Matrix elements of the Lie superalgebra $gl(m|n)$

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A comment about Representation Theory...

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From

$$[E_{ab}, E_{cd}] = \delta_{bc} E_{ad} - (-1)^{((a)+(b))((c)+(d))} \delta_{ad} E_{cb},$$

to

$$(N_r^p)^2 = \prod_{k \neq r = 1}^{m} \left(\frac{(\alpha_{k,p} - \alpha_{r,p} - (-1)^{(r)})(\alpha_{k,p} - \alpha_{r,p} + 1 - (-1)^{(r)})}{(\alpha_{k,p-1} - \alpha_{r,p} - (-1)^{(r)})(\alpha_{k,p+1} - \alpha_{r,p} + 1 - (-1)^{(r)})} \right)$$
$$\times \left(\frac{\prod_{k=m+1}^{p-1} (\alpha_{k,p-1} - \alpha_{r,p} - 1 - (-1)^{(r)})}{\prod_{k \neq r = m+1}^{p} (\alpha_{r,p} - \alpha_{k,p} + (-1)^{(r)})} \right)$$
$$\times \left(\frac{\prod_{k=m+1}^{p+1} (\alpha_{k,p+1} - \alpha_{r,p} - (-1)^{(r)})}{\prod_{k \neq r = m+1}^{p} (\alpha_{r,p} - \alpha_{k,p} + (-1)^{(r)})} \right)$$

Scheunert, Nahm & Rittenberg (1976) Complete reducibility of star (generalized Hermitian) representations

Kac (1978) - Classifications

Molev, Palev, Stoilova & Van der Jeugt (1987 - 2011) Matrix element formulae for certain classes of representations

And in parallel with the above...

Bracken, Jarvis, Green & Gould (70s and 80s) - characteristic (polynomial) identities

Gould & R.B.Zhang (1990) - Classification of f.d. unitary representations of $gl(m|n)$ into type 1 and type 2 (which are in fact dual).

Gradings

- The general linear Lie superalgebra $gl(m|n)$ is a $\mathbb{Z}_2$ graded algebra with a graded operation (graded commutator)

$$[X, Y] = XY - (-1)^{(X)(Y)}YX$$

- This grading extends to all objects - morphisms, vectors etc.
- We define a grading (or parity) on indices to be $(a) = 0,\ 1 \leq a \leq m$ and $(a) = 1,\ m+1 \leq a \leq m+n$.
- For an object X we have $(X) = 0, 1$ if X is even/odd respectively.

Define the graded commutator to be

$$[X, Y] = XY - (-1)^{(X)(Y)}YX$$

Lie Superalgebras

The superalgebra $gl(m|n)$ is the complex Lie algebra with basis (generators,operators) $E_{pq}(1 \leq p, q \leq m + n)$ that satisfy the commutation relations

$$[E_{pq}, E_{rs}] = \delta_{qr} E_{ps} - (-1)^{((p)+(q))((r)+(s))} \delta_{ps} E_{rq}$$

where the commutator is given by

$$[X, Y] = XY - (-1)^{(X)(Y)} YX$$

where the parity of a generator is given by

$$(E_{pq}) = (p) + (q)$$

Generators

It is convenient to place the generators in a matrix. For example, the $gl(2|3)$ generators are...

$$\begin{pmatrix} E_{11} & E_{12}| & E_{13} & E_{14} & E_{15} \\ E_{21} & E_{22}| & E_{23} & E_{24} & E_{25} \\ -- & -- & -- & -- & -- \\ E_{31} & E_{32}| & E_{33} & E_{34} & E_{35} \\ E_{41} & E_{42}| & E_{43} & E_{44} & E_{45} \\ E_{51} & E_{52}| & E_{53} & E_{54} & E_{55} \end{pmatrix}$$

Note that we have the subalgebra chain

$$gl(m|n) \supset gl(m|n-1) \supset \ldots \supset gl(m|1) \supset gl(m) \supset \ldots \supset gl(1)$$

The Gelfand-Tsetlin basis for $gl(n)$

A Gelfand-Tsetlin basis of a $gl(n)$ module $V(\Lambda)$ is given by *all possible* triangular arrays $(\lambda_{i,j})_{n\geqslant i\geqslant j\geqslant 1}$ of integers

$$\begin{array}{ccccccccc} \lambda_{n,1} & & \lambda_{n,2} & & \cdots & & \lambda_{n,n-1} & & \lambda_{n,n} \\ & \lambda_{n-1,1} & & \lambda_{n-1,2} & & \cdots & & \lambda_{n-1,n-1} & \\ & & \cdots & & \cdots & & \cdots & & \\ & & & \lambda_{2,1} & & \lambda_{2,2} & & & \\ & & & & \lambda_{1,1} & & & & \end{array}$$

where the top row is fixed $\{\lambda_{n,1}, \lambda_{n,2}, ..., \lambda_{n,n}\} = \Lambda$, each row obeys lexicality, and the lower rows are subject to the betweeness conditions

$$\lambda_{i+1,j} \leqslant \lambda_{i,j} \leqslant x_{i+1,j+1}.$$

The Gelfand-Tsetlin basis for $gl(2|3)$

A Super Gelfand-Tsetlin basis of a $gl(m|n)$ module $V(\Lambda)$ is given by a triangular array $(\lambda_{p,i})$ $1 \leq i \leq p \leq m+n$ of integers

$$\begin{matrix} \lambda_{5,1} & & \lambda_{5,2} & & \lambda_{5,3} & & \lambda_{5,4} & & \lambda_{5,5} \\ & \lambda_{4,1} & & \lambda_{4,2} & & \lambda_{4,3} & & \lambda_{4,4} & \\ & & \lambda_{3,1} & & \lambda_{3,2} & & \lambda_{3,3} & & \\ & & & \lambda_{2,1} & & \lambda_{2,2} & & & \\ & & & & \lambda_{1,1} & & & & \end{matrix}$$

where the top row is fixed $\{\lambda_{m+n,1}, \lambda_{m+n,2}, ..., \lambda_{m+n,m+n}\} = \Lambda$. The even labels have the same branching conditions as the $gl(m)$ case. The odd labels are subject to the super branching conditions

$$\lambda_{p+1,i} - \lambda_{p,i} \in \{0, 1\}.$$

Our goal

To calculate the matrix elements $\langle p|E_{ij}|q\rangle$ for all $gl(m|n)$ generators E_{ij} and all GT basis states $\langle p|$ and $|q\rangle$ we will take these steps

- Construct a matrix composed of the generators
- Notice that this matrix satisfies a polynomial identity.
- This identity allows us to define a set of projection operators.
- We then construct a vector operator ψ using the $gl(m|n)$ generators.
- Each component ψ^i of this vector operator can be projected out into 'shift components' via the use of the projection operators.

The adjoint matrix

Consider a matrix whose elements are generators of $gl(m|n)$.
We will call this the $gl(m|n)$ adjoint matrix:

$$\mathcal{A}_{\alpha}^{\ \beta} = -(-1)^{(\alpha)(\beta)} E_{\beta\alpha}.$$

$\mathcal{A}$ may be regarded as an operator on the tensor product representation $V_{\epsilon_1} \otimes V(\Lambda)$ where V_{ϵ_1} is the vector rep. Note that we have

$$V_{\epsilon_1} \otimes V(\Lambda) = \oplus_{i=1}^{m+n} V(\Lambda + \epsilon_i)$$

Characteristic identities

Theorem (Gould,1987): The adjoint matrix $\mathcal{A}$ satisfies the characteristic identity

$$\prod_{i=1}^{m}(\mathcal{A}-\alpha_i)\prod_{\mu=1}^{n}(\mathcal{A}-\alpha_\mu)=0 \tag{1}$$

when acting on an irreducible $gl(m|n)$ module $V(\Lambda)$ where the even and odd adjoint roots are given by

$$\alpha_i = i-1-\Lambda_i.$$

and

$$\alpha_\mu = \Lambda_\mu + m + 1 - \mu.$$

Projection operators

Projection operators can be constructed using the characteristic identity

$$P[a] = \prod_{k \neq a}^{m+n} \left(\frac{\mathcal{A} - \alpha_k}{\alpha_a - \alpha_k} \right)$$

We can immediately see that they are orthogonal

$$P[a]P[b] = 0 \quad \text{for } a \neq b$$

and in fact

$$P[a]P[b] = \delta_{ab} P[a]$$

also

$$\sum_a P[a] = 1$$

Vector operators

In the superalgebra setting, a general vector operator ψ is defined as a set of operators ψ^r $(1 \leq r \leq m+n)$ that obey

$$[E_{pq}, \psi^r] = (-1)^{(\psi)((p)+(q))} \pi_{\varepsilon_1}(E_{pq})_{sr} \psi^s$$
$$= (-1)^{(\psi)((p)+(q))} \delta^r_q \psi^p$$

Conveniently, the right hand column of the adjoint matrix defined earlier forms a vector operator ψ

$$\psi^p = (-1)^{(p)} E_{p,m+n+1}$$

Shift operators

Each component of ψ can be resolved into shift components

$$\psi^p = \sum_{i=1}^{m} \psi[i]^p + \sum_{\mu=1}^{n} \psi[\mu]^p.$$

where

$$\psi[r]^p = \psi^q P[r]_q^p \quad 1 \leq a \leq m+n \tag{2}$$

The shift components cause the following changes to the representation labels Λ

$$\psi[i] : \Lambda_j \to \Lambda_j + \delta_{ij} \quad (1 \leq i, j \leq m),$$
$$\psi[\mu] : \Lambda_\nu \to \Lambda_\nu + \delta_{\nu\mu} \quad (1 \leq \mu, \nu \leq n).$$

Factorization of the matrix elements

With a little work we notice that

$$\left\langle \begin{pmatrix} \Lambda \\ s \end{pmatrix} \middle| \psi^\dagger[a]^\beta \psi[a]^\alpha \middle| \begin{pmatrix} \Lambda \\ p \end{pmatrix} \right\rangle$$
$$= |\langle \Lambda + \varepsilon_a || \psi || \Lambda \rangle|^2 \left\langle \begin{pmatrix} \Lambda \\ s \end{pmatrix} \middle| P[a]_\beta{}^\alpha \middle| \begin{pmatrix} \Lambda \\ p \end{pmatrix} \right\rangle$$

And with more work (starting from the characteristic identity) we obtain...

The matrix element equation for elementary generators

$$E_{p,p+1}|\mu\rangle = \sum_{r=1}^{p} N_p^r |\mu\rangle_{rp}.$$

$$(N_p^r)^2 = \prod_{k\neq r=1}^{m} \left(\frac{(\alpha_{k,p} - \alpha_{r,p} - [r])(\alpha_{k,p} - \alpha_{r,p} + 1 - [r])}{(\alpha_{k,p-1} - \alpha_{r,p} - [r])(\alpha_{k,p+1} - \alpha_{r,p} + 1 - [r])} \right)$$
$$\times \left(\frac{\prod_{k=m+1}^{p-1}(\alpha_{k,p-1} - \alpha_{r,p} - 1 - [r]) \prod_{k=m+1}^{p+1}(\alpha_{k,p+1} - \alpha_{r,p} - [r])}{\prod_{k\neq r=m+1}^{p}(\alpha_{k,p} - \alpha_{r,p} - 1 - [r])(\alpha_{r,p} - \alpha_{k,p} + [r])} \right)$$

where

$$\alpha_{i,p\pm 1} = i - 1 - \lambda_{i,p\pm 1}, \quad \alpha_{\mu,p\pm 1} = \lambda_{\mu,p\pm 1} + m + 1 - \mu$$

and

$$[r] = (-1)^{(r)} \in \{+1, -1\}$$

The matrix element equation for the non-elementary generators

$$N[u_p u_{p-1} \dots u_l] = \pm \prod_{r=l}^{p} N^{r}_{u_r} \prod_{s=l+1}^{p} \left[-(\bar{\beta}_{u_s} - \bar{\alpha}_{u_{s-1}} + 1)^{-1} (\bar{\beta}_{u_s} - \bar{\alpha}_{u_{s-1}})^{-1} \right]^{1/2} \tag{3}$$

Future work

- Matrix elements of type 2 modules
- Generalizations to the quantized superalgebra $U_q(gl(m|n))$
- Generalizations to arbitary tensor products - the pattern calculus.

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Thank You!