bullet\circ\dots} = T_i \circ \Psi_{\dots\circ\bullet\dots}$$
$$t_0^{-1}\Psi_{\circ\dots} = T_0 \circ \Psi_{\bullet\dots},$$
$$t_L^{-1}\Psi_{\dots\bullet} = T_L \circ \Psi_{\dots\circ}.$$

The state $\psi_{\bullet\dots\bullet}$ is highest weight:

$$T_i \circ \psi_{\bullet\dots\bullet} = t^{1/2}\psi_{\bullet\dots\bullet},$$
$$T_L T_{L-1} \dots T_1 T_0 \circ \psi_{\bullet\dots\bullet} = (t_L t_0)^{-1} t^{1-L}\psi_{\bullet\dots\bullet}.$$

Action of Hecke algebra gives all other components in terms of hghest weight state.

Explicit solutions

$$\psi_{\bullet} = \frac{1}{x_1} + A$$

$$\psi_{\bullet\bullet} = \left(\frac{1}{x_1 x_2}\right) + A\left(\frac{1}{x_1} + \frac{1}{x_2}\right) + B$$

$$\psi_{\bullet\bullet\bullet} = \left(\frac{1}{x_1 x_2 x_3}\right) + A\left(\frac{1}{x_1 x_2} + \frac{1}{x_1 x_3} + \frac{1}{x_2 x_3}\right) + B\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right) + C$$

where the coefficients A and B are as follows,

$$A = \frac{d - c(ad + bd - 1)}{abcd - 1}$$

$$B = \frac{d^2 - cd(ad + bd - 1)\left(1 + t^2\right)}{(abcd - 1)\left(abcdt^2 - 1\right)} + \frac{c^2\left(1 + a^2 d^2 t^2 + b^2 d^2 t^2 - bd\left(1 + t^2\right) + ad(bd - 1)\left(1 + t^2\right)\right)}{(abcd - 1)\left(abcdt^2 - 1\right)}$$

$$\Psi_{\tau_1,\ldots,\tau_n}(x_1,\ldots,x_n) = \langle W|X_1\cdots X_N|V\rangle$$

Solution turns out to be simple:

$$X_i = \begin{cases} E - 1 + x_i & \text{if } \tau_i = \circ \\ D - 1 + x_i^{-1} & \text{if } \tau_i = \bullet. \end{cases}$$

$$\begin{aligned} DE - tED &= (1-t)(D+E) \\ \langle W|(E + abD) &= \langle W|(1-a)(1-b) \\ (cdE + D)|V\rangle &= (1-c)(1-d)|V\rangle. \end{aligned}$$

Normalisation

The normalisation is defined by

$$Z(x_1,\dots,x_L) = \sum_{\tau} \Psi_{\tau} = \langle W| \prod_{i=1}^{L} (D + E - 2 + x_i + x_i^{-1}) |V\rangle.$$

$Z(x_1,\dots,x_L)$ is the symmetric Koornwinder polynomial of shape 1^L.

Matrix product akin to vertex operator approach, but more practical

The evaluation $Z(1,\dots,1)$ is equal to the homogeneous ASEP normalisation.

Evaluations $x_i = 1$ are poorly understood in the theory of symmetric polynomials, deserve more attention!