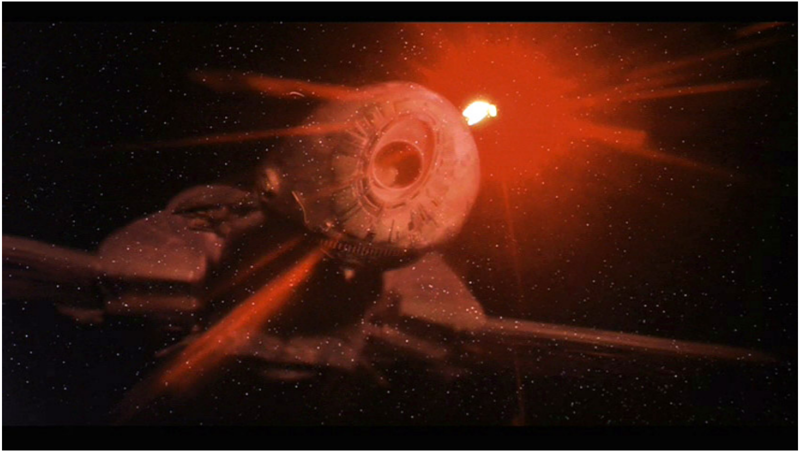
# Transformation Optics and the mathematics of invisibility

Robert T. Thompson

Department of Mathematics and Statistics
University of Otago

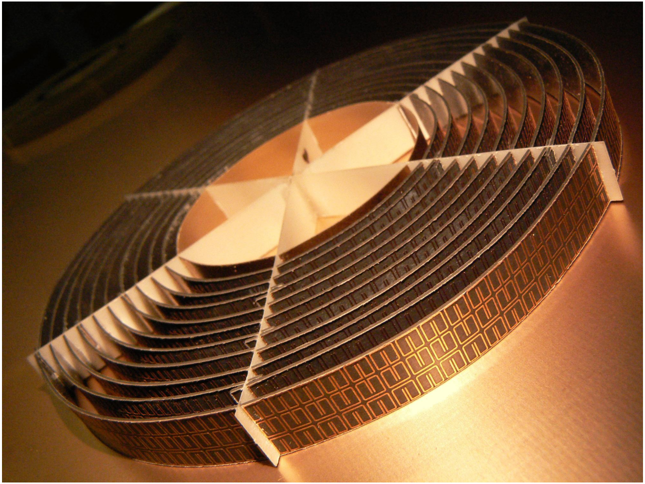
$2^{nd}$ ANZAMP meeting, 27 November 2013

NOT CAMOUFLAGE



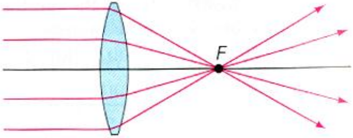
NOT SCIENCE FICTION

NOT (HOLLYWOOD) MAGIC

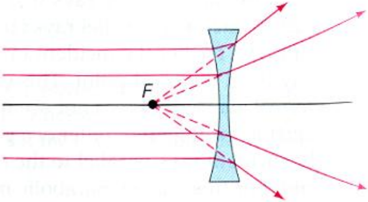


CYLINDRICAL ELECTROMAGNETIC CLOAK

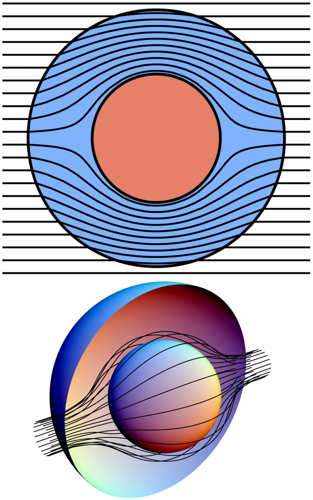
D. Smith, D. Schurig, S. Cummer




Converging lens




Diverging lens

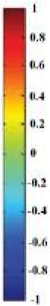


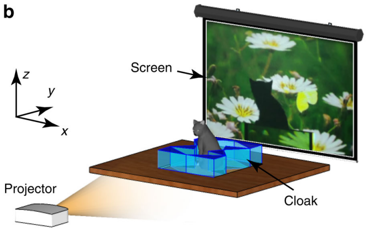
Pendry, et. al. 2006

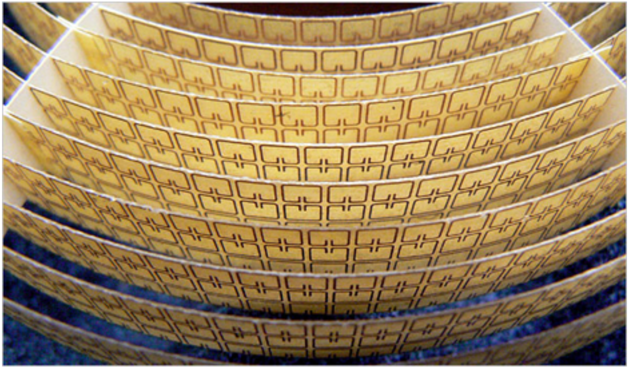
BENDS LIGHT LIKE A LENS. NO SHADOW/REFLECTION.

a) Measured free field
b) Measured scattering without cloak
c) Full parameter simulation
d) Reduced parameter simulation
e) Measured scattering with cloak

Scale: Instantaneous field intensity

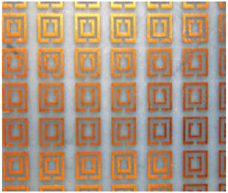



b
z
y
x
Screen
Projector
Cloak



# Metamaterials

- Engineered materials
- Construct and embed electric and magnetic dipoles
- Only works for wavelengths larger than dipole size
- Tailor dipole arrangement as desired
- Total control over electromagnetic response of the material
- Need not be isotropic or homogeneous
- Allows for bizarre material properties (e.g. negative refractive index)

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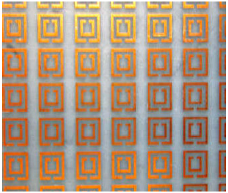

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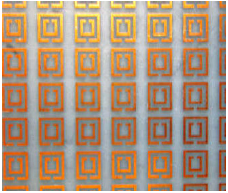

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 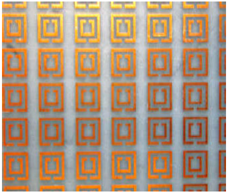

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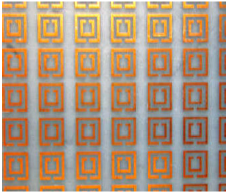

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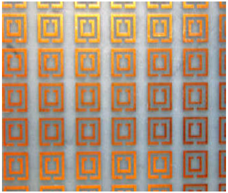
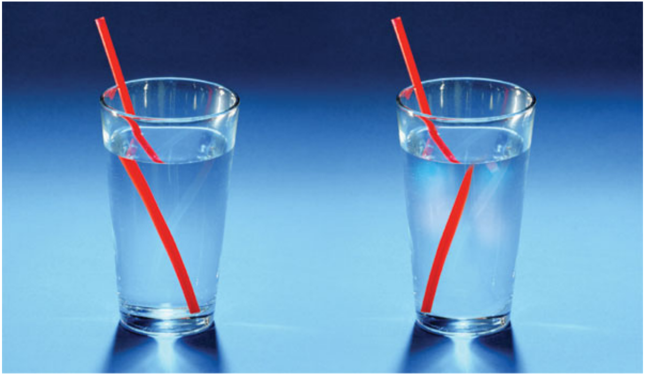
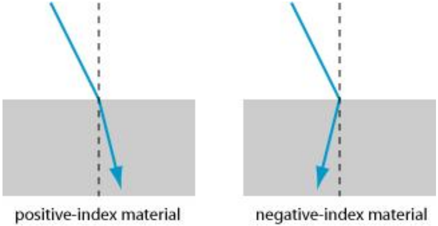
 




positive-index material
negative-index material

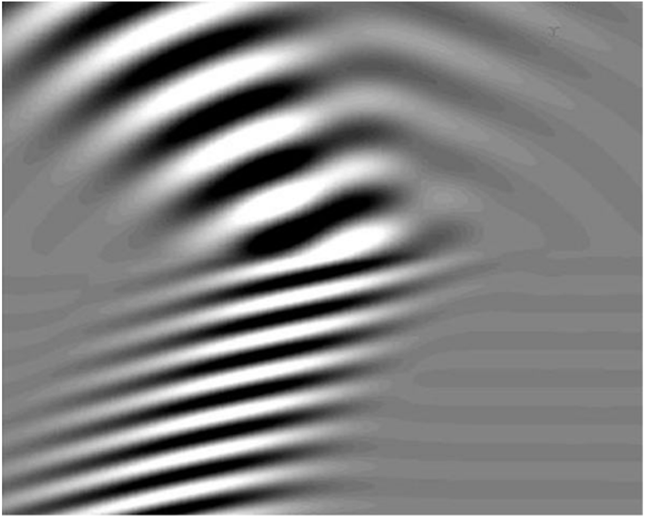
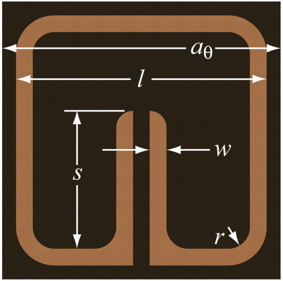
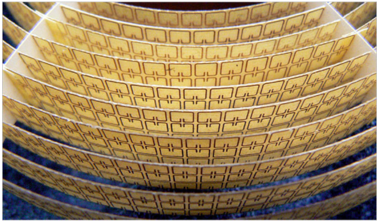


Image: Anthony Hoffman

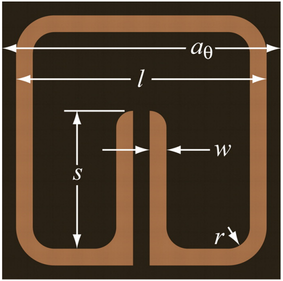



| cyl. | $r$ | $s$ | $\mu_r$ |
|---|---|---|---|
| 1 | 0.260 | 1.654 | 0.003 |
| 2 | 0.254 | 1.677 | 0.023 |
| 3 | 0.245 | 1.718 | 0.052 |
| 4 | 0.230 | 1.771 | 0.085 |
| 5 | 0.208 | 1.825 | 0.120 |
| 6 | 0.190 | 1.886 | 0.154 |
| 7 | 0.173 | 1.951 | 0.188 |
| 8 | 0.148 | 2.027 | 0.220 |
| 9 | 0.129 | 2.110 | 0.250 |
| 10 | 0.116 | 2.199 | 0.279 |

Schurig, et. al. 2006



- Precise design and engineering of complicated dipole arrangement
- Inverse problem: Given desired field behavior, what are required material parameters?

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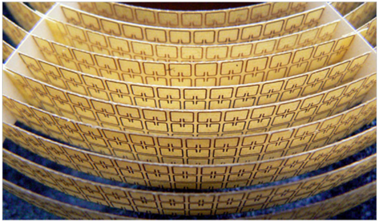

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Typical 3-dimensional vector representation of electrodynamics:

### Maxwell's Equations

$$\nabla \cdot \mathbf{D} = \rho, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}, \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

### Potentials

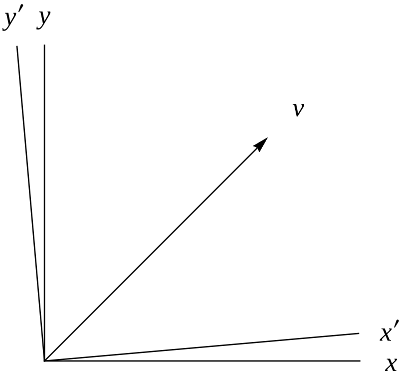
$$\mathbf{E} = \nabla \varphi, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

### Constitutive Relations

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}$$

Transformation Optics is based on:

1. The covariance of Maxwell's equations
2. Passive vs. Active transformations



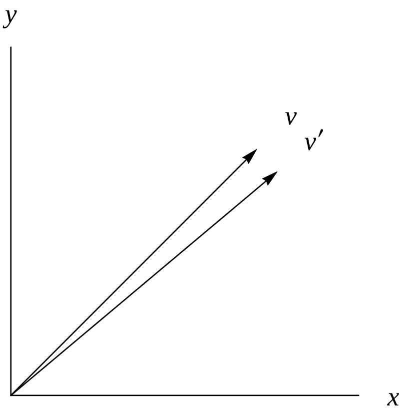

## Transformed Maxwell Eqs.

$$\nabla' \cdot \mathbf{B}' = 0, \quad \nabla' \times \mathbf{E}' + \frac{\partial \mathbf{B}'}{\partial t'} = 0$$

$$\nabla' \cdot \mathbf{D}' = \rho', \quad \nabla' \times \mathbf{H}' - \frac{\partial \mathbf{D}'}{\partial t'} = \mathbf{J}'$$

## Transformed Constitutives

$$\mathbf{D}' = \varepsilon' \mathbf{E}', \quad \mathbf{B}' = \mu' \mathbf{H}'$$




## Transformed Maxwell Eqs.

$$\nabla \cdot \mathbf{B}' = 0, \quad \nabla \times \mathbf{E}' + \frac{\partial \mathbf{B}'}{\partial t} = 0$$

$$\nabla \cdot \mathbf{D}' = \rho', \quad \nabla \times \mathbf{H}' - \frac{\partial \mathbf{D}'}{\partial t} = \mathbf{J}'$$

## Transformed Constitutives

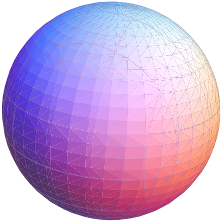
$$\mathbf{D}' = \varepsilon' \mathbf{E}', \quad \mathbf{B}' = \mu' \mathbf{H}'$$

Question: Given an active transformation that produces a new set of fields, can we find parameters $\varepsilon'$ and $\mu'$ such that the new fields are a solution?

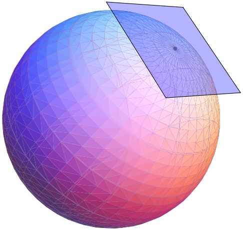
1. A Crash Course in Differential Geometry
   - manifolds, tangent and cotangent spaces
   - tensor products
   - exterior derivative
   - metric
   - volume
   - Hodge dual
   - geometry summary
2. Classical Electrodynamics in Vacuum
3. Classical Electrodynamics in Linear Dielectrics
4. Transformation Optics
5. Extensions of the Transformation method

For our purposes a manifold is a collection of points

- May have some intuitive shape

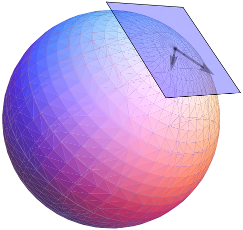


Can attach a flat "tangent space" to each point $p$, called $T_p(M)$



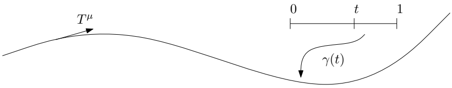
- Tangent space has same dimension as $M$
- Linear approximation of the manifold

## $T_p(M)$ is a vector space



- Tangent vectors live in $T_p(M)$
- Each point has its own tangent space

A parametric curve $\gamma(t)$ on $M$ is the image of $\gamma : \mathbb{R} \to M$.




- Tangent to the curve at $p$ is $T = \left.\frac{d\gamma}{dt}\right|_p$
- Tangent vectors at $p \leftrightarrow$ directional derivatives at $p$.
- $\left\{\frac{\partial}{\partial x^\mu}\right\}$ forms basis for $T_p(M)$
- Collection of $T_p(M)\ \forall\ p \in M$ is labeled $T(M)$
- $V \in T(M)$ is a *vector field*

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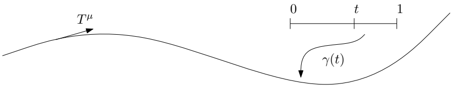


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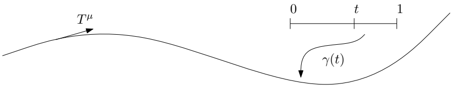


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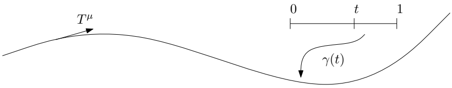


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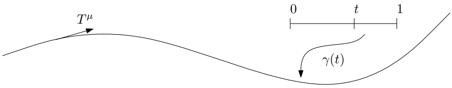


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Cotangent Space: $T_p^*(M) =$ adjoint of $T_p(M)$

- Space of $\mathbb{R}$-valued functions on $T_p(M)$
- For $\alpha \in T_p^*(M)$, $v \in T_p(M)$, then $\alpha(v) = r$ for $r \in \mathbb{R}$
- $m$-dimensional vector space
- If $\{\partial_\mu\}$ is basis of $T_p(M)$, then $\{\mathrm{d}x^\mu\}$ is basis of $T_p^*(M)$
  - $\mathrm{d}x^\mu(\partial_\nu) = \mathrm{d}x^\mu \frac{\partial}{\partial x^\nu} = \delta^\mu_\nu$
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- $\alpha \in T^*(M)$ called a *differential 1-form*

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Tensor products generalize multiplication between vector spaces $\mathbf{v}, \mathbf{u}, \mathbf{w} \in T_p(M)$

## Tensor Product

General bilinear operation

- $(\mathbf{v}+\mathbf{u}) \otimes \mathbf{w} = \mathbf{v} \otimes \mathbf{w} + \mathbf{u} \otimes \mathbf{w}$
- $\mathbf{v} \otimes (\mathbf{u}+\mathbf{w}) = \mathbf{v} \otimes \mathbf{u} + \mathbf{v} \otimes \mathbf{w}$
- $a(\mathbf{v} \otimes \mathbf{u}) = (a\mathbf{v}) \otimes \mathbf{u} = \mathbf{v} \otimes (a\mathbf{u})$

## Wedge Product

*Alternating* bilinear operation

- Also require $\mathbf{v} \wedge \mathbf{u} = -\mathbf{u} \wedge \mathbf{v}$
- $\mathbf{u} \wedge \mathbf{u} = 0$
- Generalizes cross product

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Given $\mathbf{v} = v^1 e_1 + v^2 e_2 + v^3 e_3$, and $\mathbf{u} = u^1 e_1 + u^2 e_2 + u^3 e_3$

$$\mathbf{v} \wedge \mathbf{u} = (v^1 u^2 - v^2 u^1)(e_1 \wedge e_2) + (v^1 u^3 - v^3 u^1)(e_1 \wedge e_3) + (v^2 u^3 - v^3 u^2)(e_2 \wedge e_3)$$

- $\mathbf{v} \wedge \mathbf{u} \in \wedge^2 T_p(M)$ ($2^{nd}$ exterior product)
- $\wedge^2 T_p(M)$ has basis $\{(e_1 \wedge e_2), (e_1 \wedge e_3), (e_2 \wedge e_3)\}$
- Extend to $\wedge^k T_p(M)$ ("alternating $k$-vectors")
- Similarly $\wedge^k T_p^*(M)$ ("alternating $k$-covectors")
- Bundled into $\wedge^k T(M)$ and $\wedge^k T^*(M)$
- Alternating tensors of rank $\binom{k}{0}$ or $\binom{0}{k}$

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Metric: symmetric, bilinear 2-form, $\mathbf{g} \in (T^*)^2(M)$. Defines inner product on $T(M)$.

- Basically a function that takes two tangent vectors and returns a number

$$\mathbf{g}(\mathbf{V}, \mathbf{U}) = r$$

- $\mathbf{g}(\mathbf{V}, *)$ is a function that takes one tangent vector and returns a number
  - But this is a 1-form!
- So a metric induces a map

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by

$$g_{\mu\nu} V^{\mu} = V_{\nu} \Rightarrow V_{\nu} U^{\nu} = g_{\mu\nu} V^{\mu} U^{\nu}$$

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For $m = Dim(M)$, the vector space $\wedge^m T^*(M)$ is 1D.

- Implies any $\alpha \in \wedge^m T^*(M) \propto$ some $\omega \in \wedge^m T^*(M)$
- $\omega$ called the *volume form*
- in local coordinates, a natural, covariant choice is

$$\omega = \sqrt{|g|}(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m),$$

- $\omega$ induces a unique map

$$\omega : \wedge^{m-k} T(M) \to \wedge^k T^*(M).$$

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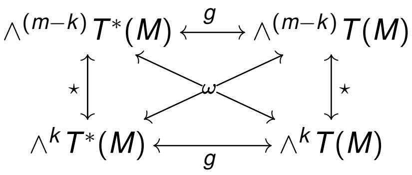


- $\mathbf{g} : \binom{k}{0} \leftrightarrow \binom{0}{k}$ alt. tensors
- $\omega : \binom{k}{0} \leftrightarrow \binom{0}{m-k}$ alt. tensors
- $\star = \omega \circ g : \binom{m-k}{0} \leftrightarrow \binom{k}{0}$ alt. tensors and $\binom{0}{m-k} \leftrightarrow \binom{0}{k}$ alt. tensors
- $\star$ called “Hodge dual”

Want to describe elctrodynamics on manifolds

- A manifold is a collection of points
  - tangent & cotangent space at each point
  - alternating ($\wedge$) products of tangent/cotangent spaces
  - metric **g** defines inner product (symmetric matrix)
  - canonical volume form $\omega$
- "$\wedge$" constructs alternating $k$-vector fields and $k$-forms
  - Represented as skew-symmetric matrices
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## Classical electrodynamics in vacuum

- Combine $(\varphi, \vec{A})$ into 1-form $\mathbf{A} = A_\mu$
- The *field strength* tensor $\mathbf{F} \in \wedge^2 T^*(M)$ encodes $\vec{E}$ and $\vec{B}$

$$\mathbf{F} = \mathrm{d}\mathbf{A} \Rightarrow F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$

- In local frame (or Minkowski space)

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

- Recall $\mathrm{d}(\mathrm{d}\mathbf{A}) = 0$ for any 1-form $\mathbf{A}$
- $\mathrm{d}\mathbf{F} = 0 \quad \Leftrightarrow \quad$ Homogeneous Maxwell Eqs.

## Inhomogeneous eqs. $\Rightarrow$ require action

$$S = \int_M \mathcal{L}$$

- $\mathcal{L}$ must be a 4-form constructed from **A** or **F**
- Use only operations $\wedge$, $\mathrm{d}$, and $\star$.
- $\mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A} = 0$ by antisymmetry of $\wedge$
- $\mathbf{F} \wedge \mathbf{F}$ is total divergence $\to$ No good!
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$$\int_M (\mathbf{F} \wedge \star\mathbf{F}) = \int_M d^4x \sqrt{|g|}(F^{\mu\nu} F_{\mu\nu})$$

$$(\star\mathbf{F})_{\mu\nu} = \frac{1}{2}\sqrt{|g|}\,\epsilon_{\mu\nu\alpha\beta} g^{\alpha\gamma} g^{\beta\delta} F_{\gamma\delta}, \quad (\star\mathbf{F})_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}$$

- The field strength **F** encodes information about the fields:
  - Electric field strength and magnetic flux.
- Let the *excitation tensor* **G** encode information about
  - Electric flux and magnetic field strength.
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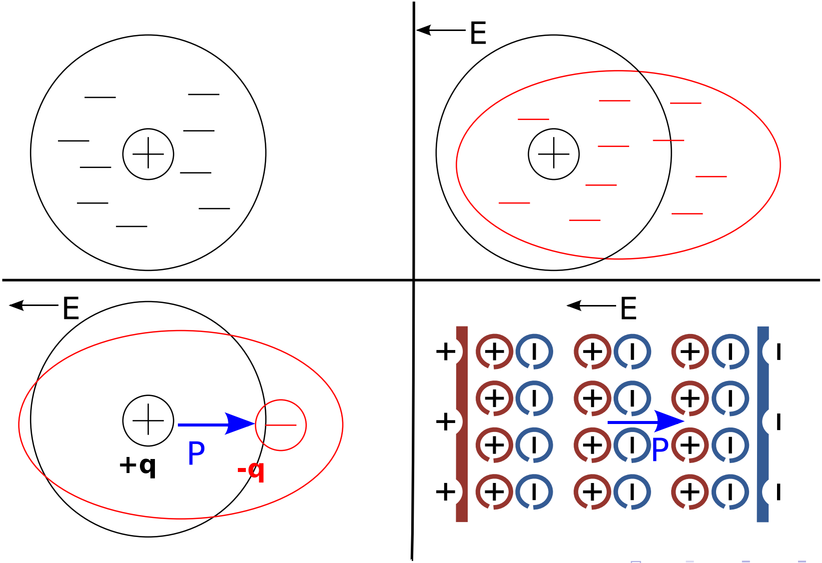
The action is generalized to

$$S = \int \frac{1}{2}\mathbf{F} \wedge \mathbf{G} + \mathbf{J} \wedge \mathbf{A}$$

Vary with respect to **A**

$$\mathrm{d}\mathbf{G} = \mathbf{J} \qquad \text{Inhomogeneous Maxwell Eqs.}$$


E
E
+q
P
-q
E
P

Effective theory accounts for average atomic response to applied fields.

- Applied $\vec{E}$ induces dipole far field $\vec{P}$

$$\vec{E}_{\text{net}} = \vec{E}_{\text{applied}} + \vec{P} = \vec{E}_{\text{applied}} + \bar{\bar{\chi}}_E \vec{E}_{\text{applied}} = (\bar{\bar{1}} + \bar{\bar{\chi}}_E)\vec{E}_{\text{applied}}$$

- New constitutive relation

$$\vec{D}_{\text{net}} = (\bar{\bar{1}} + \bar{\bar{\chi}}_E)\vec{E}_{\text{applied}} = \bar{\bar{\varepsilon}}\vec{E}_{\text{applied}}$$

- Macroscopic equations contain material-dependent set of constitutive relations
- Take the minimal approach: Extend vacuum relations to more general linear map

$$\mathbf{G} = \star(\chi \mathbf{F})$$

$$G_{\mu\nu} = \star_{\mu\nu}{}^{\alpha\beta}(\chi \mathbf{F})_{\alpha\beta}$$

- Properties of $\chi$:
  - Antisymmetric on $1^{st}$ and $2^{nd}$ sets of indices
  - In vacuum, $\chi_{vac}(\mathbf{F}) = \mathbf{F}$
  - Maximum of 36 independent components

$\chi_{vac}\mathbf{F} = \mathbf{F}$ is sufficient to specify all components of $\chi_{vac}$

$$(\chi_{vac})_{\gamma\delta}{}^{\sigma\rho} = \frac{1}{2}\begin{pmatrix}
\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&1&0&0\\-1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&1&0\\0&0&0&0\\-1&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\-1&0&0&0 \end{pmatrix} \\
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\end{pmatrix}$$

- $\chi_{vac}$ is unique, independent of coordinate choice

Components of $\mathbf{G} = \star(\chi \mathbf{F})$ can be collected as

$$\vec{D} = \bar{\bar{\varepsilon}}^c \vec{E} + {}^b\bar{\bar{\gamma}}^c \vec{B}, \quad \vec{H} = \bar{\bar{\mu}}^c \vec{B} + {}^e\bar{\bar{\gamma}}^c \vec{E}$$

$$\vec{D} = \bar{\bar{\varepsilon}} \vec{E} + {}^h\bar{\bar{\gamma}} \vec{H}, \quad \vec{B} = \bar{\bar{\mu}} \vec{H} + {}^e\bar{\bar{\gamma}} \vec{E}$$

Components of $\mathbf{G} = \star(\chi \mathbf{F})$ can be collected as

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Rearrange to usual representation:

$$\vec{D} = \bar{\bar{\varepsilon}} \vec{E} + {}^h\bar{\bar{\gamma}} \vec{H}, \quad \vec{B} = \bar{\bar{\mu}} \vec{H} + {}^e\bar{\bar{\gamma}} \vec{E}$$

$$\vec{D} = \bar{\bar{\varepsilon}}^{c} \vec{E} + {}^{b}\bar{\bar{\gamma}}^{c} \vec{B}, \quad \vec{H} = \bar{\bar{\mu}}^{c} \vec{B} + {}^{e}\bar{\bar{\gamma}}^{c} \vec{E}$$

$$\vec{D} = \bar{\bar{\varepsilon}} \vec{E} + {}^{h}\bar{\bar{\gamma}} \vec{H}, \quad \vec{B} = \bar{\bar{\mu}} \vec{H} + {}^{e}\bar{\bar{\gamma}} \vec{E}$$

Easily switch back and forth with:

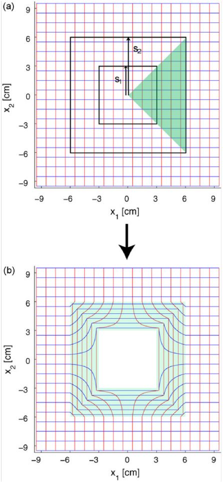
$$\bar{\bar{\mu}} = \left(\bar{\bar{\mu}}^{c}\right)^{-1}, \; \bar{\bar{\varepsilon}} = \bar{\bar{\varepsilon}}^{c} - \left({}^{b}\bar{\bar{\gamma}}^{c}\right) \bar{\bar{\mu}} \left({}^{e}\bar{\bar{\gamma}}^{c}\right), \; {}^{h}\bar{\bar{\gamma}} = \left({}^{b}\bar{\bar{\gamma}}^{c}\right) \bar{\bar{\mu}}, \\ {}^{e}\bar{\bar{\gamma}} = -\bar{\bar{\mu}} \left({}^{e}\bar{\bar{\gamma}}^{c}\right)$$

- Essentially equivalent representations
- $3 \times 3$ matrices are NOT tensors

$$
{\chi_{\gamma\delta}}^{\sigma\rho} = \frac{1}{2}
\begin{pmatrix}
\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & * & * & * \\
\begin{pmatrix} 0 & -\varepsilon^c_{xx} & -\varepsilon^c_{xy} & -\varepsilon^c_{xz} \\ \varepsilon^c_{xx} & 0 & {}^b\gamma^c_{xz} & -{}^b\gamma^c_{xy} \\ \varepsilon^c_{xy} & -{}^b\gamma^c_{xz} & 0 & {}^b\gamma^c_{xx} \\ \varepsilon^c_{xz} & {}^b\gamma^c_{xy} & -{}^b\gamma^c_{xx} & 0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & * & * \\
\begin{pmatrix} 0 & -\varepsilon^c_{yx} & -\varepsilon^c_{yy} & -\varepsilon^c_{yz} \\ \varepsilon^c_{yx} & 0 & {}^b\gamma^c_{yz} & -{}^b\gamma^c_{yy} \\ \varepsilon^c_{yy} & -{}^b\gamma^c_{yz} & 0 & {}^b\gamma^c_{yx} \\ \varepsilon^c_{yz} & {}^b\gamma^c_{yy} & -{}^b\gamma^c_{yx} & 0 \end{pmatrix} & \begin{pmatrix} 0 & {}^e\gamma^c_{zx} & {}^e\gamma^c_{zy} & {}^e\gamma^c_{zz} \\ -{}^e\gamma^c_{zx} & 0 & -\mu^c_{zz} & \mu^c_{zy} \\ -{}^e\gamma^c_{zy} & \mu^c_{zz} & 0 & -\mu^c_{zx} \\ -{}^e\gamma^c_{zz} & -\mu^c_{zy} & \mu^c_{zx} & 0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & * \\
\begin{pmatrix} 0 & -\varepsilon^c_{zx} & -\varepsilon^c_{zy} & -\varepsilon^c_{zz} \\ \varepsilon^c_{zx} & 0 & {}^b\gamma^c_{zz} & -{}^b\gamma^c_{zy} \\ \varepsilon^c_{zy} & -{}^b\gamma^c_{zz} & 0 & {}^b\gamma^c_{zx} \\ \varepsilon^c_{zz} & {}^b\gamma^c_{zy} & -{}^b\gamma^c_{zx} & 0 \end{pmatrix} & \begin{pmatrix} 0 & -{}^e\gamma^c_{yx} & -{}^e\gamma^c_{yy} & -{}^e\gamma^c_{yz} \\ {}^e\gamma^c_{yx} & 0 & \mu^c_{yz} & -\mu^c_{yy} \\ {}^e\gamma^c_{yy} & -\mu^c_{yz} & 0 & \mu^c_{yx} \\ {}^e\gamma^c_{yz} & \mu^c_{yy} & -\mu^c_{yx} & 0 \end{pmatrix} & \begin{pmatrix} 0 & {}^e\gamma^c_{xx} & {}^e\gamma^c_{xy} & {}^e\gamma^c_{xz} \\ -{}^e\gamma^c_{xx} & 0 & -\mu^c_{xz} & \mu^c_{xy} \\ -{}^e\gamma^c_{xy} & \mu^c_{xz} & 0 & -\mu^c_{xx} \\ -{}^e\gamma^c_{xz} & -\mu^c_{xy} & \mu^c_{xx} & 0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}
\end{pmatrix}
$$

- $*$ indicates entries that are antisymmetric on either the $1^{st}$ or $2^{nd}$ set of indices

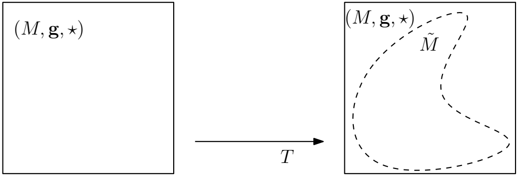
- Potential 1-form **A**
- Field strength tensor $\mathbf{F} = \mathrm{d}\mathbf{A}$ (alt. 2-form)
  - Electric field strength, $E$
  - Magnetic flux, $B$
- Excitation tensor **G** (alt. 2-form)
  - Magnetic field strength, $H$
  - Electric flux, $D$
- Constitutive relation $\mathbf{G} = \star(\chi \mathbf{F})$
  - Vacuum is trivial dielectric s.t. $\chi_{vac}\mathbf{F} = \mathbf{F}$
- Maxwell's equations
  - $\mathrm{d}\mathbf{F} = 0$
  - $\mathrm{d}\mathbf{G} = \mathbf{J}$



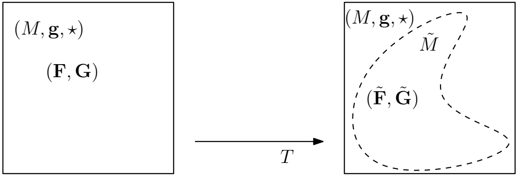

Rahm, et. al. 2008

Typical picture of transformation optics

- Start with empty Minkowski space
- Perform a coord. transformation
- “Open a hole in space”
- Fields dragged with coord. points
- Fields can’t get into the hole
- Find equivalent material




- Imagine $T : M \to \tilde{M} \subseteq M$, **g** unaffected
- Initial $(\mathbf{F}, \mathbf{G})$ dragged to new $(\tilde{\mathbf{F}}, \tilde{\mathbf{G}})$
  - Supported only on $\tilde{M}$
- New field configuration must be supported by new $\tilde{\chi}$
  - Physically: change fields $\Leftrightarrow$ change material, (e.g. dielectric slab in parallel plate capacitor)
  - $\chi_{\text{initial}}$ may be vacuum, not necessary!




- Imagine $T : M \to \tilde{M} \subseteq M$, $\mathbf{g}$ unaffected
- Initial $(\mathbf{F}, \mathbf{G})$ dragged to new $(\tilde{\mathbf{F}}, \tilde{\mathbf{G}})$
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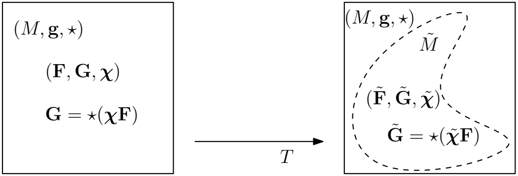


- Imagine $T : M \to \tilde{M} \subseteq M$, $\mathbf{g}$ unaffected
- Initial $(\mathbf{F}, \mathbf{G})$ dragged to new $(\tilde{\mathbf{F}}, \tilde{\mathbf{G}})$
  - Supported only on $\tilde{M}$
- New field configuration must be supported by new $\tilde{\chi}$
  - Physically: change fields $\Leftrightarrow$ change material, (e.g. dielectric slab in parallel plate capacitor)
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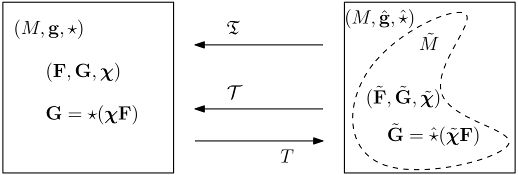


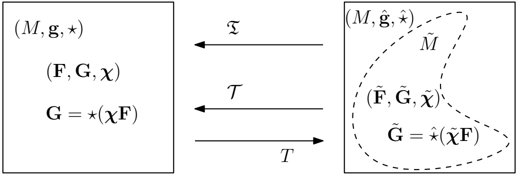

Some subtleties involved:

- To be rigorous, need $\mathfrak{T}$ to transform $\mathbf{g}$
- Let $\mathfrak{T}$ be identity

$$\begin{cases} \mathfrak{T} : M \to M \\ \mathfrak{T}^*(\mathbf{g}) = \hat{\mathbf{g}} \end{cases}$$

- Fields $(\mathbf{F}, \mathbf{G})$ NOT transformed by $T$
- Need $\mathcal{T}$ to transform $(\mathbf{F}, \mathbf{G})$
  - If $\exists\ T^{-1}$, then we can use $\mathcal{T} = T^{-1}$
  - $\mathcal{T}^*$ called the *pullback* of $\mathcal{T}$ (did not discuss)

$$\begin{cases} \mathcal{T} : \tilde{M} \subseteq M \to M \\ \mathcal{T}^*(\mathbf{F}) = \tilde{\mathbf{F}} \end{cases}$$

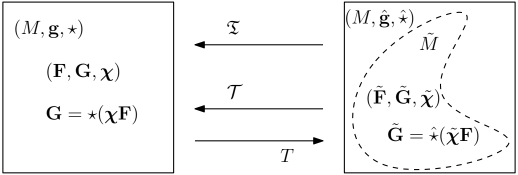



$\tilde{\mathbf{G}} = \mathcal{T}^*\mathbf{G}$, so at $x \in \tilde{M}$

$$\tilde{\mathbf{G}}_x = \mathcal{T}^*\left(\mathbf{G}_{\mathcal{T}(x)}\right) = \mathcal{T}^*\left(\star_{\mathcal{T}(x)} \circ \chi_{\mathcal{T}(x)} \circ \mathbf{F}_{\mathcal{T}(x)}\right)$$

But from $\tilde{\mathbf{G}} = \hat{\star}(\tilde{\chi}\tilde{\mathbf{F}})$ we also have

$$\tilde{\mathbf{G}}_x = \hat{\star}_x \circ \tilde{\chi}_x \circ \mathcal{T}^*\left(\mathbf{F}_{\mathcal{T}(x)}\right)$$

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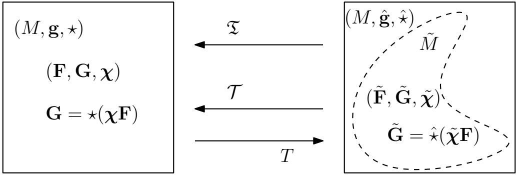


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But from $\tilde{\mathbf{G}} = \hat{\star}(\tilde{\chi}\tilde{\mathbf{F}})$ we also have

$$\tilde{\mathbf{G}}_x = \hat{\star}_x \circ \tilde{\chi}_x \circ \mathcal{T}^*\left(\mathbf{F}_{\mathcal{T}(x)}\right)$$

Since $\tilde{\mathbf{G}}_x = \tilde{\mathbf{G}}_x$

$$\hat{\star}_x \circ \tilde{\chi}_x \circ \mathcal{T}^*\left(\mathbf{F}_{\mathcal{T}(x)}\right) = \mathcal{T}^*\left(\star_{\mathcal{T}(x)} \circ \chi_{\mathcal{T}(x)} \circ \mathbf{F}_{\mathcal{T}(x)}\right)$$

$$\hat{\star}_x \circ \tilde{\boldsymbol{\chi}}_x \circ \mathcal{T}^* \left(\mathbf{F}_{\mathcal{T}(x)}\right) = \mathcal{T}^* \left(\star_{\mathcal{T}(x)} \circ \boldsymbol{\chi}_{\mathcal{T}(x)} \circ \mathbf{F}_{\mathcal{T}(x)}\right)$$

Can be solved for $\tilde{\chi}$:

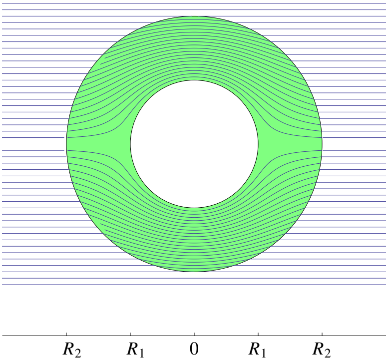
$$\tilde{\chi}_{\eta\tau}{}^{\pi\theta}(x) = -\hat{\star}_{\eta\tau}{}^{\lambda\kappa}\Big|_x \Lambda^{\alpha}{}_{\lambda}\Lambda^{\beta}{}_{\kappa} \star_{\alpha\beta}{}^{\mu\nu}\Big|_{\mathcal{T}(x)} \chi_{\mu\nu}{}^{\sigma\rho}\Big|_{\mathcal{T}(x)} (\Lambda^{-1})^{\pi}{}_{\sigma} (\Lambda^{-1})^{\theta}{}_{\rho}$$

Thompson 2010

## Features

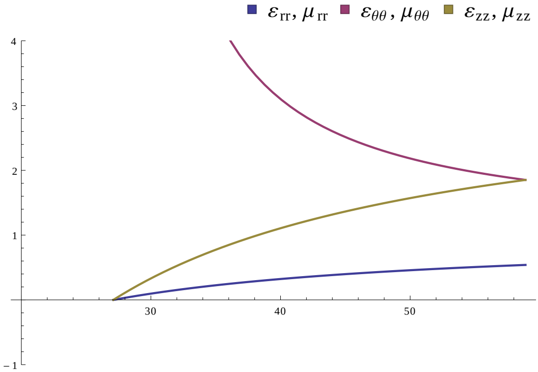
- $\boldsymbol{\Lambda}$ is Jacobian matrix of $\mathcal{T}$
- $\boldsymbol{\Lambda}^{-1}$ is matrix inverse of $\boldsymbol{\Lambda}$
- Initial $\chi$ can be non-vacuum
- $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}^{-1}$ evaluated at $x$
- $\tilde{\chi}$ undetermined for $x \notin \tilde{M}$
- Can be non-Minkowskian

$$\tilde{\chi}_{\eta\tau}{}^{\pi\theta}(x) = -\hat{\star}_{\eta\tau}{}^{\lambda\kappa}\Big|_{x} \Lambda^{\alpha}{}_{\lambda}\Lambda^{\beta}{}_{\kappa} \star_{\alpha\beta}{}^{\mu\nu}\Big|_{\mathcal{T}(x)} \chi_{\mu\nu}{}^{\sigma\rho}\Big|_{\mathcal{T}(x)} (\Lambda^{-1})^{\pi}{}_{\sigma} (\Lambda^{-1})^{\theta}{}_{\rho}$$




$$\mathcal{T}(t,r,\theta,z) = \left(t, \frac{(r-R_1)R_2}{(R_2-R_1)}, \theta, z\right)$$

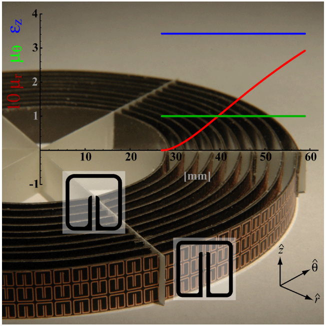
$$\bar{\bar{\varepsilon}} = \bar{\bar{\mu}} = \begin{pmatrix} 1-\frac{R_1}{r} & 0 & 0 \\ 0 & \left(1-\frac{R_1}{r}\right)^{-1} & 0 \\ 0 & 0 & \left(1-\frac{R_1}{r}\right)\left(\frac{R_2}{R_2-R_1}\right)^2 \end{pmatrix}$$




- Some parameters $< 1$
- Complicated, anisotropic medium. How to realize?

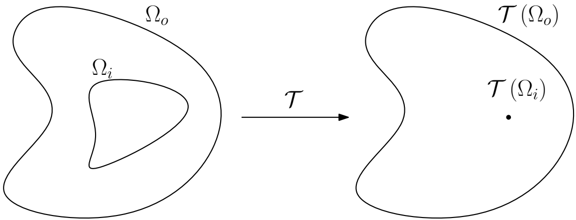
# Parameter reduction: trade performance for fabrication

$$\nabla \cdot \varepsilon(\mathbf{x})\vec{E} = 0$$
$$\nabla \times \mu^{-1}(\mathbf{x})\vec{B} - \varepsilon(\mathbf{x})\frac{\partial \vec{E}}{\partial t} = 0$$




- $\mu \to f(x)\mu, \quad \varepsilon \to f^{-1}(x)\varepsilon$
- Rescale so $\mu_{\theta\theta} = 1$.
- Single polarization.

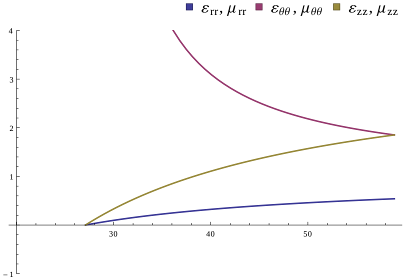
- $\mathcal{T}(t, r, \theta, z) = \left(t, \frac{(r - R_1)R_2}{(R_2 - R_1)}, \theta, z\right)$ was linear choice
- Not unique



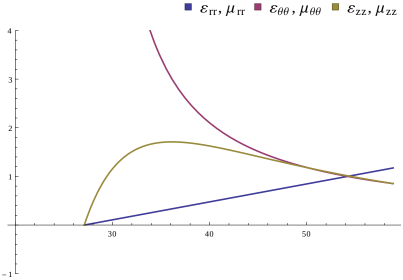

- Could let boundary determine cloak
- e.g. let $\mathcal{T}$ be harmonic map (Hu et. al. 2009)

Assume $\mathcal{T}$ harmonic with $\mathcal{T}(R_2) = R_2$ and $\mathcal{T}(R_1) = 0$

$$\nabla^2 \mathcal{T} = 0 \quad \Rightarrow \quad \mathcal{T}(r) = \frac{R_2^2}{R_2 - R_1}\left(1 - \frac{R_1}{r}\right)$$




$\mathcal{T}$ linear




$\mathcal{T}$ harmonic

$$\tilde{\chi}_{\eta\tau}{}^{\pi\theta}(x) = -\hat{\star}_{\eta\tau}{}^{\lambda\kappa}\Big|_x \Lambda^\alpha{}_\lambda \Lambda^\beta{}_\kappa \star_{\alpha\beta}{}^{\mu\nu}\Big|_{\mathcal{T}(x)} \chi_{\mu\nu}{}^{\sigma\rho}\Big|_{\mathcal{T}(x)} (\Lambda^{-1})^\pi{}_\sigma (\Lambda^{-1})^\theta{}_\rho$$

- Time-mixing transformations (Cummer and Thompson, 2011)

$$\mathcal{T}(t,x,y,z) = \left(\frac{t}{ax+b}, x, y, z\right), \quad x \neq 0$$

- Applications in relative motion via boost (Thompson, et. al. 2011)
- Non-vacuum prior media, $\chi_{\text{initial}} \neq \chi_{vac}$ (Thompson 2010)
- Non-Minkowskian applications (e.g. Earth orbit) (Thompson 2012)
- Analog space-times (Thompson and Frauendiener 2010)

One path to acoustic cloaking

Pressure perturbations:

$$\ddot{p} = B \frac{1}{\sqrt{\gamma}} \partial_i \left( \sqrt{\gamma} \rho^{ij} \partial_j p \right)$$

- $B$ = bulk modulus
- $\rho^{ij}$ = inverse density matrix
- $\gamma$ = spatial metric
- Form invariant under spatial coordinate transformations
- Can do transformation acoustics *with spatial transformations*

Potential function $\phi_1$ for velocity perturbation $v_1 = \nabla\phi_1$:

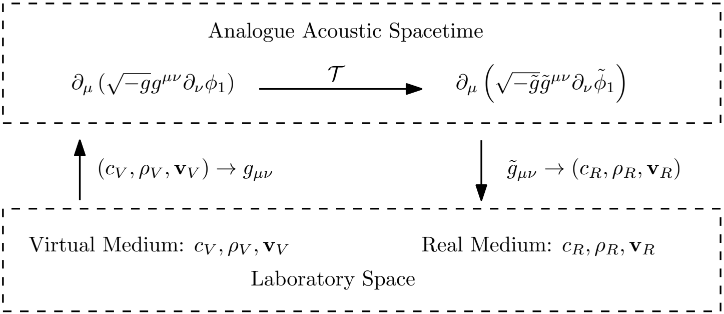
$$-\partial_t\left(\rho c^{-2}(\partial_t\phi_1 + \mathbf{v}\cdot\nabla\phi_1)\right) + \nabla\cdot\left(\rho\nabla\phi_1 - \rho c^{-2}(\partial_t\phi_1 + \mathbf{v}\cdot\nabla\phi_1)\mathbf{v}\right) = 0$$

- $\mathbf{v}$ = background fluid velocity
- $\rho$ = isotropic mass density
- $c = \sqrt{\frac{B}{\rho}}$ local sound speed

$$\Updownarrow$$

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi_1) \qquad g_{\mu\nu} = \frac{\rho}{c}\begin{pmatrix} -c^2 + v^i v^j\gamma_{ij} & \vdots & -v^j\gamma_{ij} \\ \cdots\cdots & \cdot & \cdots \\ -v^j\gamma_{ij} & \vdots & \gamma_{ij} \end{pmatrix}$$

- $\phi_1$ described by massless KG eq. in *acoustic analogue spacetime*




- Enables expanded set of transformations
- Time transformations $\Rightarrow$ frequency shifts

$$\rho(\mathbf{x})c(\mathbf{x})\frac{\partial u}{\partial t} = \nabla \cdot (\kappa(\mathbf{x}\nabla u) + s(\mathbf{x}, t))$$

- $\rho(\mathbf{x}) =$ density
- $c(\mathbf{x}) =$ specific heat capacity
- $\kappa(\mathbf{x}) =$ matrix-valued thermal conductivity

- Invariant to *spatial transformations*
- Can do transformation thermodynamics
- Thermal cloak

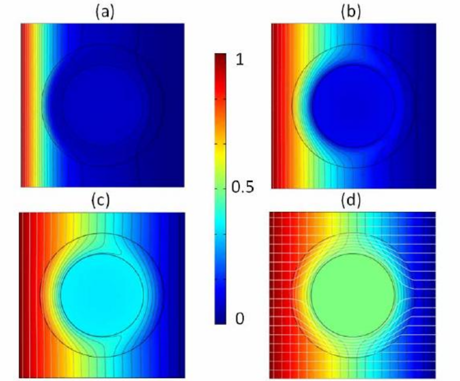



Fig. 2. Diffusion of heat from the left on a cloak with $R_1 = 2.10^{-4}$m and $R_2 = 3.10^{-4}$m. The temperature is normalized throughout time on the left side of the cell. Snapshots of temperature distribution at $t = 0.001s$ (a), $t = 0.005s$ (b), $t = 0.02s$ (c), $t = 0.05s$ (d). Streamlines of thermal flux are also represented with white color in panel (d). The mesh formed by streamlines and isothermal values illustrates the deformation of the transformed thermal space: the central disc ('invisibility region') is a hole in the metric, which is curved smoothly around it.

1. A Crash Course in Differential Geometry
2. Classical Electrodynamics in Vacuum
3. Classical Electrodynamics in Linear Dielectrics
4. Transformation Optics
5. Extensions of the Transformation method
6. Conclusions
   - future directions

Outstanding issues:

- Complicated, anisotropic, inhomogeneous media
  - Conformal transformations $\rightarrow$ isotropic, inhomogeneous
  - Quasi-conformal transformations $\rightarrow$ neglectable anisotropy
  - Other classes of restricted transformations?
  - Other optimization tools?
- Perfect transformation media unrealistic
  - Incorporate dispersion, dissipation
  - Geometrically? Covariantly?

- Transformation method based on
  - invariance of system of equations
  - active transformations
- Trans. Optics great potential for future application
- Many avenues still to explore
- Trans. Acoustics and Trans. thermodynamics popular too