Accurate lower bounds on two-dimensional constraint capacities

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INTRODUCTION

In the old days before cloud computing and flash drives...

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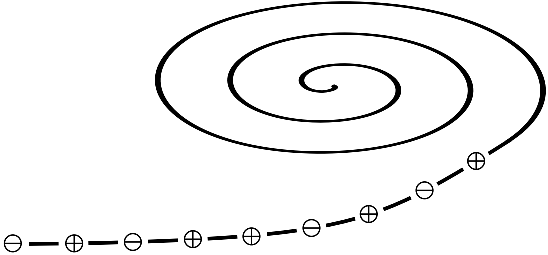
In the old days before cloud computing and flash drives... there was tape.

(Taken from `laamc.wordpress.com`)

Magnetic tape was (and is) a reliable means of data storage and backup.

TAPE

Let's consider a very simple model of tape: magnetic spins which are aligned sequentially along the tape.

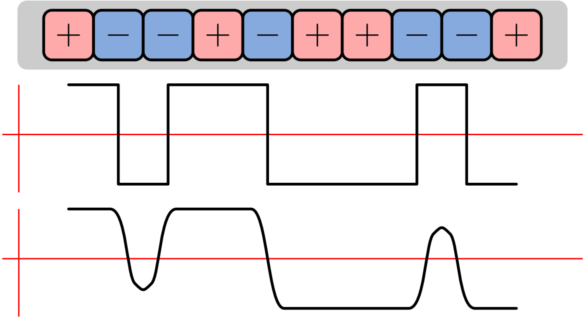


These spins can be 'pointed' in two 'directions' — $\oplus$ or $\ominus$.

The data stored in the tape is encoded in the orientations of the spins.

Constraints

Due to hardware limitations, not all sequences of spins are available.



Here the tape head doesn't like having two adjacent $\ominus$ spins.

Capacity

How much data can this tape store?

CAPACITY

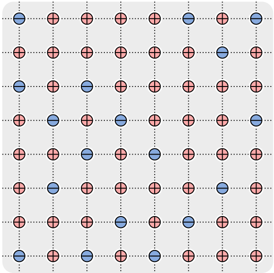
How much data can this tape store? In other words, how many valid configurations are there?

It is simple to show that the number of valid configurations on N spins is the $(N+1)$th Fibonacci number, which grows asymptotically as $\left(\frac{1+\sqrt{5}}{2}\right)^N$. So the tape stores $\log_2\left(\frac{1+\sqrt{5}}{2}\right) \times N$ bits.

We call $\log_2\left(\frac{1+\sqrt{5}}{2}\right)$ the *capacity* of the constraint, while $\frac{1+\sqrt{5}}{2} = 1.618\ldots$ is the *growth constant* (denoted κ).

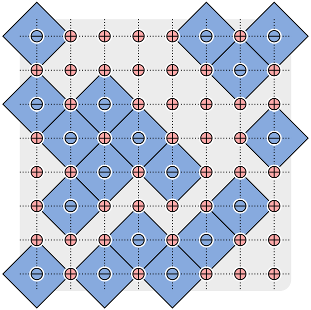
Two-dimensional constraints

Now imagine this constraint in two dimensions, where ⊖ spins cannot be horizontally or vertically adjacent.



TWO-DIMENSIONAL CONSTRAINTS

Now imagine this constraint in two dimensions, where ⊖ spins cannot be horizontally or vertically adjacent.



This is the hard squares lattice gas model.

The hard squares model

What is the capacity of *this* constraint?

We can find the capacity by calculating the more general partition function:

$$Z_N = \sum_{\text{configurations on } N \text{ spins}} z^{\#\text{ of } \ominus \text{ spins in the configuration}}.$$

The growth constant follows from the evaluation of the partition function at $z = 1$:

$$\kappa = \lim_{N \to \infty} Z_N^{1/N}\big|_{z=1}.$$

BOUNDING THE CAPACITY

The previous best lower bound on hard squares was by Friedland, Lundow and Markström (2010):

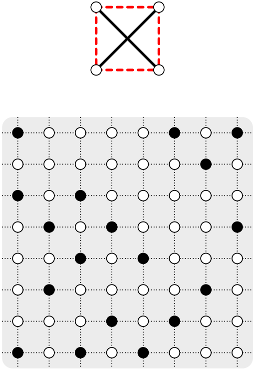
$$\kappa \geq 1.503\,048\,082\,475\,332\,264\,3220\ldots.$$

We set out to improve this bound and bounds on the capacity of many other constraints!

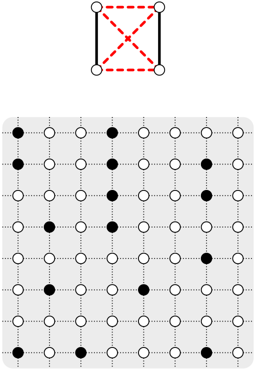
Louidor and Marcus (2010) used Rayleigh quotients of transfer matrices on several of the models we analyse later.

Exclusion models

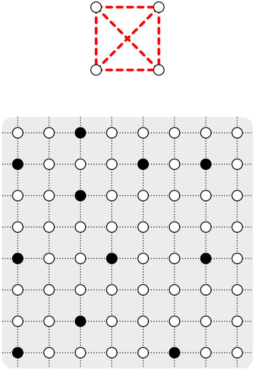
The *exclusion* models forbid certain local arrangements of $\ominus$ spins.



Hard squares



Read-write
isolated memory



Non-attacking kings

PARITY MODELS

The *parity* models read the spins across or down the lines of the lattice.

Look at the number of $\oplus$ spins between every pair of $\ominus$ spins. If this number is:

- always even, the configuration satisfies the *even* constraint;
- always odd, the configuration satisfies the *odd* constraint.

Louidor and Marcus (2010) showed that the odd model has growth constant exactly $\sqrt{2}$.

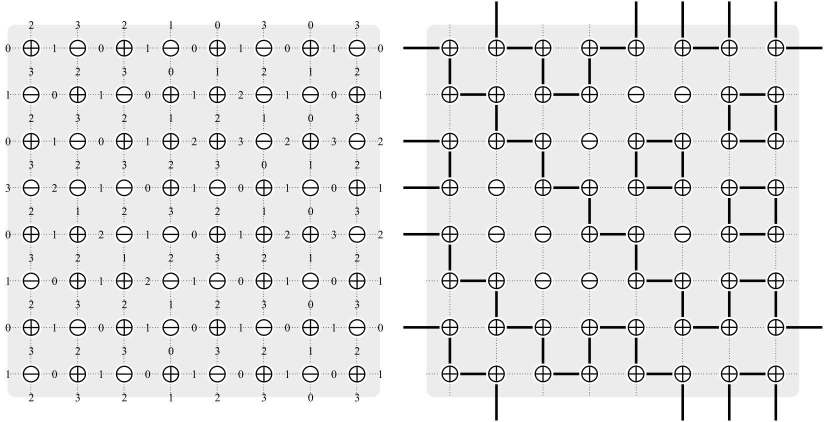
CHARGE MODELS

The *charge*(q) models also reads the spins across or down the lines of the lattice.

If the partial sum of the $\oplus$s and $\ominus$s always stays between 0 and q, then the configuration satisfies the charge(q) constraint.

- Charge(1) is trivial;
- Louidor and Marcus showed that charge(2) has growth constant $2^{2^{-d}}$ in d dimensions;
- We look at charge(3).

CHARGE(3) AND EVEN MODELS



Colouring models

The *q-colouring* models label spins from 0 to $q-1$ (for integer q). No two adjacent spins can have the same label (*colour*).

- 2-colouring is trivial (there are exactly 2 possibilities);
- 3-colouring is known to have an exact growth constant of $(4/3)^{3/2} = 1.5396\ldots$, proved by Lieb (1967);
- We look at 4- and 5-colouring.

Corner transfer matrix formalism

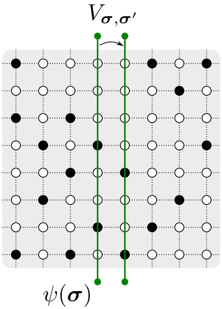
We use corner transfer matrix formalism to calculate lower bounds on the growth constant. This formalism was created by Baxter (1968).

It can be (and was) used to numerically estimate physical properties of a variety of models (including hard squares).

Baxter also used it to generate series coefficients of these properties.

Transfer matrices

Earlier bound-finding techniques used column transfer matrices.



The growth constant follows immediately from the largest eigenvalue of the transfer matrix:

$$\kappa = \lim_{m\to\infty} \Lambda_m^{1/m}.$$

Reformulating the eigenvalue problem

Finding the largest eigenvalue of the transfer matrix can be reformulated as a maximisation problem:

$$\Lambda = \max_{\psi} \frac{\psi^T V \psi}{\psi^T \psi},$$

and so if we choose *any* ψ we obtain a lower bound for Λ.

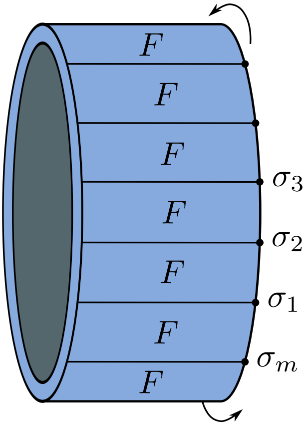
Here we use Baxter's Ansatz and choose a ψ which satisfies

$$\psi(\sigma_1, \ldots, \sigma_m) = \text{Tr}\ [F(\sigma_1, \sigma_2) F(\sigma_2, \sigma_3) \ldots F(\sigma_m, \sigma_1)]$$

for a set of $n \times n$ matrices $F(a, b)$.

Baxter's Ansatz

F represents the transfer matrix of half a row, while ψ is a half-plane partition function.



$$= \psi(\sigma_1, \ldots, \sigma_m)$$

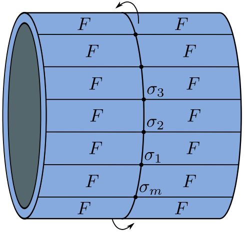
Further reformulation

We can re-write the expression for Λ by means of

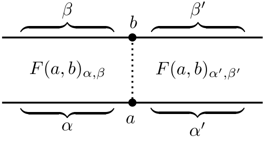
$$\begin{aligned} \psi^T \psi &= \operatorname{Tr} R^m, \\ \psi^T V \psi &= \operatorname{Tr} S^m, \end{aligned}$$

where R and S are full-row transfer matrices of different sizes.

FURTHER REFORMULATION

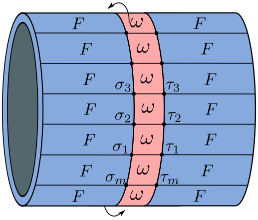


$= \mathrm{Tr} R^m$

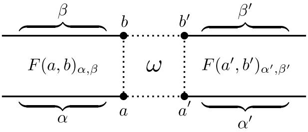


$R_{(\alpha|a|\alpha'),(\beta|b|\beta')}$

FURTHER REFORMULATION



$= \mathrm{Tr} S^m$



$S_{(\alpha|a,a'|\alpha'),(\beta|b,b'|\beta')}$

CTM EQUATIONS

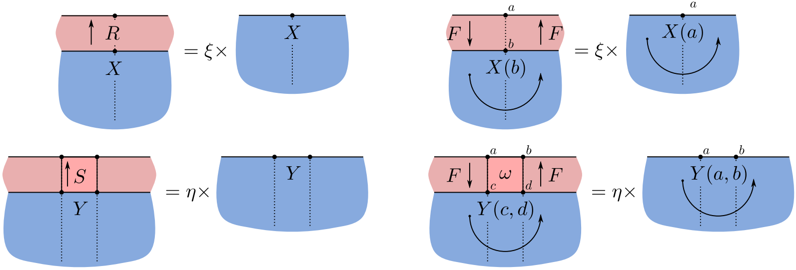
In the thermodynamic limit, the right-hand sides depend only on the dominant eigenvalues of R and S, denoted ξ and η.

The eigenvalue equations can be reformulated in terms of F and the eigenvectors contained in the matrix families X and Y:

$$\begin{aligned} \sum_b F(a,b)X(b)F(b,a) &= \xi X(a), \\ \sum_{a,b,c,d} \omega \begin{pmatrix} a & b \\ c & d \end{pmatrix} F(a,c)Y(c,d)F(d,b) &= \eta Y(a,b). \end{aligned}$$

(ω is 1 if the face is valid and 0 otherwise.)

CTM EQUATIONS



Obtaining a lower bound

To obtain a lower bound, we first find an appropriate set of Fs. Then we use the power method implicitly, iterating the CTM equations to calculate ξ and η.

The lower bound is

$$\kappa \geq \lim_{m\to\infty} \left(\frac{\psi^T V \psi}{\psi^T \psi} \right)^{1/m} = \lim_{m\to\infty} \left(\frac{\text{Tr } S^m}{\text{Tr } R^m} \right)^{1/m} = \frac{\eta}{\xi}.$$

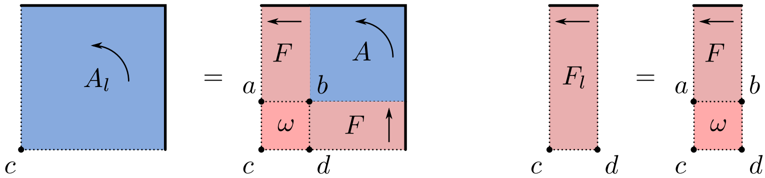
The corner transfer matrix renormalisation group method

In order to obtain good Fs, we use a heuristic known as the corner transfer matrix renormalisation group method, devised by Nishino and Okunishi (1996).

This method makes the finite-size Fs as close to the infinite-size transfer matrices as possible.

It does this by repeatedly expanding and reducing them in a way which preserves important characteristics.

THE CORNER TRANSFER MATRIX RENORMALISATION GROUP METHOD



$$
\begin{aligned}
A(a) &= P^T(a)A_l(a)P(a) \\
F(a,b) &= P^T(a)F_l(a,b)P(b)
\end{aligned}
$$

Convergence

The method is (uniquely) claimed to be sub-exponential; more precisely that computing n digits requires $\exp(\alpha\sqrt{n})$ computing time.

This assertion can be tested empirically; the error in our estimates is given by the eigenvalue spectrum of $X^2(a)$.

Our results support the asymptotic form.

Convergence

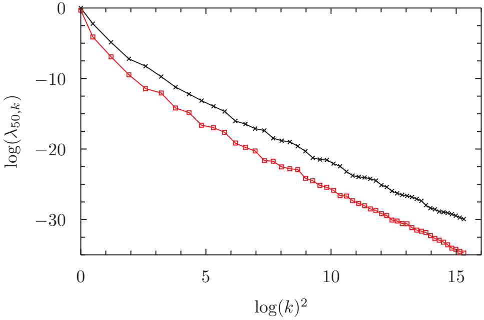


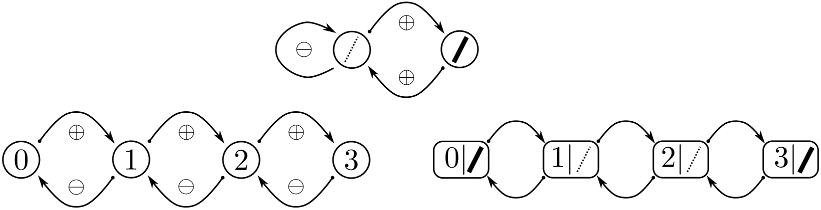
TABLE OF RESULTS

Model	Matrix size	Lower bound on (and estimate of) κ
Hard squares	256	1.503 048 082 475 332 264 322 066 329 475 553 689 385 781 038 610 305 062 028 101 735 933 850 396 923 440 380 463 299 47 (65)
RWIM	128	1.448 957 371 775 608 489 872 231 406 108 136 686 434 371 (7)
NAK	256	1.342 643 951 124 601 297 851 730 161 875 740 395 719 438 196 938 393 943 434 885 455 0 (1)
Even	128	1.357 587 502 184 123 (5)
Charge(3)	74	(1.357 587 50)
4-Colouring	96	2.336 056 641 041 133 656 814 01 (4)
5-Colouring	64	3.250 404 923 167 119 143 819 73 (6)

THE CHARGE(3) AND EVEN MODELS

Why do the charge(3) and even models have similar growth constants? Are they actually equal?

We can prove they are equal on a one-dimensional lattice by a 2-to-1 mapping of states.



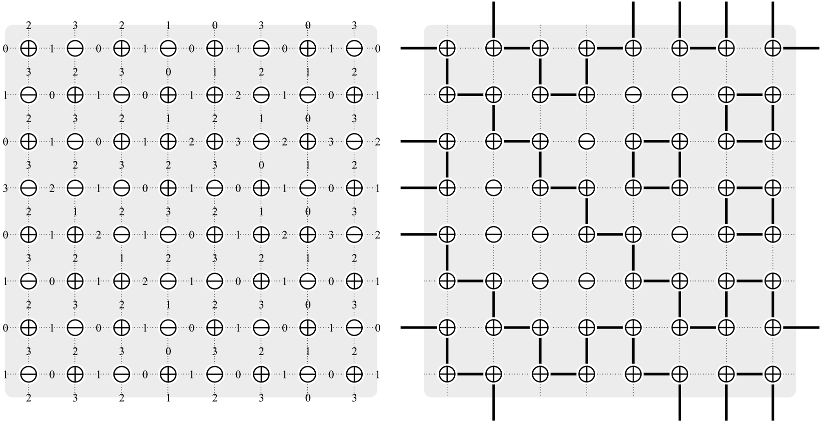
The charge(3) and even models

This mapping can be used in two dimensions to show that $cap(charge(3)) \geq cap(even)$.

If the parity of the N and W edges are identical and determined in every face then the mapping is a bijection.

However, it is possible to have faces where this doesn't happen, so this doesn't prove that they are equal.

THE CHARGE(3) AND EVEN MODELS



We have non-rigorous arguments for the capacities being equal.

Conclusion

- We use statistical mechanical techniques to examine a information theory problem.
- Using corner transfer matrix formalism, we derive very accurate lower bounds on the capacity of several models.
- Our lower bounds improve on all known proved lower bounds.
- They also match our estimates for all but the last one or two digits.
- This leads to a remarkable conjecture that the charge(3) and even constraints have the same capacity.

Future work

- We want to *prove* that the charge(3) and even constraints have the same capacity.
- We can apply this methodology to more models — any model which can be expressed using a local face weight.
- We're looking for upper bounds of comparable quality. Linear algebra theorems which bound the error in an eigenvalue estimate may help.
- A recent extension to the CTMRG allows it to calculate series, including the generating function for self-avoiding polygons.

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