The master T-operator and Baxter Q-operators for quantum integrable systems

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based on:

• V. Kazakov, S. Leurent, Z. Tsuboi, Commun. Math. Phys. 311(2012) 787,

• A. Alexandrov, V. Kazakov, S. Leurent, Z. Tsuboi, A. Zabrodin,
JHEP 1309 (2013) 064,

• A. Alexandrov, S. Leurent, Z. Tsuboi, A. Zabrodin, arXiv:1306.1111 [math-ph],

• Z. Tsuboi, arXiv:1205.1471,

• S. Khoroshkin, Z. Tsuboi, in preparation (2013).

Introduction

The Baxter Q-operators were introduced by Baxter when he solved the 8-vertex model. His method of the Q-operators is recognized as one of the most powerful tools in quantum integrable systems.

Baxter Q-operator: operator whose eigenvalues gives the Baxter Q-functions.

Baxter Q-function: polynomial (or power series) whose roots gives the roots of the Bethe ansatz equations.

Introduction

Our goals are

1. to construct Baxter Q-operators systematically
2. to write the T-operators (transfer matrices) in terms of the Q-operators: Wronskian-like determinant formulas
3. to establish functional relations among them: T-system, TQ-relations, QQ-relations

Introduction

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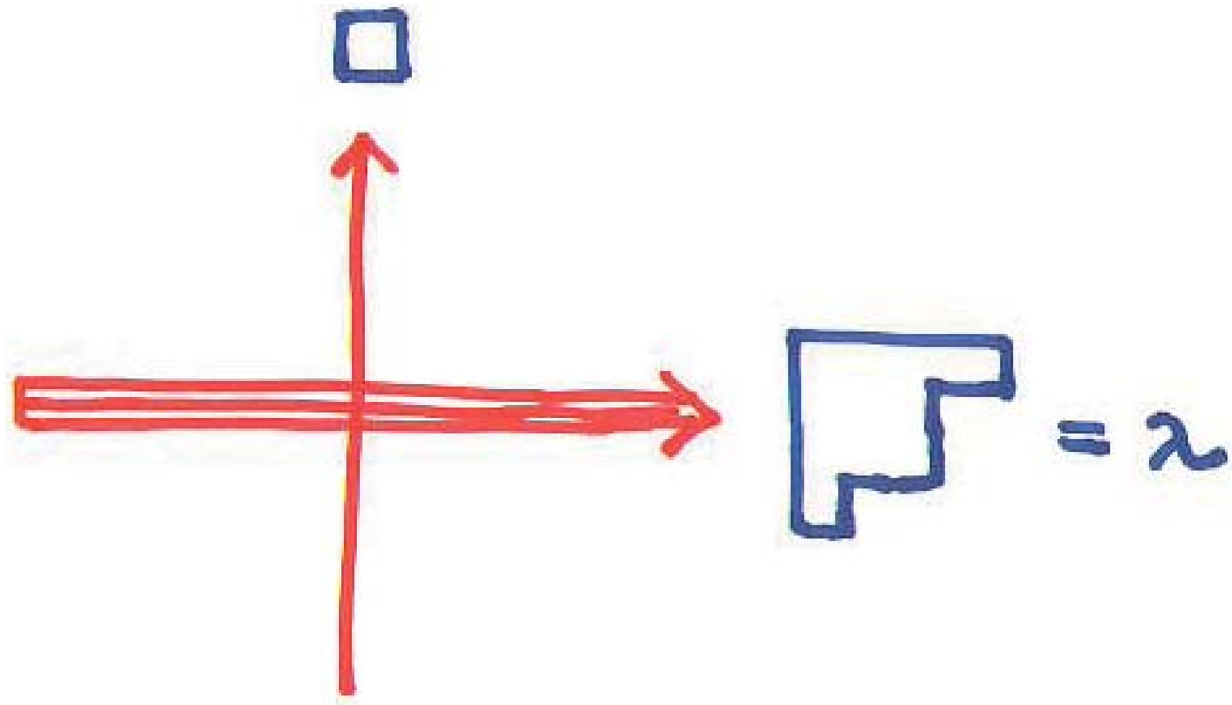
For these purposes, we consider an embedding of the quantum integrable system into the soliton theory. The key object is the master T-operator.

- a kind of a generating function of the transfer matrices
- τ-function in the soliton theory

The R-matrix for the irreducible representation π_λ of $gl(N)$

$$R^{\lambda}(u) = u + \sum_{ij} e_{ij} \otimes \pi_{\lambda}(e_{ji}),$$

$(e_{ij})_{ab} = \delta_{ia}\delta_{jb}$, $e_{ij} \in gl(N)$



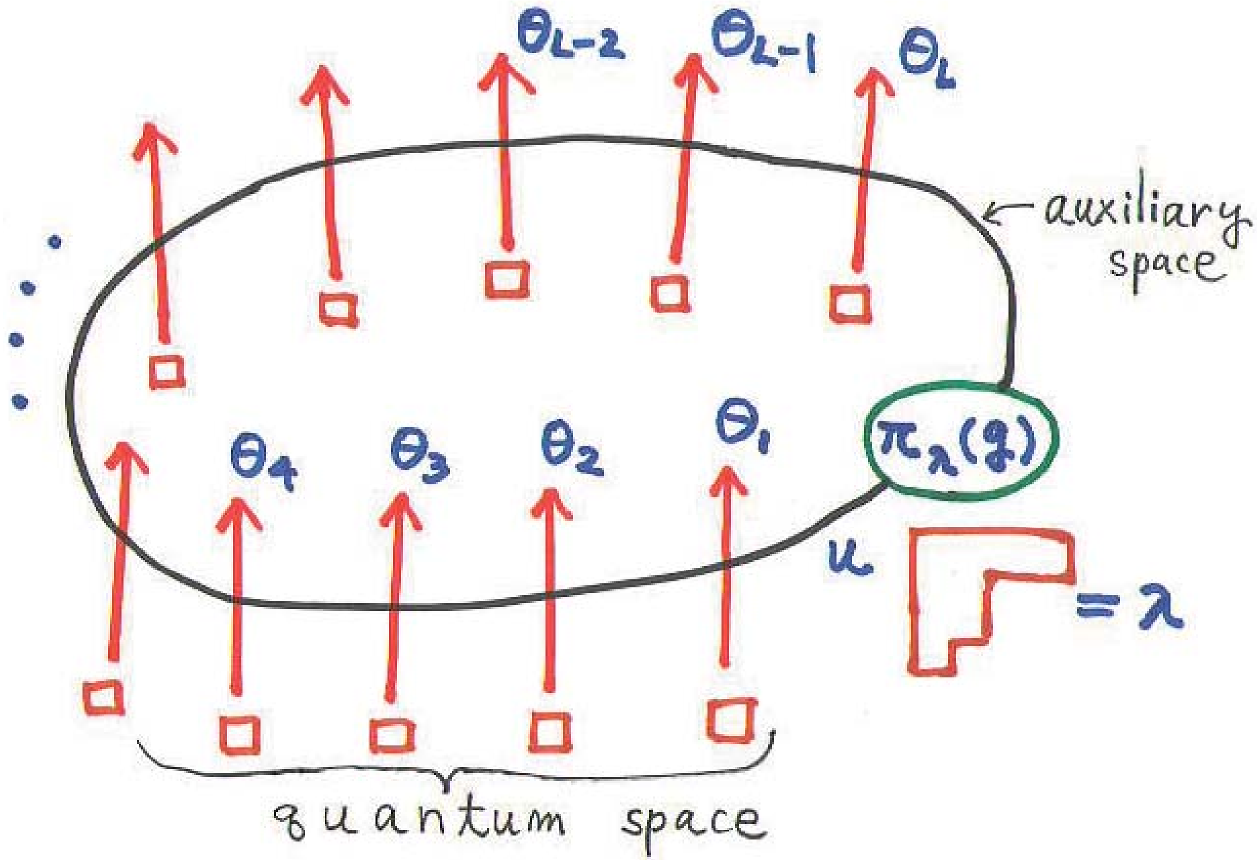
The transfer matrix

$$T^\lambda(u) = \mathrm{Tr}_0\left[R^\lambda_{L0}(u-\theta_L)\cdots R^\lambda_{20}(u-\theta_2)R^\lambda_{10}(u-\theta_1)(1^{\otimes L}\otimes \pi_\lambda(g))\right]$$

$R^\lambda_{j0}(u)$ is a R-matrix (j: quantum space $\pi_{(1)}$, 0: auxiliary space π_λ),

$g \in GL(N)$ is a boundary twist matrix,

$\theta_j \in \mathbb{C}$ is an inhomogeneity on the spectral parameter u.



From the Yang-Baxter equation and $GL(N)$-invariance,

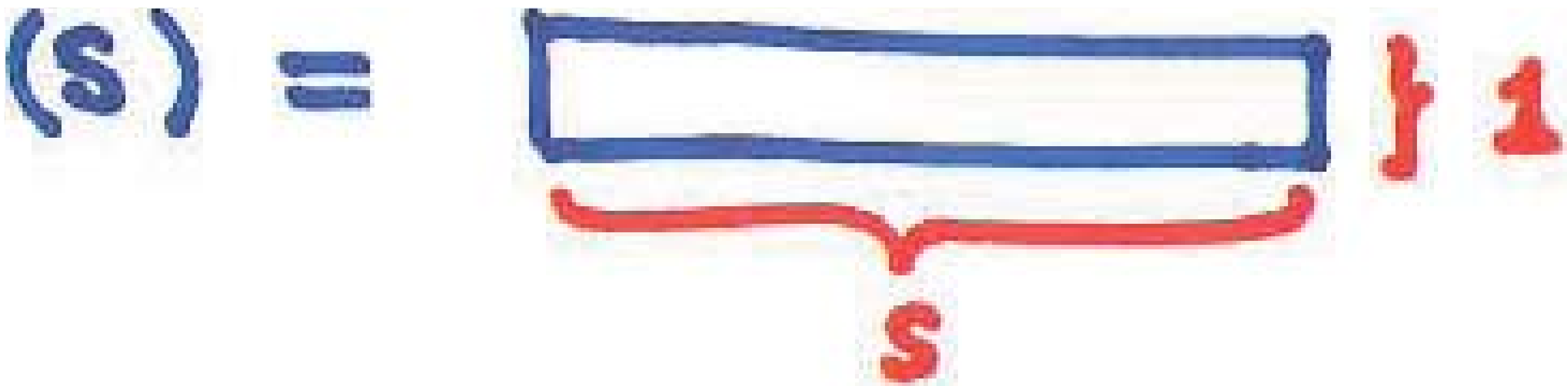
$$[T^{\lambda}(u), T^{\mu}(v)] = 0$$

for a fixed $g \in GL(N)$

Characters

Generating function for the characters of the symmetric tensor representations of $g \in GL(N)$

$$w(z) = \det(1 - gz^{-1})^{-1} = \frac{1}{(1 - x_1 z^{-1})(1 - x_2 z^{-1}) \cdots (1 - x_N z^{-1})} = \sum_{s=0}^{\infty} \chi_{(s)} z^{-s}$$



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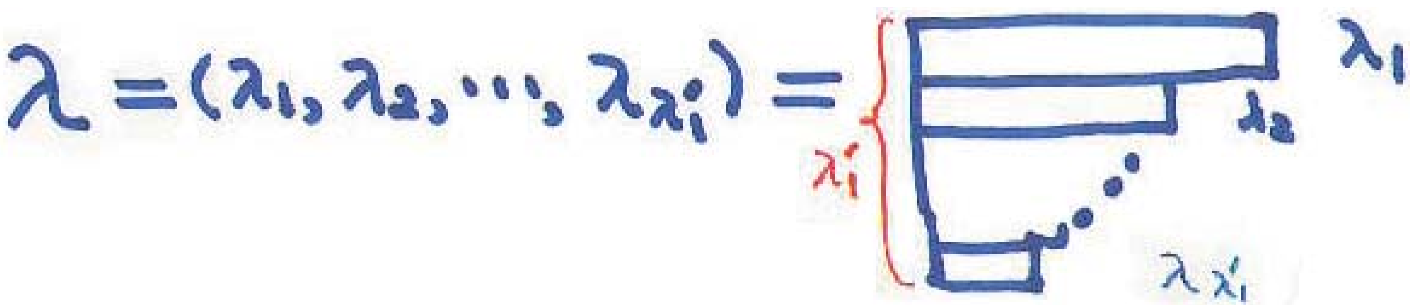
Jacobi-Trudi formula for the Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\lambda'_1})$:

$$\chi_\lambda = \det_{1 \le i,j \le \lambda'_1} \left(\chi_{(\lambda_i - i + j)} \right)$$



Transfer matrices (T-operators)

T-operators are generalization of the characters

Cherednik-Bazhanov-Reshetikhin formula (analogue of Jacobi-Trudi formula):

$$\mathsf{T}^{\lambda}(u) = \det_{1 \le i,j \le \lambda'_1} \left(\mathsf{T}^{(s-i+j)}(u - j + 1) \right)$$

The co-derivative [Kazakov,Vieira '07]

$$\hat{D} \otimes f(g) = \frac{\partial}{\partial \phi} \otimes f(e^{\phi \cdot e} g) \Big|_{\phi=0},$$

$$\text{for} \quad g \in GL(N),$$

$\phi \cdot e \equiv \sum_{\alpha\beta} e_{\alpha\beta} \phi_{\alpha\beta}$,
$\frac{\partial}{\partial \phi} = \sum_{\alpha\beta} e_{\alpha\beta} \frac{\partial}{\partial \phi_{\beta\alpha}}$,
$e_{\alpha\beta} \in gl(N)$, $\phi_{\alpha\beta} \in \mathbb{C}$.

The co-derivative [Kazakov,Vieira '07]

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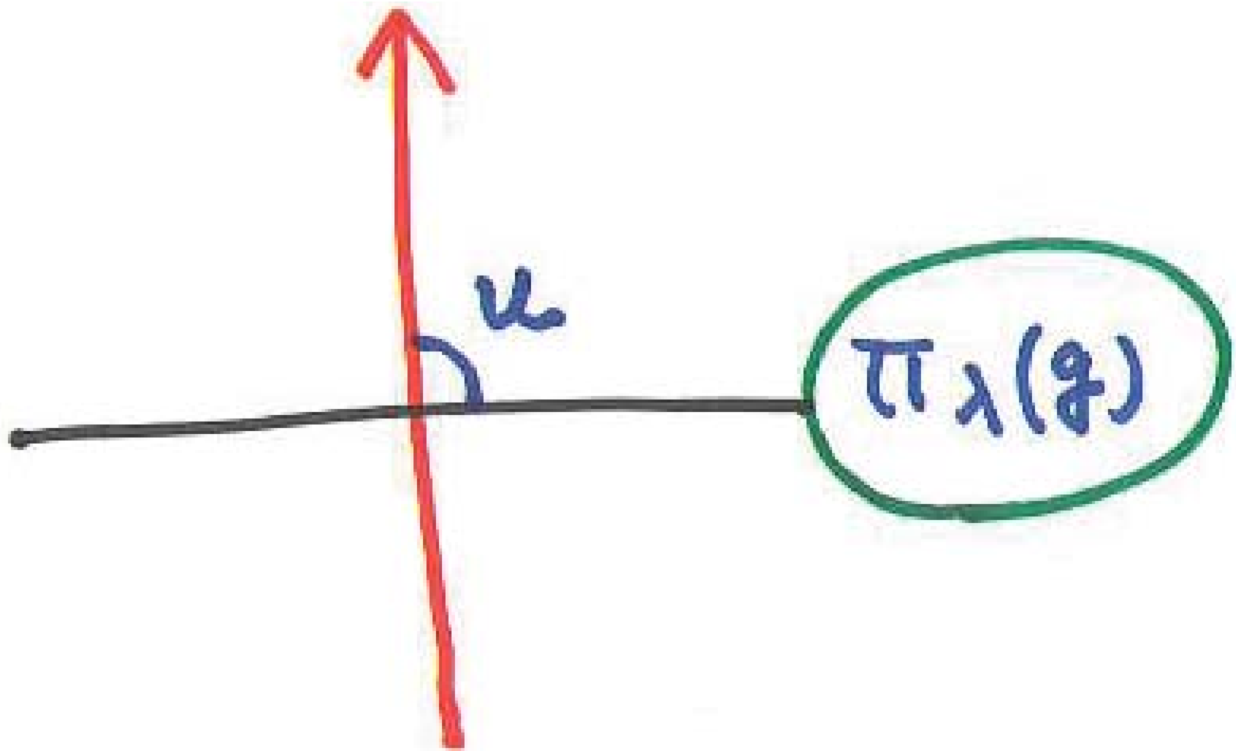
$$\hat{D} \otimes \pi_\lambda(g) = \sum_{ij} (e_{ij} \otimes \pi_\lambda(e_{ji}))\ (1 \otimes \pi_\lambda(g))$$

The R-matrix can be written in terms of the co-derivative

$$(u + \hat{D}) \otimes \pi_\lambda(g) = \left(u + \sum_{ij} e_{ij} \otimes \pi_\lambda(e_{ji}) \right) (1 \otimes \pi_\lambda(g)),$$
$$= R^\lambda(u)(1 \otimes \pi_\lambda(g))$$

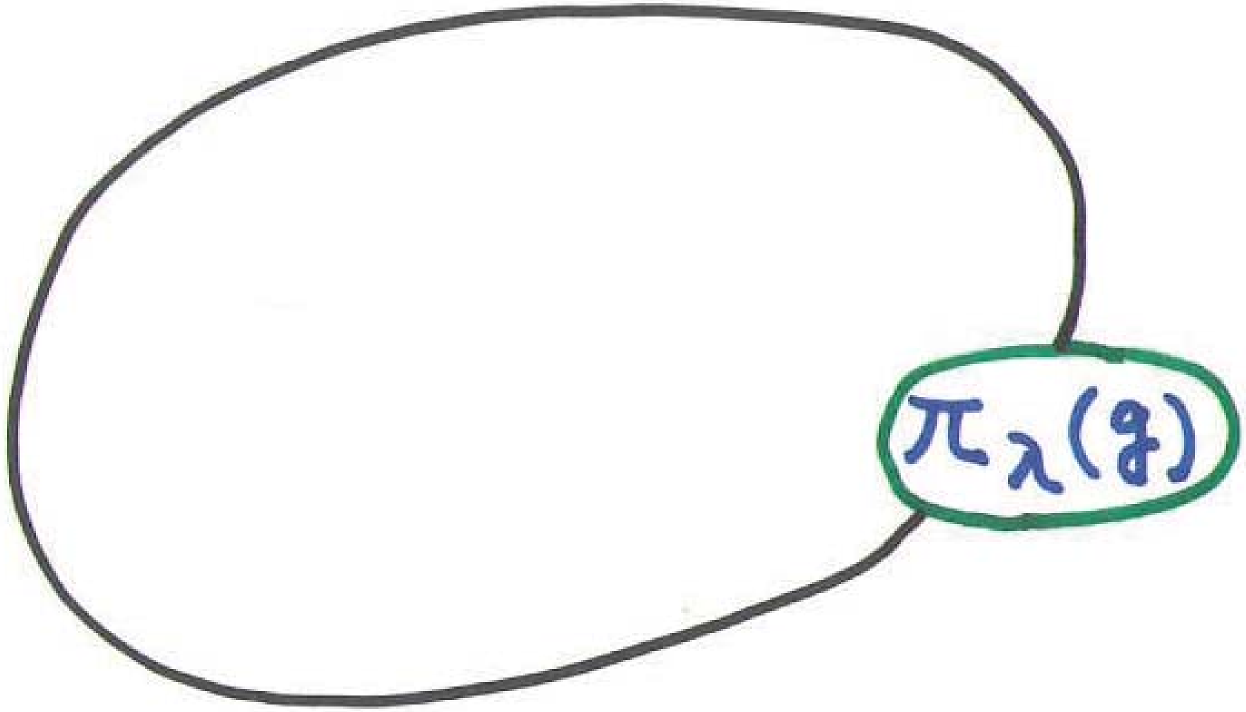


The transfer matrix in terms of the co-derivative [Kazakov,Vieira '07]

$L = 0$

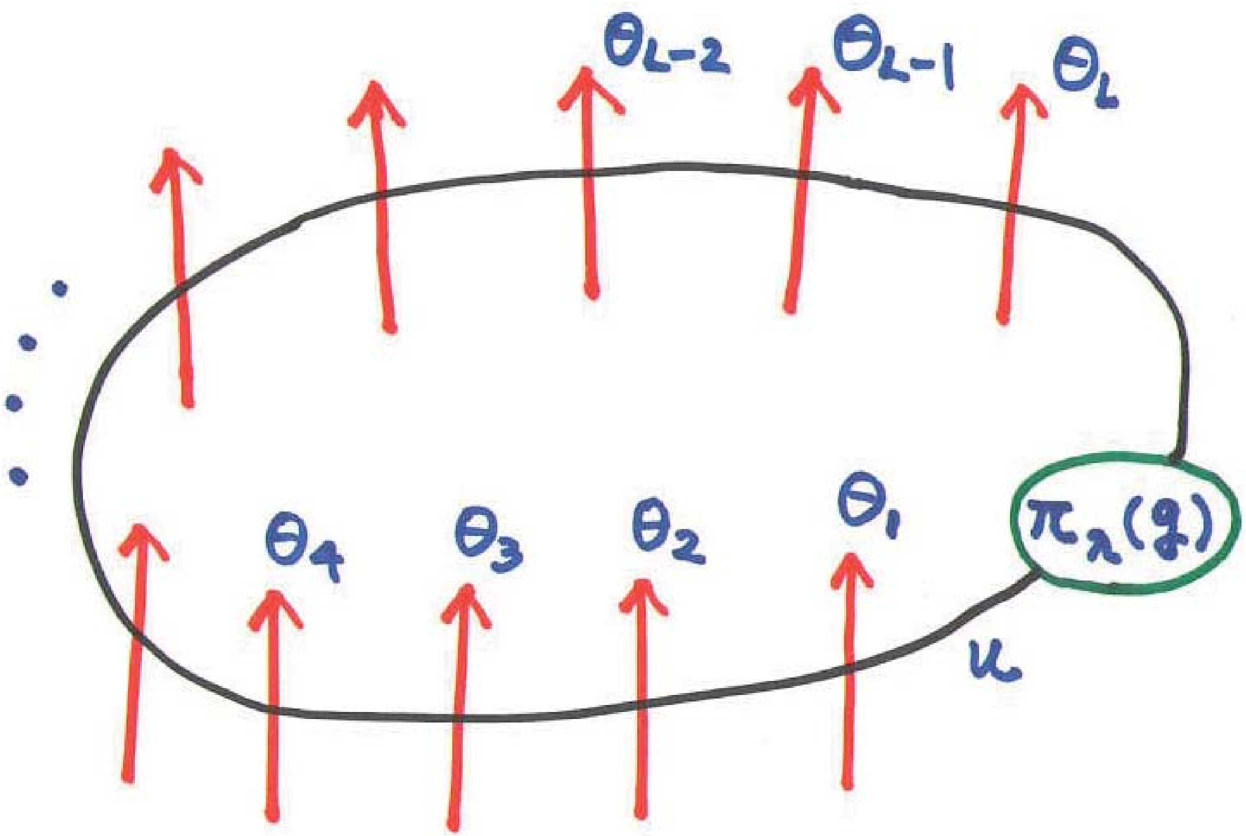
$$T^{\lambda}(u) = \mathrm{Tr}\ \pi_{\lambda}(g) = \chi_{\lambda}(g)$$

(the character of the twist g in the irrep λ)



L-site case

$$T^{\lambda}(u) = \mathrm{Tr}\ [R^{\lambda}_{L0}(u-\theta_L)\cdots R^{\lambda}_{20}(u-\theta_2)R^{\lambda}_{10}(u-\theta_1)(1^{\otimes L}\otimes\pi_\lambda(g))]$$
$$= (u_1+\hat{D})\otimes(u_2+\hat{D})\otimes\cdots\otimes(u_L+\hat{D})\ \ \chi_\lambda(g), \qquad u_i := u-\theta_i$$



The master T-operator [Alexandrov-Kazakov-Leurent-ZT-Zabrodin '11]

Schur functions in the KP-time variables $t = \{t_1, t_2, t_3, \dots\}$

$$\exp\left(\sum_{k=1}^{\infty} t_k z^{-k}\right) = \sum_{n=0}^{\infty} s_{(n)}(t) z^{-n},$$

$$s_\lambda(t) = \det_{1 \le i,j \le \lambda'_1} \left(s_{(\lambda_i - i + j)}\right)$$

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The master T-operator (τ-function)

$$T(u, \mathbf{t}) = \sum_\lambda s_\lambda(\mathbf{t}) T^\lambda(u).$$

$$T^\lambda(u) = s_\lambda(\tilde{\partial}) T(u, \mathbf{t})\Big|_{\mathbf{t}=0}, \quad \tilde{\partial} = \{\partial_{t_1}, \frac{1}{2}\partial_{t_2}, \frac{1}{3}\partial_{t_3}, \ldots\}$$

The master T-operator (τ-function)

$$T(u,\mathbf{t}) = \sum_{\lambda} s_{\lambda}(\mathbf{t}) T^{\lambda}(u).$$

- The master T-operator commutes for any u, t: $[T(u,\mathbf{t}), T(u',\mathbf{t}')] = 0$.
- The master T-operator contains Baxter Q-operators and T-operators for all levels of the nested Bethe ansatz
- The master T-operator is a τ-function of

1. KP-hierarchy with respect to times $t_1, t_2, \cdots$,
2. MKP-hierarchy with respect to times $t_0, t_1, t_2, \cdots$.

Here $t_0 = u$ plays a role of the spectral parameter in the quantum integrable system. The statement 2 is equivalent to that *the coefficients $T^{\lambda}(u)$ of the Schur function expansion obey the Cherednik-Bazhanov-Reshetikhin formula* $T^{\lambda}(u) = \det_{1 \le i,j \le \lambda'_1} (T^{(s-i+j)}(u-j+1))$.

The master T-operator in terms of the co-derivative

$$T(u,\mathbf{t}) = \sum_{\lambda} s_\lambda(\mathbf{t}) T^\lambda(u)$$
$$= (u_1 + \hat{D}) \otimes (u_2 + \hat{D}) \otimes \cdots \otimes (u_L + \hat{D})\ \chi(\mathbf{t})$$

A generating function of the characters

$$\chi(\mathbf{t}) = \sum_{\lambda} s_\lambda(\mathbf{t}) \chi_\lambda(g).$$

Bilinear identity for the master T-operator

$$\oint_C e^{\xi(\mathbf{t}-\mathbf{t}',z)} z^{u-u'} T(u,\mathbf{t}-[z^{-1}])\, T(u',\mathbf{t}'+[z^{-1}])\, dz = 0$$

$$t + [z^{-1}] = \{t_1 + z^{-1}, t_2 + \tfrac{1}{2}z^{-2}, t_3 + \tfrac{1}{3}z^{-3}, \dots\},$$

$$\xi(\mathbf{t}, z) = \sum_{n=1}^{\infty} t_n z^n$$

We can derive various bilinear equations by choosing u, u', t, t'

(KP or MKP equations).

Hirota Bilinear equation for the master T-operator (MKP)

$$u' = u - 1, \qquad t'_k = t_k - \tfrac{1}{k}(z_1^{-k} + z_2^{-k}),$$

$$z e^{\xi(\mathbf{t}-\mathbf{t}', z)} = \frac{z}{(1 - \frac{z}{z_1})(1 - \frac{z}{z_2})},$$

$$\oint_C e^{\xi(\mathbf{t}-\mathbf{t}', z)} z^{u-u'} T(u, \mathbf{t} - [z^{-1}])\, T(u', \mathbf{t}' + [z^{-1}])\, dz = 0,$$

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$$\oint_C e^{\xi(\mathbf{t}-\mathbf{t}', z)} z^{u-u'} T(u, \mathbf{t} - [z^{-1}])\, T(u', \mathbf{t}' + [z^{-1}])\, dz = 0,$$

$$z_2 T\left(u+1, \mathbf{t} + [z_1^{-1}]\right) T\left(u, \mathbf{t} + [z_2^{-1}]\right) - z_1 T\left(u+1, \mathbf{t} + [z_2^{-1}]\right) T\left(u, \mathbf{t} + [z_1^{-1}]\right)$$
$$+ (z_1 - z_2) T\left(u+1, \mathbf{t} + [z_1^{-1}] + [z_2^{-1}]\right) T\Big(u, \mathbf{t}\Big) = 0$$

The equations in the hierarchy are obtained by expanding in negative powers of z_1, z_2.

Bäcklund transformations for the master T-operator

Let us take any subset $\{i_1, i_2, \ldots, i_n\}$ of the set $\{1, 2, \ldots, N\}$. There are 2^N such sets. We define the *nested master T-operators* $T^{(i_1 \ldots i_n)}(u, \mathbf{t})$ recursively by taking the residue of the master T-operator.

$$T^{(i_1 \ldots i_n)}(u, \mathbf{t}) = \pm \mathrm{res}_{z_{i_n} = x_{i_n}} \left(z_{i_n}^{-u-1} e^{-\xi(\mathbf{t}, z_{i_n})} T^{(i_1 \ldots i_{n-1})}(u+1, \mathbf{t} + [z_{i_n}^{-1}]) \right),$$

where $\{x_1, x_2, \ldots, x_N\}$ are the eigenvalues of the boundary twist matrix $g \in GL(N)$ and $T^{\emptyset}(u, \mathbf{t}) = T(u, \mathbf{t})$. These define the undressing chain that terminates at the level N:

$$T(u, \mathbf{t}) \to T^{(i_1)}(u, \mathbf{t}) \to T^{(i_1 i_2)}(u, \mathbf{t}) \to \ldots \to T^{(12 \ldots N)}(u, \mathbf{t}) \to 0$$

and satisfy the bilinear relations (Bäcklund transformations) in the same way as the master T-operator :

$$x_j^{-1} T^{(i_1 \ldots i_n i)}(u, \mathbf{t}) T^{(i_1 \ldots i_n j)}(u+1, \mathbf{t}) - x_i^{-1} T^{(i_1 \ldots i_n j)}(u, \mathbf{t}) T^{(i_1 \ldots i_n i)}(u+1, \mathbf{t})$$
$$= \varepsilon_{ij} T^{(i_1 \ldots i_n ij)}(u, \mathbf{t}) T^{(i_1 \ldots i_n)}(u+1, \mathbf{t}),$$

where $i, j, k \in \{1, 2, \ldots, N\} \setminus \{i_1, i_2, \ldots, i_n\}$, $i \neq j, i \neq k, j \neq k$, $\varepsilon_{ij} = \pm 1$.

A general definition of the Baxter Q-operators

We define the *Baxter Q-operators* by the nested master T-operators as their restrictions to zero values of $\mathbf{t}$:

$$Q_{(i_1 \ldots i_n)}(u) = T^{(i_1 \ldots i_n)}(u, \mathbf{t} = 0).$$

QQ-relations

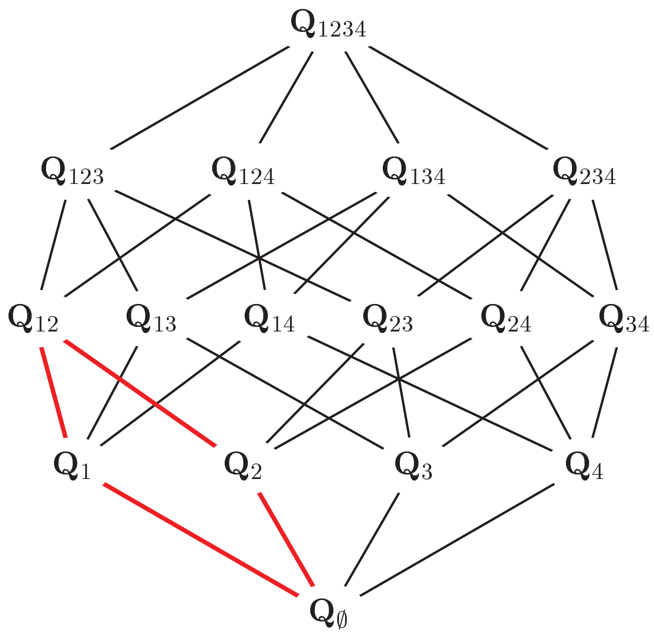
From the bilinear identity for the nested master T-operator,

$$(x_i - x_j)\mathbf{Q}_{I\cup\{i,j\}}(u+1)\mathbf{Q}_I(u) =$$

$$= x_i\mathbf{Q}_{I\cup\{j\}}(u)\mathbf{Q}_{I\cup\{i\}}(u+1) - x_j\mathbf{Q}_{I\cup\{i\}}(u)\mathbf{Q}_{I\cup\{j\}}(u+1)$$

$I \subset \{1, 2, \ldots, N\}$

Hasse diagram for Q-operators: $gl(4)$ case



- $2^4 = 16$ Q-operators
- 4-cycles correspond the QQ-relations.

$$(x_1 - x_2)\mathbf{Q}_{\{1,2\}}(u+1)\mathbf{Q}_{\{\}}(u) = x_1\mathbf{Q}_{\{2\}}(u)\mathbf{Q}_{\{1\}}(u+1) - x_2\mathbf{Q}_{\{1\}}(u)\mathbf{Q}_{\{2\}}(u+1)$$

- Any Q-operators can be expressed in terms of an elementary set of Q-operators: Wronskian-like determinant.

From the QQ-relations to Bethe equations

Q-functions (eigenvalues): $Q_I(u) = c_I \prod_{k=1}^{K_I}(u - u_k^{(I)})$

Zeros of the Q-functions: $u = u_k^{(I)}$

From the QQ-relations to Bethe equations

The (nested) Bethe equations

$$-1 = \frac{x_i}{x_j} \frac{Q_I(u_k^{(I\cup\{i\})}-1)Q_{I\cup\{i\}}(u_k^{(I\cup\{i\})}+1)Q_{I\cup\{i,j\}}(u_k^{(I\cup\{i\})})}{Q_I(u_k^{(I\cup\{i\})})Q_{I\cup\{i\}}(u_k^{(I\cup\{i\})}-1)Q_{I\cup\{i,j\}}(u_k^{(I\cup\{i\})}+1)},$$

$$k = 1, 2, \ldots, K_{I\cup\{i\}}$$

We derived these without using the Bethe ansatz.

- Generalization to $gl(N|M)$ case is relatively easy [Kazakov, Leurent, ZT'10,...].

Master T-operator for the Gaudin model

T-operator for the spin chain in terms of the co-derivative:

$$\mathsf{T}_\lambda(x) = \left(1 + \frac{\eta\,\mathsf{D}_n}{x - x_n}\right) \dots \left(1 + \frac{\eta\,\mathsf{D}_1}{x - x_1}\right) \chi_\lambda(g),$$

$$\mathsf{D}_i f(g) = \frac{\partial}{\partial \varepsilon} \sum_{ab} e_{ab}^{(i)} f(e^{\varepsilon \mathsf{e}_{ba}} g)\Big|_{\varepsilon=0}, \qquad g \in GL(N).$$

Let us consider the case $\lambda = (1)$, $\eta \to 0$.

Master T-operator for the Gaudin model

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Let us consider the case $\lambda = (1)$, $\eta \to 0$.

$$\mathsf{T}_{(1)}(x) = N + \eta\Big(\operatorname{tr} h + \sum_i \frac{1}{x - x_i}\Big) + \eta^2\left(\frac{1}{2}\operatorname{tr} h^2 + \sum_i \frac{h_i}{x - x_i} + \sum_{i<j} \frac{P_{ij}}{(x - x_i)(x - x_j)}\right) + $$

The second order term corresponds to the Hamiltonian of the Gaudin model.

In principle, commutative family of higher integral of motion will be obtained from the expansion of $\mathsf{T}_\lambda(x)$ with respect to η. But to extract non-trivial integrals of motion from this expansion is a problem.

Master T-operator for the Gaudin model

Let us modify the definition of the T-operator as

$$\tilde{\mathsf{T}}_\lambda(x) = \Big(1+\frac{\eta\,\mathsf{D}_n}{x-x_n}\Big)\dots\Big(1+\frac{\eta\,\mathsf{D}_1}{x-x_1}\Big)\chi_\lambda(g-\mathbb{I})\,, \qquad g = e^{\eta h}.$$

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Since $\chi_\lambda(g - \mathbb{I})$ is a linear combination of characters $\chi_\mu(g)$ with different μ:

$$\chi_\lambda(g - \mathbb{I}) = \sum_{\mu\subset\lambda} c_{\lambda\mu}\chi_\mu(g),$$

$\tilde{\mathsf{T}}_\lambda(x)$ is a linear combination of the $\mathsf{T}_\mu(x)$'s:

$$\tilde{\mathsf{T}}_\lambda(x) = \sum_{\mu\subset\lambda} c_{\lambda\mu}\mathsf{T}_\mu(x).$$

We define the higher Gaudin transfer matrices as

$$\mathsf{T}^G_\lambda(x) = \lim_{\eta\to 0}\Big(\eta^{-|\lambda|}\tilde{\mathsf{T}}_\lambda(x)\Big).$$

Co-derivative for the Gaudin model: from the group derivative to the Lie algebra derivative

To express the higher Gaudin transfer matrices $\mathsf{T}^G_\lambda(x) = \lim_{\eta\to 0}\Big(\eta^{-|\lambda|}\tilde{\mathsf{T}}_\lambda(x)\Big)$ expricitly, we introduce a modified co-derivative (Lie algebra derivative) on the lattice site i:

$$\mathsf{d}_i f(h) = \frac{\partial}{\partial\varepsilon}\sum_{ab} e^{(i)}_{ab} f(h+\varepsilon \mathsf{e}_{ba})\Big|_{\varepsilon=0}, \qquad h \in gl(N).$$

Examples:

$$\mathsf{d}_i\,(\mathsf{tr}\,h) = \mathbb{I}_i, \qquad \mathsf{d}_i h_j = \sum_{ab} e^{(i)}_{ab} e^{(j)}_{ba} = P_{ij}$$

$$\mathsf{T}^G_\lambda(x) = \lim_{\eta\to 0}\Big(\eta^{-|\lambda|}\tilde{\mathsf{T}}_\lambda(x)\Big),$$

$$\tilde{\mathsf{T}}_\lambda(x) = \Big(1 + \frac{\eta\,\mathsf{D}_n}{x - x_n}\Big)\dots\Big(1 + \frac{\eta\,\mathsf{D}_1}{x - x_1}\Big)\chi_\lambda(g - \mathbb{I})\,, \quad g = e^{\eta h}.$$

$$\eta^k\mathsf{D}_k\dots\mathsf{D}_1\chi_\lambda(g - \mathbb{I}) = \eta^m\,\mathsf{d}_k\dots\mathsf{d}_1\chi_\lambda(h) + O(\eta^{m+1}), \quad \eta\to 0$$

The family of commuting operators for the (twisted) Gaudin model:

$$\mathsf{T}^G_\lambda(x) = \Big(1 + \frac{\mathsf{d}_n}{x - x_n}\Big)\dots\Big(1 + \frac{\mathsf{d}_1}{x - x_1}\Big)\chi_\lambda(h)$$

or, in the polynomial normalization,

$$T^G_\lambda(x) = (x - x_n + \mathsf{d}_n)\,\dots\,(x - x_1 + \mathsf{d}_1)\chi_\lambda(h).$$

Examples:

$$\mathsf{T}^{G}_{\emptyset}(x) = 1,$$

$$\mathsf{T}^{G}_{(1)}(x) = \mathrm{tr}\, h + \sum_i \frac{1}{x - x_i},$$

$$\mathsf{T}^{G}_{(1^2)}(x) = \frac{1}{2}(\mathrm{tr}\, h)^2 + \mathrm{tr}\, h \sum_i \frac{1}{x - x_i} + \sum_{i<j} \frac{1}{(x - x_i)(x - x_j)} - H(x),$$

$$\mathsf{T}^{G}_{(2)}(x) = \frac{1}{2}(\mathrm{tr}\, h)^2 + \mathrm{tr}\, h \sum_i \frac{1}{x - x_i} + \sum_{i<j} \frac{1}{(x - x_i)(x - x_j)} + H(x),$$

Note that in this approach, the Hamiltonian $H(x)$ of the Gaudin model emerges from $\mathsf{T}_{(1^2)}(x)$ or $\mathsf{T}_{(2)}(x)$ rather than $\mathsf{T}_{(1)}(x)$.

Master T-operator for the Gaudin model

The master T-operator for the Gaudin model is defined as

$$T^G(x,\mathbf{t}) = \sum_\lambda s_\lambda(\mathbf{t}) T^G_\lambda(x),$$

In terms of the modified co-derivative,

$$T^G(x,\mathbf{t}) = (x - x_n + \mathsf{d}_n) \, \ldots \, (x - x_1 + \mathsf{d}_1) \exp\Big(\sum_{k \geq 1} t_k \, \mathsf{tr}\, h^k \Big).$$

The bilinear identity and Hirota equations

The master T-operator satisfies the bilinear identity for the KP hierarchy

$$\oint_{\infty} e^{\xi(\mathbf{t}-\mathbf{t}',z)} T^G\left(x, \mathbf{t} - [z^{-1}]\right) T^G\left(x, \mathbf{t}' + [z^{-1}]\right) dz = 0 \quad \text{for all } \mathbf{t}, \mathbf{t}'.$$

By specializing parameters, one can obtain bilinear identities for the master T-operator (KP or the differential Fay identity)

The 3-term Hirota equation (the Fay identity or KP eq)

$$
\begin{aligned}
&(z_2 - z_3)\, T^G\left(x, \mathbf{t} + [z_1^{-1}]\right) T^G\left(x, \mathbf{t} + [z_2^{-1}] + [z_3^{-1}]\right) \\
&\quad + (z_3 - z_1)\, T^G\left(x, \mathbf{t} + [z_2^{-1}]\right) T^G\left(x, \mathbf{t} + [z_1^{-1}] + [z_3^{-1}]\right) \\
&\qquad + (z_1 - z_2)\, T^G\left(x, \mathbf{t} + [z_3^{-1}]\right) T^G\left(x, \mathbf{t} + [z_1^{-1}] + [z_2^{-1}]\right) = 0.
\end{aligned}
$$

By taking the limit $z_3 \to \infty$, we obtain (the differential Fay identity)

$$
\begin{aligned}
&T^G(x, \mathbf{t} + [z_2^{-1}])\partial_x T^G(x, \mathbf{t} + [z_1^{-1}]) - T^G(x, \mathbf{t} + [z_1^{-1}])\partial_x T^G(x, \mathbf{t} + [z_2^{-1}]) \\
&+ (z_1 - z_2)\Big[T^G(\mathbf{t})T^G(x, \mathbf{t} + [z_1^{-1}] + [z_2^{-1}]) - T^G(x, \mathbf{t} + [z_1^{-1}])T^G(x, \mathbf{t} + [z_2^{-1}])\Big] = 0.
\end{aligned}
$$

Cherednik-Bazhanov-Reshetikhin (CBR) determinant formula

For the original spin chain

$$\mathsf{T}_\lambda(x) = \det_{1\le i,j\le \lambda'_1} \mathsf{T}_{(\lambda_i - i + j)}\big(x-(j-1)\eta\big),$$

For the Gaudin transfer matrices:

$$\mathsf{T}^G_\lambda(x) = \det_{1\le i,j\le \lambda'_1} \left(\sum_{k=0}^{j-1} (-1)^k \begin{pmatrix} j-1 \\ k \end{pmatrix} \partial_x^k \mathsf{T}^G_{(\lambda_i - i + j - k)}(x) \right),$$

The quantum Giambelli formula

For the original spin chain:

$$\mathsf{T}_\lambda(x) = \det_{1\le i,j\le d(\lambda)} \mathsf{T}_{\lambda_i - i, \lambda'_j - j}(x),$$

For the Gaudin model:

$$\mathsf{T}^G_\lambda(x) = \det_{1\le i,j\le d(\lambda)} \mathsf{T}^G_{\lambda_i - i, \lambda'_j - j}(x),$$

where $\mathsf{T}^G_{l,k}(x) := \mathsf{T}^G_{(l+1,1^k)}(x)$; $d(\lambda)$ is the number of boxes in the main diagonal of the Young diagram λ.

cf. Quantum Giambelli formula for $U_q(B_n^{(1)})$: [Kuniba,Ohta,Suzuki,1995]

The condition for the bilinear identity

• The original spin chain

$$T(x,\mathbf{t}) = \sum_\lambda S_\lambda(\mathbf{t}) T_\lambda(x)$$ is the tau-function of the MKP hierarchy

$$\Longleftrightarrow$$

$$\mathsf{T}_\lambda(x) = \det_{1\le i,j\le \lambda'_1} \mathsf{T}^G_{(\lambda_i - i + j)}(x - (j-1)\eta)$$ [CBR (quantum Jacobi-Trudi) formula].

• The Gaudin model

$$T^G(x,\mathbf{t}) = \sum_\lambda S_\lambda(\mathbf{t}) T^G_\lambda(x)$$ is the tau-function of the KP hierarchy

$$\Longleftrightarrow$$

$$\mathsf{T}^G_\lambda(x) = \det_{1\le i,j\le d(\lambda)} \mathsf{T}^G_{\lambda_i - i, \lambda'_j - j}(x)$$ (quantum Giambelli formula).

Q-operators for trigonometric models based on L-operators

The Q-operators can also be defined as the trace of some monodromy matrices, which are defined as product of L-operators. In general, such L-operators are image of the universal R-matrix for q-oscillator representations of the Borel subalgebra of the quantum affine algebra (for $U_q(\hat{sl}(2))$, [Bazhanov-Lukyanov-Zamolodchikov]).

$U_q(\hat{sl}(2|1))$: [Bazhanov, ZT '08]

How about $U_q(\hat{gl}(M|N))$ case ? [ZT '12].

See also,

$U_q(\hat{sl}(3))$: [Bazhanov, Hibberd, Khoroshkin '01]

$U_q(\hat{sl}(M))$ (a subset of the Q-operators): [Kojima'08]

$U_q(C(2)^{(2)})$: [Kulish, Zeitlin]

$U_q(\hat{sl}(2))$ or $U_q(\hat{sl}(3))$ (discussions on the universal R-matrix): [Boos, Gohmann, Klumper, Nirov, Razumov]

Quantum affine superalgebra

The (centerless) quantum affine superalgebra $U_q(\widehat{gl}(M|N))$ is defined by

$$L_{ij}^{(0)} = \overline{L}_{ji}^{(0)} = 0, \quad \text{for} \quad 1 \le i < j \le M+N$$
$$L_{ii}^{(0)}\overline{L}_{ii}^{(0)} = \overline{L}_{ii}^{(0)}L_{ii}^{(0)} = 1 \quad \text{for} \quad 1 \le i \le M+N,$$
$$\mathbf{R}^{23}(x,y)\mathbf{L}^{13}(y)\mathbf{L}^{12}(x) = \mathbf{L}^{12}(x)\mathbf{L}^{13}(y)\mathbf{R}^{23}(x,y),$$
$$\mathbf{R}^{23}(x,y)\overline{\mathbf{L}}^{13}(y)\overline{\mathbf{L}}^{12}(x) = \overline{\mathbf{L}}^{12}(x)\overline{\mathbf{L}}^{13}(y)\mathbf{R}^{23}(x,y),$$
$$\mathbf{R}^{23}(x,y)\mathbf{L}^{13}(y)\overline{\mathbf{L}}^{12}(x) = \overline{\mathbf{L}}^{12}(x)\mathbf{L}^{13}(y)\mathbf{R}^{23}(x,y), \quad x,y \in \mathbb{C},$$

$$\mathbf{L}(x) = \sum_{i,j} L_{ij}(x) \otimes E_{ij}, \quad \overline{\mathbf{L}}(x) = \sum_{i,j} \overline{L}_{ij}(x) \otimes E_{ij},$$
$$L_{ij}(x) = \sum_{n=0}^{\infty} L_{ij}^{(n)} x^{-n}, \quad \overline{L}_{ij}(x) = \sum_{n=0}^{\infty} \overline{L}_{ij}^{(n)} x^{n},$$

where $\mathbf{R}(x,y) = \mathbf{R} - \frac{x}{y}\overline{\mathbf{R}}$ is the R-matrix of the Perk-Schultz model; $\mathbf{R}$ and $\overline{\mathbf{R}}$ do not depend on the spectral parameter; E_{ij} is $(M+N) \times (M+N)$ matrix unit.

Quantum (finite) superalgebra

The quantum affine superalgebra $U_q(\hat{gl}(M|N))$ has a finite subalgebra $U_q(gl(M|N))$ defined by

$$L_{ij} = \overline{L}_{ji} = 0, \quad \text{for} \quad 1 \le i < j \le M+N$$

$$L_{ii}\overline{L}_{ii} = \overline{L}_{ii}L_{ii} = 1 \quad \text{for} \quad 1 \le i \le M+N,$$

$$\mathbf{R}^{23}\mathbf{L}^{13}\mathbf{L}^{12} = \mathbf{L}^{12}\mathbf{L}^{13}\mathbf{R}^{23},$$

$$\mathbf{R}^{23}\overline{\mathbf{L}}^{13}\overline{\mathbf{L}}^{12} = \overline{\mathbf{L}}^{12}\overline{\mathbf{L}}^{13}\mathbf{R}^{23},$$

$$\mathbf{R}^{23}\mathbf{L}^{13}\overline{\mathbf{L}}^{12} = \overline{\mathbf{L}}^{12}\mathbf{L}^{13}\mathbf{R}^{23},$$

$$\mathbf{L} = \sum_{i,j} L_{ij} \otimes E_{ij}, \quad \overline{\mathbf{L}} = \sum_{i,j} \overline{L}_{ij} \otimes E_{ij}.$$

There is an evaluation map from $U_q(\hat{gl}(M|N))$ to $U_q(gl(M|N))$ such that

$$\mathbf{L}(x) \mapsto \mathbf{L} - \overline{\mathbf{L}} x^{-1},$$

$$\overline{\mathbf{L}}(x) \mapsto \overline{\mathbf{L}} - \mathbf{L} x.$$

The difference between $\mathbf{L}(x)$ and $\overline{\mathbf{L}}(x)$ is not very important under the evaluation map. We will consider only $\mathbf{L}(x)$ (q-superYangian).

Contractions of $U_q(gl(M|N))$

Let us take a subset I of the set $\{1, 2, \ldots, M + N\}$ and its complement set $\overline{I} := \{1, 2, \ldots, M + N\} \setminus I$. There are 2^{M+N} choices of the subsets in this case. Corresponding to the set I, we consider 2^{M+N} kind of representations of the q-superYangian. For this purpose, we consider 2^{M+N} kind of contractions of $U_q(gl(M|N))$.

Contractions of $U_q(gl(M|N))$

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$$L_{ii}\overline{L}_{ii} = \overline{L}_{ii}L_{ii} = 1 \quad \text{for} \quad 1 \leq i \leq M+N$$

is replaced by

$$\begin{aligned} L_{ii}\overline{L}_{ii} = \overline{L}_{ii}L_{ii} = 1 \quad &\text{for} \quad i \in I, \\ \overline{L}_{aa} = 0 \qquad\qquad &\text{for} \quad a \in \overline{I}. \end{aligned}$$

Then one can obtain 2^{M+N} kind of algebraic solutions of the graded Yang-Baxter equation via the map $\mathbf{L}_I(x) = \mathbf{L} - \overline{\mathbf{L}}x^{-1}$.

In addition, we consider subsidiary contractions for the non-diagonal elements. For example, suppose the sets have the form $I = \{1\}$, $\overline{I} = \{2, \ldots, M+N\}$, then we assume

$$\begin{pmatrix} \overline{L}_{11} & \overline{L}_{12} & \ldots & \overline{L}_{1,M+N} \\ 0 & \overline{L}_{22} & \ldots & \overline{L}_{2,M+N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \overline{L}_{M+N,M+N} \end{pmatrix} \Rightarrow \begin{pmatrix} \overline{L}_{11} & \overline{L}_{12} & \ldots & \overline{L}_{1,M+N} \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}$$

Remark

A preliminary form of these contractions were previously discussed for

$U_q(gl(3))$: Bazhanov, Khoroshkin, (2001) unpulished

$U_q(gl(2|1))$: Bazhanov, ZT, (2007) talks at conferences

In terms of the q-oscillator algebra , we can realize the contracted algebra.

Example: the case $I=\{1\}$, $\overline{I}=\{2,\ldots,M+N\}$,

$$
(L_{ab}) = \left(\begin{array}{c|cccc}
q^{\mp\sum_{k=2}^{M+N}\mathbf{n}_{1,k}} & 0 & & \cdots & 0 \\
\hline
 & q^{\pm\mathbf{n}_{1,2}} & & \cdots & 0 \\
\pm\mathbf{c}_{a1}q^{\pm\sum_{k=2}^{a-1}\mathbf{n}_{i,k}} & \pm(q-q^{-1})\mathbf{c}_{a1}\mathbf{c}^{\dagger}_{1b}q^{\pm\sum_{k=b}^{a-1}\mathbf{n}_{1,k}} & & \ddots & \vdots \\
 & & & & q^{\pm\mathbf{n}_{1,M+N}}
\end{array}\right),
$$

$$
(\overline{L}_{ab}) = \left(\begin{array}{c|ccc}
q^{\pm\sum_{k=2}^{M+N}\mathbf{n}_{1,k}} & & (q-q^{-1})\mathbf{c}^{\dagger}_{1b}q^{\pm\sum_{k=b}^{M+N}\mathbf{n}_{1,k}} & \\
\hline
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{array}\right)
$$

$$
[\mathbf{c}_{ai},\mathbf{c}^{\dagger}_{jb}]_{q^{(-1)^{p(a)}\delta_{ab}\delta_{ij}}} = \delta_{ab}\delta_{ij}q^{-(-1)^{p(i)}\mathbf{n}_{ia}},\ [\mathbf{n}_{ia},\mathbf{c}_{bj}] = -\delta_{ij}\delta_{ab}\mathbf{c}_{bj},,,,
$$

$$\mathbf{L}_I(x) = \mathbf{L} - \overline{\mathbf{L}} x^{-1}$$

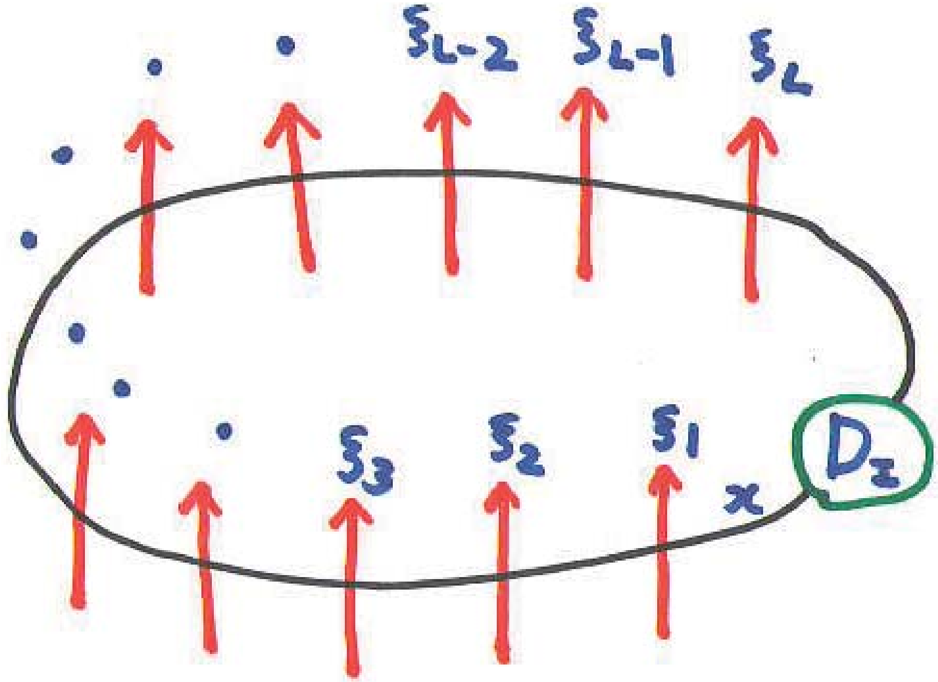
gives q-oscillator solutions of the graded Yang Baxter equations. This also gives q-oscillator representations of the q-superYangian.

After some renormalization, $\mathbf{L}_I(x)$ reduces to the L-operator similar to the one for rational models [Bazhanov, Frassek, Lukowski, Meneghelli, Staudacher '10] in the limit $q \to 1$.

Other approaches to L-operators: [S.E.Derkachov, A.N.Manashov], [D.Chicherin, S.Derkachov, A.P.Isaev '12],....etc.

Q-operators

$$\mathbf{Q}_I(x) = \frac{\mathrm{Str}_{\mathcal{F}_I}\left[\mathbf{L}_I^{0L}(\xi_L/x)\cdots\mathbf{L}_I^{02}(\xi_2/x)\mathbf{L}_I^{01}(\xi_1/x)(\mathbf{D}_I\otimes 1^{\otimes L})\right]}{\mathrm{Str}_{\mathcal{F}_I}\left[(\mathbf{D}_I\otimes 1^{\otimes L})\right]}.$$



T-operators can be expressed in terms of Q-operators: Wronskian-like determinant formula.

Universal R-matrix and factorization of the L-operator **[Khoroshkin, ZT '13]**

L-operators for infinite dimensional Verma modules (of the quantum affine algebra or Yangian) factorize with respect to L-operators for the Q-operators. Examples for such factorization formulas were given by a number of people. [Derkachov, Manashov, Bazhanov, Frassek, Lukowski, Meneghelli, Staudacher,,.....]

We reconsidered this phenomenon in relation to the universal R-matrix and obtained a '**universal factorization formula**', which are **independent** of the quantum space.

Quantum affine algebra $U_q(\hat{sl}(2))$: generated by e_i, f_i, h_i $(i = 0, 1,\ h_0 + h_1 = 0)$

It contains subalgebras:

Borel subalgebras
$\mathcal{B}_+$: e_i, h_i,
$\mathcal{B}_-$: f_i, h_i,

(finite) quantum algebra $U_q(sl(2))$: E, F, H.

The universal R-matrix $\mathcal{R} \in \mathcal{B}_+ \otimes \mathcal{B}_-$ is defined by

$$\Delta'(a)\ \mathcal{R} = \mathcal{R}\ \Delta(a) \qquad \text{for} \quad \forall\ a \in U_q(\hat{sl}(2))\,,$$

$$(\Delta \otimes 1)\,\mathcal{R} = \mathcal{R}^{13}\,\mathcal{R}^{23}$$

$$(1 \otimes \Delta)\,\mathcal{R} = \mathcal{R}^{13}\,\mathcal{R}^{12}$$

q-oscillator algebra

We introduce two kind of oscillator algebras Osc_i $(i = 1, 2)$, which are generated by the generators $\mathsf{h}_i, \mathsf{e}_i, \mathsf{f}_i$ with the relations:

$$[\mathsf{h}_1, \mathsf{h}_1] = 0, \qquad [\mathsf{h}_1, \mathsf{e}_1] = 2\mathsf{e}_1, \qquad [\mathsf{h}_1, \mathsf{f}_1] = -2\mathsf{f}_1,$$

$$\mathsf{f}_1\mathsf{e}_1 = q\frac{1 - q^{\mathsf{h}_1}}{(q - q^{-1})^2}, \qquad \mathsf{e}_1\mathsf{f}_1 = q\frac{1 - q^{\mathsf{h}_1 - 2}}{(q - q^{-1})^2},$$

$$[\mathsf{h}_2, \mathsf{h}_2] = 0, \qquad [\mathsf{h}_2, \mathsf{e}_2] = 2\mathsf{e}_2, \qquad [\mathsf{h}_2, \mathsf{f}_2] = -2\mathsf{f}_2,$$

$$\mathsf{f}_2\mathsf{e}_2 = q^{-1}\frac{1 - q^{-\mathsf{h}_2}}{(q - q^{-1})^2}, \qquad \mathsf{e}_2\mathsf{f}_2 = q^{-1}\frac{1 - q^{-\mathsf{h}_2 + 2}}{(q - q^{-1})^2},$$

Osc_i can be obtained by taking a limit $(\mu \to \pm\infty)$ of the Verma module of $U_q(sl(2))$ with highest weight μ.

Note that Osc_1 and Osc_2 can be swapped one another by the transformation $q \mapsto q^{-1}$.

There are evaluation maps $\rho_x^{(i)} : \mathcal{B}_+ \mapsto \mathrm{Osc}_i$

$$\rho_x^{(i)}(e_0) = \mathsf{f}_i, \quad \rho_x^{(i)}(e_1) = x\mathsf{e}_i, \quad \rho_x^{(i)}(h_0) = -\mathsf{h}_i, \quad \rho_x^{(i)}(h_1) = \mathsf{h}_i,$$

$$i = 1, 2.$$

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$$\exp_{q^{-2}}^{-1}(\lambda \mathsf{e}_1 \otimes \mathsf{f}_2) \Big((\rho_{xq^{\mu}}^{(1)} \otimes \rho_{xq^{-\mu}}^{(2)}) \Delta(\mathsf{a}) \Big) \exp_{q^{-2}}(\lambda \mathsf{e}_1 \otimes \mathsf{f}_2) = \\ = (\mathsf{ev}_x^{(1)} \otimes \mathsf{ev}_x^{(2)}) \Delta(\mathsf{a}), \quad \mathsf{a} \in \mathcal{B}_+,$$

$\lambda = q - q^{-1}$, $\mu, x \in \mathbb{C}$.

There are evaluation maps $\rho_x^{(i)} : \mathcal{B}_+ \mapsto \mathrm{Osc}_i$

$$\rho_x^{(i)}(e_0) = \mathsf{f}_i, \quad \rho_x^{(i)}(e_1) = x\mathsf{e}_i, \quad \rho_x^{(i)}(h_0) = -\mathsf{h}_i, \quad \rho_x^{(i)}(h_1) = \mathsf{h}_i, \quad i = 1, 2,$$

$$\exp_{q^{-2}}^{-1}(\lambda \mathsf{e}_1 \otimes \mathsf{f}_2)\left((\rho_{xq^{\mu}}^{(1)} \otimes \rho_{xq^{-\mu}}^{(2)})\Delta(\mathsf{a})\right)\exp_{q^{-2}}(\lambda \mathsf{e}_1 \otimes \mathsf{f}_2) = \\ = (\mathsf{ev}_x^{(1)} \otimes \mathsf{ev}_x^{(2)})\Delta(\mathsf{a}), \quad \mathsf{a} \in \mathcal{B}_+,$$

$\lambda = q - q^{-1}$, $\mu, x \in \mathbb{C}$,
Evaluation map $\mathsf{ev}_x^{(1)} : \mathcal{B}_+ \mapsto U_q(sl(2))$,

$$\mathsf{ev}_x^{(1)}(e_0) = F_1, \quad \mathsf{ev}_x^{(1)}(e_1) = xE_1, \quad \mathsf{ev}_x^{(1)}(h_0) = -H_1, \quad \mathsf{ev}_x^{(1)}(h_1) = H_1,$$

$E_1 = (q^{\mu} - q^{-\mu-\mathsf{h}_1})\mathsf{e}_1$, $F_1 = \mathsf{f}_1$, $H_1 = \mathsf{h}_1 + \mu$ realize Verma module with the highest weight μ on an appropriate Fock space.

$$\mathsf{ev}_x^{(2)}(e_0) = 0, \quad \mathsf{ev}_x^{(2)}(e_1) = x\mathsf{e}_2, \quad \mathsf{ev}_x^{(2)}(h_0) = \mu - \mathsf{h}_2, \quad \mathsf{ev}_x^{(2)}(h_1) = \mathsf{h}_2 - \mu.$$

A universal factorization formula

$$\exp_{q^{-2}}^{-1}(\lambda \mathsf{e}_1 \otimes \mathsf{f}_2 \otimes 1)\left((\rho^{(1)}_{xq^{\mu}} \otimes 1 \otimes 1)\mathcal{R}_{13}\right)\left((1 \otimes \rho^{(2)}_{xq^{-\mu}} \otimes 1)\mathcal{R}_{23}\right)\exp_{q^{-2}}(\lambda \mathsf{e}_1 \otimes \mathsf{f}_2 \otimes 1) =$$
$$= \left((\mathsf{ev}^{(1)}_{x} \otimes 1 \otimes 1)\mathcal{R}_{13}\right)\exp_{q^{-2}}(\lambda \otimes x\mathsf{e}_2 \otimes f_1)\, q^{\frac{1 \otimes (\mathsf{h}_2 - \mu) \otimes h_1}{2}},$$

$f_1, h_1 \in \mathcal{B}_-$, $\lambda = q - q^{-1}$, $\mu, x \in \mathbb{C}$.

($\mathcal{R}$ for q – oscillator) $\times$ ($\mathcal{R}$ for q – oscillator) $\simeq$ ($\mathcal{R}$ for Verma module)$(\dots)$

The third space (quantum space) is arbitrarily.

Example for fundamental representation on the quantum space

$$\exp^{-1}_{q^{-2}}(\lambda \mathsf{e}_1 \otimes \mathsf{f}_2 \otimes 1)\left((\rho^{(1)}_{xq^{\mu}} \otimes 1 \otimes 1)\mathcal{R}_{13}\right)\left((1 \otimes \rho^{(2)}_{xq^{-\mu}} \otimes 1)\mathcal{R}_{23}\right)\exp_{q^{-2}}(\lambda \mathsf{e}_1 \otimes \mathsf{f}_2 \otimes 1) =$$
$$= \left((\mathsf{ev}^{(1)}_x \otimes 1 \otimes 1)\mathcal{R}_{13}\right)\exp_{q^{-2}}(\lambda \otimes x\mathsf{e}_2 \otimes f_1)\, q^{\frac{1\otimes(\mathsf{h}_2-\mu)\otimes h_1}{2}}.$$

$$f_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h_1 = -h_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\exp^{-1}_{q^{-2}}(\lambda \mathsf{e}_1 \otimes \mathsf{f}_2) \begin{pmatrix} q^{\frac{\mathsf{h}_1}{2}} & \lambda \mathsf{f}_1 q^{-\frac{\mathsf{h}_1}{2}} \\ \lambda x \mathsf{e}_1 q^{\frac{\mathsf{h}_1}{2}+\mu} & q^{-\frac{\mathsf{h}_1}{2}} - xq^{\frac{\mathsf{h}_1}{2}+\mu-1} \end{pmatrix} \begin{pmatrix} q^{\frac{\mathsf{h}_2}{2}} - xq^{-\frac{\mathsf{h}_2}{2}-\mu-1} & \lambda \mathsf{f}_2 q^{-\frac{\mathsf{h}_2}{2}} \\ \lambda x \mathsf{e}_2 q^{\frac{\mathsf{h}_2}{2}-\mu} & q^{-\frac{\mathsf{h}_2}{2}} \end{pmatrix}$$
$$\times \exp_{q^{-2}}(\lambda \mathsf{e}_1 \otimes \mathsf{f}_2) =$$
$$= \phi(x) \begin{pmatrix} q^{\frac{H_1}{2}} - q^{-1}xq^{-\frac{H_1}{2}} & \lambda F_1 q^{-\frac{H_1}{2}} \\ \lambda x E_1 q^{\frac{H_1}{2}} & q^{-\frac{H_1}{2}} - q^{-1}xq^{\frac{H_1}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda x \mathsf{e}_2 & 1 \end{pmatrix} \begin{pmatrix} q^{\frac{\mathsf{h}_2-\mu}{2}} & 0 \\ 0 & q^{-\frac{\mathsf{h}_2-\mu}{2}} \end{pmatrix}$$

Concluding remarks

• We defined the master T-operator for quantum integrable spin chains. It corresponds to the tau-function for MKP-hierarchy and allows an embedding of quantum integrable system into the solition theory. The Baxter Q-operators are defined as residue of the master T-operator. Functional relations among T-and Q-operators follow from the master identity (MKP equation). In the case of the Gaudin model, the master T-operator is the tau-function of the KP-hierarchy.

• For the trigonometric models related to $U_q(\hat{gl}(M|N))$, we proposed L-operators for the Q-operators based on the contraction of the algebra.

• A universal factorization formula of the L-operator was derived based on the universal R-matrix.

• It is desirable to unify these approaches and construct Q-operators systematically for any quantum integrable systems.