Integrable lattice equations, slow degree growth and possible signatures over finite fields

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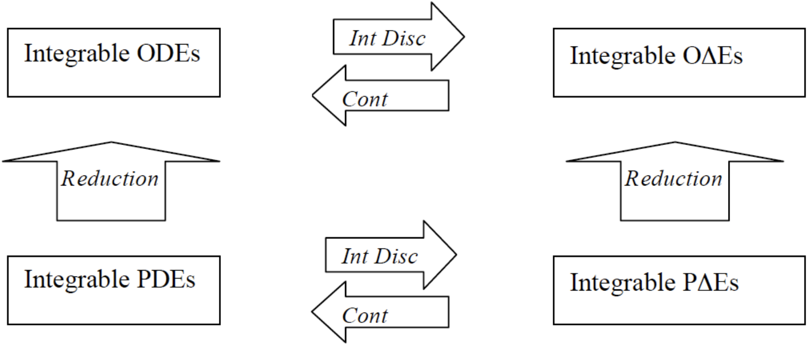
Introduction

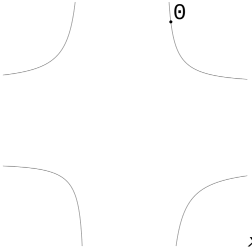
- What is integrability? Answer: many definitions.
- Algebraic entropy: integrability associated with "low complexity" - vanishing entropy (Bellon,Viallet, Hietarinta, Tremblay, Ramani, Grammaticos, Halburd...Arnold, Friedland,Milnor, Diller, Favre, Bedford, Kim, Hasselblatt, Propp, Silverman, Blanc and Cantat)
- Integrable lattice equations also have slow growth of degrees heuristically, i.e. a lot of cancellations
- Can we prove it?
- Can you see slow growth over finite fields?

Integrable equations: continuous and discrete



Integrable OΔEs: McMillan Integrable Map

$$B = 2x^2y^2 + (3K+6)x^2 + (3K+6)y^2 - \left(\tfrac{283}{500}K^2 + \tfrac{2897}{500}K + \tfrac{1577}{250}\right)xy$$
$$- 8x - \tfrac{1}{5}y + K + 1 - t = 0. \qquad (1)$$



$$x' = -x - \frac{\delta y^2 + \epsilon y + \xi}{\alpha y^2 + \beta y + \gamma};\ y' = -y - \frac{\beta x'^2 + \epsilon x' + \lambda}{\alpha x'^2 + \delta x' + \kappa}$$

$(K, t) = (-7, 100)$; $(x_0, y_0) \approx (3, 6.71172)$

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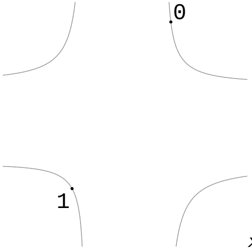

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$$(K,t) = (-7, 100);\ (x_0, y_0) \approx (3, 6.71172),\ (x_1, y_1) \approx (-3.47585, -4.21816)$$

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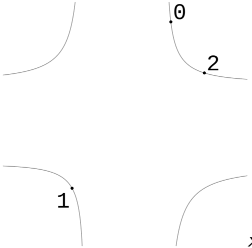

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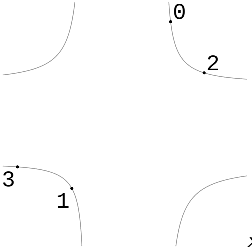

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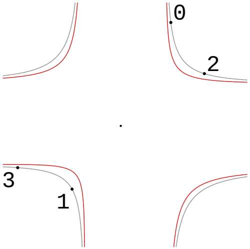
$(K,t) = (-7,100)$; $(x_0,y_0) \approx (3, 6.71172)$, $(x_1,y_1) \approx (-3.47585, -4.21816)$, $(x_2,y_2) \approx (5.19964, 3.35615)$, $(x_3,y_3) \approx (-7.04204, -2.80869)$.

Integrable OΔEs: McMillan Integrable Map

$$\begin{aligned} B = 2x^2y^2 + (3K+6)x^2 + (3K+6)y^2 - \left(\tfrac{283}{500}K^2 + \tfrac{2897}{500}K + \tfrac{1577}{250}\right)xy \\ - 8x - \tfrac{1}{5}y + K + 1 - t = 0. \end{aligned} \tag{5}$$



$K = -7;\ t = 100,\ t \approx -4.86693$

Integrable OΔEs: McMillan Integrable Map

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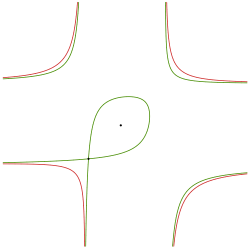

$K = -7;\ t \approx -4.86693,\ -55.17923$

Integrable OΔEs: McMillan Integrable Map

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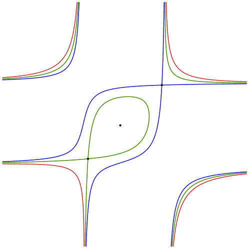

$K = -7;\ t \approx -4.86693,\ -55.17923,\ -94.90509$

Integrable OΔEs: McMillan Integrable Map

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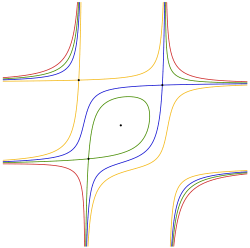

$K = -7$; $t \approx -4.86693, -55.17923, -94.90509, -149.03582$

Integrable OΔEs: McMillan Integrable Map

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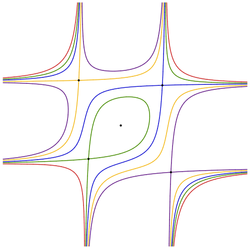


$K = -7$; $t \approx -4.86693, -55.17923, -94.90509, -149.03582, -196.17473.$

- McMillan maps are generalised by the 18 parameter family of integrable birational maps known as QRT maps (Quispel+R+Thompson, 1988)

What is algebraic entropy?

Given a rational map in an n dimensional space.

- Write it in projective coordinates $(x_0 : x_1 : \ldots : x_n)$, the map has the form

$$x_i \mapsto \phi_i(x_0, x_1, \ldots, x_n),$$

  where ϕ_i is a homogeneous polynomial.
- Cancelling common polynomial factors, the degree of the map and its iterates is well-defined.
- Let d_k be the degree of ϕ^k. The entropy is defined as

$$\epsilon := \lim_{k \to \infty} \frac{1}{k} \log(d_k)$$

- The limit exists, is invariant under birational conjugacy
- If $\epsilon = 0$, the growth is polynomial

Claim: Algebraic entropy is vanishing for integrable maps.

Question: What about lattice equations i.e. partial difference equations?

Plan

In this talk: we study integrable lattice rules and their perturbations

- We conjecture the gcd when we iterate the rules.
- We provide a recursive formula of the actual degrees (a linear partial difference equation with constant coefficients)
- We "prove" vanishing entropy for certain rules.
- We look for the signature of slow growth over finite fields and we use it as an integrability detector

List of integrable equations

We denote $x = u_{l,m}, x_1 = u_{l+1,m}, x_2 = u_{l,m+1}$ and $x_{12} = x_{l+1,m+1}$. The ABS list (Adler-Bobenko-Suris, 2003):

- List Q:

$$
\begin{aligned}
(Q_1) \quad & \alpha(x - x_2)(x_1 - x_{12}) - \beta(x - x_1)(x_2 - x_{12}) + \delta^2\alpha\beta(\alpha - \beta) = 0, \\
(Q_2) \quad & \alpha(x - x_2)(x_1 - x_{12}) - \beta(x - x_1)(x_2 - x_{12}) + \\
& \alpha\beta(\alpha - \beta)(x + x_1 + x_2 + x_{12}) \\
(Q_3) \quad & (\beta^2 - \alpha^2)(xx_{12} + x_1x_2) + \beta(\alpha^2 - 1)(xx_1 + x_2x_{12}) - \\
& \alpha(\beta^2 - 1)(xx_2 + x_1x_{12}) - \delta^2(\alpha^2 - \beta^2)(\alpha^2 - 1)(\beta^2 - 1)/(4\alpha\beta) = 0.
\end{aligned}
$$

- List H:

$$
\begin{aligned}
(H_1) \quad & (x - x_{12})(x_1 - x_2) + \beta - \alpha = 0, \\
(H_2) \quad & (x - x_{12})(x_1 - x_2) + (\alpha - \beta)(x + x_1 + x_2 + x_{12}) + \beta^2 - \alpha^2 = 0 \\
(H_3) \quad & \alpha(xx_1 + x_2x_{12}) - \beta(xx_2 + x_1x_{12}) + \delta(\alpha^2 - \beta^2) = 0.
\end{aligned}
$$

Viallet equation: Q_V (Viallet, 2009)

We also consider Q_V:

$$\begin{aligned} p_1 x x_1 x_2 x_{12} + p_2 \left(x x_1 x_2 + x x_1 x_{12} + x x_2 x_{12} + x_1 x_2 x_{12} \right) \\ + p_3 (x x_1 + x_2 x_{12}) + p_4 (x x_{12} + x_1 x_2) + p_5 (x x_2 + x_1 x_{12}) \\ + p_6 (x + x_1 + x_2 + x_{12}) + p_7 = 0 \end{aligned}$$

Equations in the ABS list can be obtained from this equations by choosing appropriate parameters. Note this equation has D_4 symmetry.

Other equations

$$
\begin{aligned}
(mKdV)\quad & xx_2 - x_1x_{12} + \alpha xx_1 - \beta x_2x_{12} = 0,\\
(sG)\quad & xx_1x_2x_{12} + \alpha\,(xx_{12} - x_1x_2) - \beta = 0\\
E16\quad & xx_1p_1 + xx_2p_5(p_1p_3 + p_2) + (xx_{12} + x_1x_2)p_2\\
& \quad + x_1x_{12}p_6 + x_2x_{12}p_3(p_5p_6 - p_2) = 0\\
E25\quad & xx_{12} + x_1x_2 + (x_1x + x_2x_{12})p_3 - (xx_2 + x_1x_{12})(p_3 + 1)\\
& + (x_{12} - x)r_4 + (x_1 - x_2)r_2 - (s(p_3 + 1) + r_4)\,(sp_3 + r_4) + sr_2 = 0
\end{aligned}
$$

Last two equations and others were suggested numerically by Hietarinta and Viallet to have vanishing entropy [*Searching for integrable lattice maps using factorization*, J Phys A **40** (2007) 12629-12643]

Setting

We consider an equation which is multi-affine on the square

$$Q(u_{l,m}, u_{l+1,m}, u_{l,m+1}, u_{l+1,m+1}) = 0. \tag{10}$$

Solve this equation for $u_{l+1,m+1} = P(u_{l,m}, u_{l+1,m}, u_{l,m+1})$.
Introduce projective coordinates $u_{l,m} = \frac{x_{l,m}}{z_{l,m}}$ so rule becomes

$$\begin{aligned} x_{l+1,m+1} &= f\left(x_{l,m}, x_{l+1,m}, x_{l,m+1}, z_{l,m}, z_{l+1,m}, z_{l,m+1}\right), \\ z_{l+1,m+1} &= g\left(x_{l,m}, x_{l+1,m}, x_{l,m+1}, z_{l,m}, z_{l+1,m}, z_{l,m+1}\right), \end{aligned}$$

where f and g are homogeneous polynomials of degree 3.

Remark

Given a multi-affine equation on the quad graph, the projective coordinates $x_{l+1,m+1}$ and $z_{l+1,m+1}$ of the top right corner are homogeneous polynomials where each term includes exactly one projective coordinate from each of the remaining 3 vertices of the square.

Factorization

- Boundary values are given as polynomials in $\mathbb{Z}[w]$ along horizontal and vertical axes in the first quadrant (both components with same degree).
- We iterate the rule with integer coefficients and complete the vertices in the first quadrant .
- We factor $x_{l,m}(w)$ and $z_{l,m}(w)$ over $\mathbb{Z}$.

$$\mathrm{gcd}_{l,m}(w) = \mathrm{gcd}(x_{l,m}(w), z_{l,m}(w)).$$

$$\begin{aligned} x_{l,m}(w) &= \mathrm{gcd}_{l,m}(w)\, \bar{x}_{l,m}(w), \\ z_{l,m}(w) &= \mathrm{gcd}_{l,m}(w)\, \bar{z}_{l,m}(w). \end{aligned}$$

$$\begin{aligned} d_{l,m} &= \max(\deg(x_{l,m}), \deg(z_{l,m})) \geq 0 \\ \bar{d}_{l,m} &= \max(\deg(\bar{x}_{l,m}), \deg(\bar{z}_{l,m})) \geq 0 \\ g_{l,m} &= \deg(\mathrm{gcd}_{l,m}) \end{aligned}$$

Algebraic entropy

Viallet and others: Calculate *reduced* degrees $\bar{d}_{l,m}$ of the $x_{l,\underline{m}}$ at each vertex in the square $[8 \times 8]$. Extract diagonal entries $\bar{d}_{m,m}$; assume from generating function; fit with univariate rational function and find asymptotics from closest singularity. Define algebraic entropy for lattice map to be

$$\epsilon = \lim_{m\to\infty} \frac{1}{m} \log(\bar{d}_{m,m}).$$

- The key issue for algebraic entropy relates to the growth of $g_{l,m}$. It has been done for reductions of H_1 by Van der Kamp
- Our approach: find exact upper bound for $d_{l,m}$, conjecture lower bound for $g_{l,m} \Longrightarrow$ upper bound for $\bar{d}_{l,m}$, hence for entropy.

Some properties of degrees

We have

- $0 \leq d_{l+1,m+1} \ \leq \ d_{l,m} + d_{l+1,m} + d_{l,m+1}$
- $\text{gcd}_{l,m}(w)\,\text{gcd}_{l+1,m}(w)\,\text{gcd}_{l,m+1}(w) \mid \text{gcd}_{l+1,m+1}(w)$

  Therefore we get,

$$\begin{aligned} g_{l+1,m+1} &\geq g_{l,m} + g_{l+1,m} + g_{l,m+1}, \\ g_{l+1,m} &\geq g_{l,m}, \\ g_{l,m+1} &\geq g_{l,m}, \\ g_{l+1,m+1} &\geq 3g_{l,m} \end{aligned}$$

The last property shows that $g_{l,m}$ grows exponentially if there exists l, m such that $g_{l,m} > 0$.

Upper bound for degrees $d_{l,m} = \max(\deg(x_{l,m}), \deg(z_{l,m}))$

$$D_c = \begin{bmatrix} 0 & 1 & 17 & 145 & 833 & 3649 & 13073 & 40081 & 108545 & 265729 \\ 0 & 1 & 15 & 113 & 575 & 2241 & 7183 & 19825 & 48639 & 108545 \\ 0 & 1 & 13 & 85 & 377 & 1289 & 3653 & 8989 & 19825 & 40081 \\ 0 & 1 & 11 & 61 & 231 & 681 & 1683 & 3653 & 7183 & 13073 \\ 0 & 1 & 9 & 41 & 129 & 321 & 681 & 1289 & 2241 & 3649 \\ 0 & 1 & 7 & 25 & 63 & 129 & 231 & 377 & 575 & 833 \\ 0 & 1 & 5 & 13 & 25 & 41 & 61 & 85 & 113 & 145 \\ 0 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For affine corner boundary conditions, the maximal degree table follows from removing the first column and the last row.

Upper bound for degrees

Let $d_{l,m}$ be the maximal degree of $x_{l,m}$ and $z_{l,m}$. Recall that

$$d_{l+1,m+1} \leq d_{l,m} + d_{l+1,m} + d_{l,m+1}$$

.

Theorem

Consider the linear partial difference equation with constant coefficients

$$a_{l+1,m+1} = a_{l,m} + a_{l+1,m} + a_{l,m+1} \tag{11}$$

for all $l, m \geq 0$*. Let* $a_{l,0} = a_{0,m} = 0$ *for all* $l, m > 0$ *and* $a_{0,0} = 1$*, i.e. this is case of previous slide. Then* $a_{l,m}$ *is the coefficient of* x^{m-1} *in the Taylor expansion of* $g_l(x)$ *around* 0*, i.e,*

$$g_l(x) = \frac{(1+x)^{l-1}}{(1-x)^l} = \sum_{m=1}^{\infty} a_{l,m} x^{m-1}.$$

Exact upper bound for degree growth

- $$a_{l,m} = \sum_{i+j=m-1} \binom{l-1}{i}\binom{j+l-1}{j}.$$
- $a_{l,m}$ well known, Delannoy numbers. Asymptotics of sequence??
- Generating functions for full double sequence and diagonal (central) sequence with presented boundary conditions are known

$$\begin{aligned}
F(x,y) &= \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} a_{l,m}\, x^l y^m = \frac{1}{1-x-y-xy} \\
D(x) &= \sum_{m=0}^{\infty} a_{m,m}\, x^m = \frac{1}{\sqrt{1-6x+x^2}} \\
a_{m,m} &\sim \frac{\cosh(\frac{\log 2}{4})}{\sqrt{\pi}} (3+2\sqrt{2})^m \frac{1}{\sqrt{m}} \\
\Rightarrow \epsilon &= \lim_{m\to\infty} \frac{1}{m} \log(d_{m,m}) \leq \epsilon_{max} = \log(3+2\sqrt{(2)}) = 1.76..
\end{aligned}$$

Exponentially growing $g_{l,m} = \deg(\gcd_{l,m})$

We calculate the degree of the gcd at each point for some known integrable lattice maps. For the case of constant axis values except for affine in w at the origin we have

$$\begin{bmatrix} 0 & 0 & 14 & 140 & 826 & 3640 & 13062 & 40068 & 108530 & 265712 \\ 0 & 0 & 12 & 108 & 568 & 2232 & 7172 & 19812 & 48624 & 108530 \\ 0 & 0 & 10 & 80 & 370 & 1280 & 3642 & 8976 & 19812 & 40068 \\ 0 & 0 & 8 & 56 & 224 & 672 & 1672 & 3642 & 7172 & 13062 \\ 0 & 0 & 6 & 36 & 122 & 312 & 672 & 1280 & 2232 & 3640 \\ 0 & 0 & 4 & 20 & 56 & 122 & 224 & 370 & 568 & 826 \\ 0 & 0 & 2 & 8 & 20 & 36 & 56 & 80 & 108 & 140 \\ 0 & 0 & 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Exponentially growing $g_{l,m} = \deg(\mathrm{gcd}_{l,m})$

For the case of affine boundary values in w, the table is

0	0	144	1104	5568	22272	75408	224016	598272	1462400
0	0	112	784	3584	12992	39984	108432	265600	598272
0	0	84	532	2184	7112	19740	48540	108432	224016
0	0	60	340	1240	3592	8916	19740	39984	75408
0	0	40	200	640	1632	3592	7112	12992	22272
0	0	24	104	288	640	1240	2184	3584	5568
0	0	12	44	104	200	340	532	784	1104
0	0	4	12	24	40	60	84	112	144
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

Spontaneous gcd

Recall that

$$gcd_{l,m}\ gcd_{l+1,m}\ gcd_{l,m+1}\ \mid gcd_{l+1,m+1},$$

so we can write

$$gcd_{l+1,m+1} = gcd_{l,m}\ gcd_{l+1,m}\ gcd_{l,m+1}\ \overline{gcd}_{l+1,m+1},$$

and we call $\overline{gcd}_{l+1,m+1}$ the spontaneous gcd at that point.
We have

$$\overline{g}_{l+1,m+1} = g_{l+1,m+1} - g_{l,m} - g_{l+1,m} - g_{l,m+1}$$

For our convenience we take $\overline{g}_{l,0} = \overline{g}_{0,m} = 0$.

Spontaneous gcd

For the first case, we obtain

$$
\begin{bmatrix}
0 & 0 & 2 & 6 & 10 & 14 & 18 & 22 & 26 & 28 \\
0 & 0 & 2 & 6 & 10 & 14 & 18 & 22 & 24 & 26 \\
0 & 0 & 2 & 6 & 10 & 14 & 18 & 20 & 22 & 22 \\
0 & 0 & 2 & 6 & 10 & 14 & 16 & 18 & 18 & 18 \\
0 & 0 & 2 & 6 & 10 & 12 & 14 & 14 & 14 & 14 \\
0 & 0 & 2 & 6 & 8 & 10 & 10 & 10 & 10 & 10 \\
0 & 0 & 2 & 4 & 6 & 6 & 6 & 6 & 6 & 6 \\
0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

Spontaneous gcd

For the second case, we have

$$\begin{bmatrix} 0 & 0 & 32 & 64 & 96 & 128 & 160 & 192 & 224 & 256 \\ 0 & 0 & 28 & 56 & 84 & 112 & 140 & 168 & 196 & 224 \\ 0 & 0 & 24 & 48 & 72 & 96 & 120 & 144 & 168 & 192 \\ 0 & 0 & 20 & 40 & 60 & 80 & 100 & 120 & 140 & 160 \\ 0 & 0 & 16 & 32 & 48 & 64 & 80 & 96 & 112 & 128 \\ 0 & 0 & 12 & 24 & 36 & 48 & 60 & 72 & 84 & 96 \\ 0 & 0 & 8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 \\ 0 & 0 & 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

Observation

For the rule H_1, for the first case of boundary values the degrees of spontaneous gcd and the actual degrees are given (respectively) as follows

$$
\begin{bmatrix}
0 & 0 & 2 & 6 & 10 & 14 & 18 & 22 & 26 & 28 \\
0 & 0 & 2 & 6 & 10 & 14 & 18 & 22 & 24 & 26 \\
0 & 0 & 2 & 6 & 10 & 14 & 18 & 20 & 22 & 22 \\
0 & 0 & 2 & 6 & 10 & 14 & 16 & 18 & 18 & 18 \\
0 & 0 & 2 & 6 & 10 & 12 & 14 & 14 & 14 & 14 \\
0 & 0 & 2 & 6 & 8 & 10 & 10 & 10 & 10 & 10 \\
0 & 0 & 2 & 4 & 6 & 6 & 6 & 6 & 6 & 6 \\
0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\quad
\begin{bmatrix}
0 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 \\
0 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 15 \\
0 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 13 & 13 \\
0 & 1 & 3 & 5 & 7 & 9 & 11 & 11 & 11 & 11 \\
0 & 1 & 3 & 5 & 7 & 9 & 9 & 9 & 9 & 9 \\
0 & 1 & 3 & 5 & 7 & 7 & 7 & 7 & 7 & 7 \\
0 & 1 & 3 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
0 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

We found that

$$\overline{g}_{l,m} + \overline{g}_{l+1,m+1} = 2(\overline{d}_{l,m-1} + \overline{d}_{l-1,m}).$$

A source of spontaneous gcd on each 2 x 2 lattice square

- Generalizing an observation of Hietarinta and Viallet (2007), we find that for many integrable lattice equations (ABS list, Q_V, sG, mKdV, E16,E25), there is a common factor $A_{l,m}$ of $x_{l+1,m+1}$ and $z_{l+1,m+1}$ that depends on coordinates at vertices $(l-1,m)$ and $(l,m-1)$. For example, for rule H_1:

$$A_{l,m} = (x_{l,m-1}z_{l-1,m} - z_{l,m-1}x_{l-1,m})^2.$$

- This creates an ongoing spontaneous gcd as we iterate.
- The common factor has the same degree $2(d_{l-1,m} + d_{l,m-1})$ for all these equations.

A recurrence that generates $gcd_{l,m}$

At the point $(l+1, m+1)$ we know that

- $gcd_{l,m}\ gcd_{l+1,m}\ gcd_{l,m+1} \mid gcd_{l+1,m+1}$ and $A_{l,m} \mid gcd_{l+1,m+1}$.

Therefore, we have

$$\frac{A_{l,m}\ gcd_{l,m}\ gcd_{l+1,m}\ gcd_{l,m+1}}{gcd\left(A_{l,m}, gcd_{l,m}\ gcd_{l+1,m}\ gcd_{l,m+1}\right)} \mid gcd_{l+1,m+1}.$$

Let $G_{l,m} = 1$ for all $l, m \leq 1$. We introduce the recurrence

$$G_{l+1,m+1} = \frac{A_{l,m}\ G_{l-1,m-1} G_{l+1,m} G_{l,m+1}}{G_{l-1,m} G_{l,m-1}}, \tag{12}$$

For the ABS equations, sG, mKdV, E16 and E25 we have evidence for the following conjecture.

Conjecture (Enabling)

Given arbitrary boundary conditions satisfying $gcd_{l,m} = 1$ *for all* $l, m \leq 1$, *then* $G_{l,m} = gcd_{l,m}$ *('up to a constant'), so* $\deg(G_{l,m}) = \deg(gcd_{l,m})$.

Polynomial growth of integrable lattice rules

Proposition

When $\deg(A_{l,m}) = 2\,(d_{l-1,m} + d_{l,m-1})$*, the degrees* $g_{l,m} = \deg(\mathrm{gcd}_{l,m})$ *of the common factors satisfy the following linear partial difference equation with constant coefficients*

$$g_{l+1,m+1} = 2(d_{l,m-1}+d_{l-1,m})+g_{l-1,m-1}+g_{l+1,m}+g_{l,m+1}-g_{l-1,m}-g_{l,m-1}.$$

Recalling that $d_{l,m} = \max(\deg(x_{l,m}), \deg(z_{l,m}))$ *satisfy*

$$d_{l+1,m+1} = d_{l,m} + d_{l+1,m} + d_{l,m+1},$$

the linear partial difference equation with constant coefficients for $\overline{d}_{l,m} = \max(\deg(\bar{x}_{l,m}), \deg(\bar{z}_{l,m}))$ *is*

$$\overline{d}_{l+1,m+1} = \overline{d}_{l+1,m} + \overline{d}_{l,m+1} + \overline{d}_{l-1,m-1} - \overline{d}_{l,m-1} - \overline{d}_{l-1,m}.$$

Polynomial growth of integrable lattice rules

Introduce $v_{l,m} = \overline{d}_{l+1,m+1} - \overline{d}_{l,m}$. We have

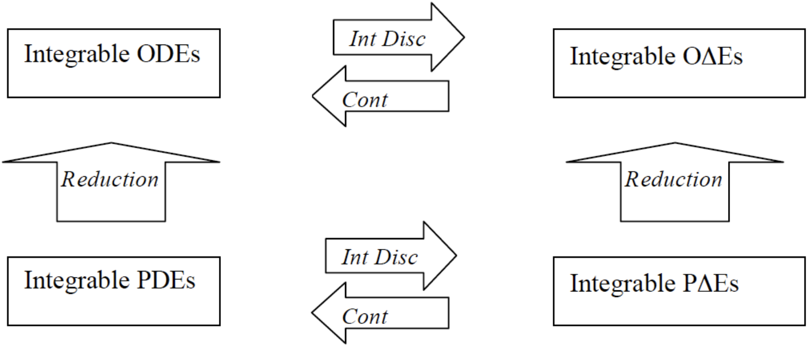
$$v_{l,m} + v_{l-1,m-1} = v_{l-1,m} + v_{l,m-1}.$$

We consider ABS equations (including Q_V), sG, mKdV, E16 and E25. Along the diagonal (from (l,m) to $(l+1,m+1)$ we obtain:

- These equations have linear growth of $\overline{d}_{l,m}$ if $x_{l,0}, z_{l,0}$ and $x_{0,m}, z_{0,m}$ are constant for $l,m > 0$ and $x_{0,0}$ and $z_{0,0}$ are affine in w.
- Quadratic growth if $x_{l,0}(w), z_{l,0}(w)$ and $x_{0,m}(w), z_{0,m}(w)$ are degree-one-polynomials in w.
- Cubic growth if $x_{l,0}(w), z_{l,0}(w)$ and $x_{0,m}(w), z_{0,m}(w)$ are polynomials of degree $l+1$ and $m+1$ in w.

Remark: $\deg(A_{l,m}) = 2\,(d_{l-1,m} + d_{l,m-1})$. If omit 2, not enough cancellation and "prove" exponential growth of reduced degrees $\overline{d}_{l,m}$.

Integrable equations: continuous and discrete



- RHS: When the OΔEs and PΔEs are defined by rational functions with rational coefficients, they make sense over any field e.g. $\mathbb{F}_p$, where p is a prime number

Integrable maps over finite fields

[R+Vivaldi 2003, Jogia+R+Vivaldi 2006]

- Area-preserving birational map of $\mathbb{R}^2$ with rational integral $I(x', y') = I(x, y)$ and $I(x, y) = x^2y^2 + x^2 + y^2 + 2xy + x + y$

$$L^{McM} : x' = y, \quad y' = -x - \frac{1+2y}{1+y^2}$$

- Level sets are elliptic curves in general. Reduce L^{McM} for $p \equiv 3 \pmod 4$ gives a permutation. Number of points on elliptic curve $\pmod p$ bounded by $HW(p) = p + 1 + 2\sqrt{p}$.

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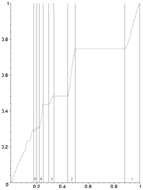
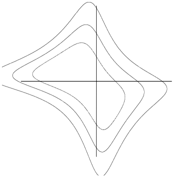


$$\mathcal{D}_p(x) = \frac{\#\{z \in \mathbb{F}_p^2 : T(z) \leq \kappa x\}}{\#\mathbb{F}_p^2},$$
$$\kappa = HW(p).$$

- Case $p = 1019$ shows quantised periods around $1/n, n = 1, 2, \ldots$

Integrability signatures over finite fields

- The signatures on reduction to finite fields can be used as necessary conditions to detect algebraic properties in parametrised families of birational maps (parameter space is also finite over $\mathbb{F}_p$).
- Over finite fields, being close to integrable is the same as being far from integrable.



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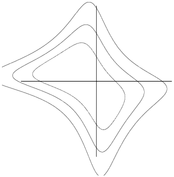

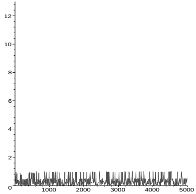

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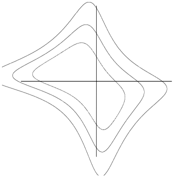

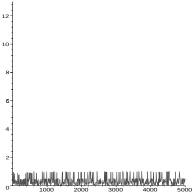

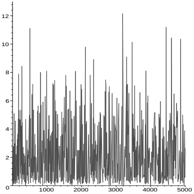

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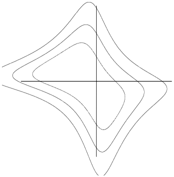

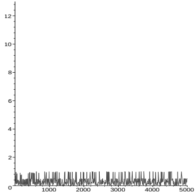

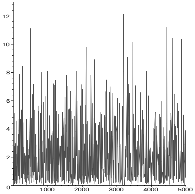

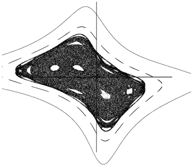

Lattice rule over finite fields

Recall that over $\mathbb{Z}$, we wrote

$$x_{l,m}(w) = \mathrm{gcd}_{l,m}(w)\,\bar{x}_{l,m}(w), \quad z_{l,m}(w) = \mathrm{gcd}_{l,m}(w)\,\bar{z}_{l,m}(w).$$

We repeat our experiments over $\mathbb{F}_p$, where p is a prime number. We take

$$x^p_{l,m}(w) \equiv x_{l,m}(w) \pmod{p}, \quad z^p_{l,m}(w) \equiv z_{l,m}(w) \pmod{p}.$$

We divide and factorize polynomials now over the finite field so that

$$x^p_{l,m}(w) = \mathrm{gcd}^p_{l,m}(w)\,\bar{x}^p_{l,m}(w) \pmod{p}, \tag{13}$$

$$z^p_{l,m}(w) = \mathrm{gcd}^p_{l,m}(w)\,\bar{z}^p_{l,m}(w) \pmod{p} \tag{14}$$

$$\mathrm{gcd}_{l,m}(w) \pmod{p} \mid \mathrm{gcd}^p_{l,m}(w)$$

but $\mathrm{gcd}^p_{l,m}(w)$ may be bigger. Faster computationally but still restricted to 11×11 square.

Polynomials over finite fields

Can go much further if cap the degrees of polynomials by using the Fermat's little theorem

$$w^p = w \pmod p. \tag{15}$$

We denote

$$\mathrm{roots}^{\mathrm{p}}_{\mathrm{l,m}} := \{w : x^p_{l,m}(w) = z^p_{l,m}(w) = 0 \pmod p)\} \tag{16}$$

We use $\mathrm{roots}^{\mathrm{p}}_{\mathrm{l,m}}$ as analogue of $\mathrm{gcd}_{l,m}$ to measure "commonality" between $x^p_{l,m}(w)$ and $z^p_{l,m}(w)$.

Remark

The non-negative integer sequence $\#\,\mathrm{roots}^{\mathrm{p}}_{\mathrm{l,m}}$ is non-decreasing as we move to the right and/or upwards on the lattice.

We consider the sequence $r^p(m) := \#\,\mathrm{roots}^{\mathrm{p}}_{\mathrm{m,m}} \leq \mathrm{p}$ along the diagonal.

Roots of integrable rules and their perturbations over $\mathbb{F}_p$

Computationally fast to calculate, with random constant initial conditions along the corner axes and $x^p_{0,0}(w)$ and $z^p_{0,0}(w)$ affine in w.

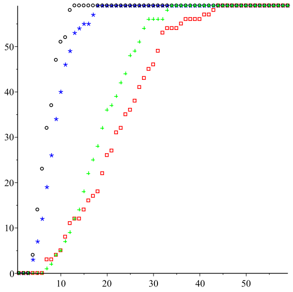

Figure: # roots along the diagonal over $\mathbb{F}_{59}$ for KdV (blue) and its perturbation (red) together with H_3 (black) and its perturbation (green).

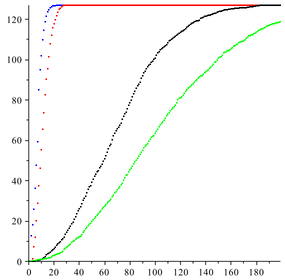

Figure: # roots along the diagonal over $\mathbb{F}_{127}$ for Q_1 (blue) and its perturbation (black) together with Q_2 (red) and its perturbation (green). Each curve is average of 10 simulations.

Observations

Over $\mathbb{F}_p$, m is called a saturation point if for all $w \in \{0, 1, ..., p-1\}$, we get $x^p_{m,m}(w) = z^p_{m,m}(w) = 0$, i.e. $r^p(m) := \#\mathrm{roots^p_{m,m}} = \mathrm{p}$. We observe the following.

- Saturation always occurs for the sequences $r^p(m)$ derived from integrable lattice rules and for many non-integrable lattice rules. The growth of $r^p(m)$ is markedly faster for integrable rules as compared to their non-integrable perturbations.
- The first saturation point $m^*(p)$ for an integrable lattice rule is much lower than the corresponding one for non-integrable perturbations.
- Integrable rules rise from 0 quickly compared to non-integrable perturbations.

Building models to explain the observations

Recall: At vertex (l, m), we have $x^p_{l,m}(w) \equiv x_{l,m}(w) \pmod{p}$ and $z^p_{l,m}(w) \equiv z_{l,m}(w) \pmod{p}$. Common roots passed to the right and upwards.

Fact: Over $\mathbb{F}_p$, a polynomial has on average 1 root independent of its degree!!

Assumption: On average, common root over $\mathbb{F}_p$ appears every T vertices. T depends on common factors of $x^p_{l,m}(w)$ and $z^p_{l,m}(w)$.

- $T = 1$ if one common factor over $\mathbb{Z}$;
- $T = \frac{1}{2}$ if two common factors over $\mathbb{Z}$;
- $T = \frac{p+1}{2}$ or $T = p$ if no common factors over $\mathbb{Z}$.

The models

- **Model 1**: Assume j distinct roots produced from m vertices. Expected number of vertices to add next new root (Bernoulli trial) is $\frac{p}{p-j}T$. Expected number of vertices to see i roots:

$$F(T,p,i) = T + \frac{p}{p-1}T + \frac{p}{p-2}T + \dots \frac{p}{p-(i-1)}T$$

At (i,i), there are $(i-1)^2-1$ or i^2 vertices that can contribute to the process.

- **Model 2**: At (i,i), let L_i be # common roots. Assume $L_2 = 0$ and

$$L_{i+1} = L_i + E(p, \frac{2i \mp 1}{T})(\frac{p - L_i}{p})$$

$E(p,N) = \frac{Np}{N+p-1}$: expected # distinct values from N choices of $\{0,\dots,p-1\}$ with replacement

Saturation points: Data versus model 1

$F(T, p, p)$ gives number of vertices to see p roots, i.e. saturation

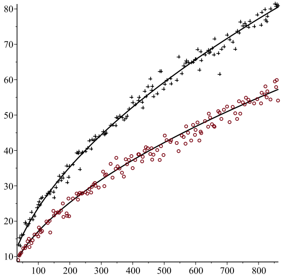

Figure: Saturation points of H_1 (cross) and H_3 (circle) vs prime. Higher curve and lower curve represent expected saturation points from Model 1 for $T = 1$ and $T = 1/2$

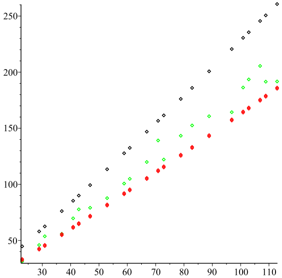

Figure: Saturation points of a perturbation of Q_2 (green) vs prime. Higher point curve (black) and lower point curve (red) are saturation points from Model 1 with $T = p$ and $T = (p + 1)/2$.

Roots along the diagonal: Data versus models 1 and 2

Recall the # roots $r^p(m) := \#\,\mathrm{roots}^{\mathrm{p}}_{\mathrm{m,m}}$ along the diagonal.

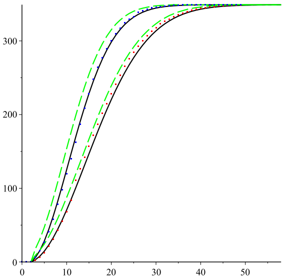

Figure: Average # roots along the diagonal over $\mathbb{F}_{349}$ for ABS equations Q_1, H_2, H_3 (blue) and Q_2, Q_3, Q_4, H_1 (red) vs predictions from Model 1 (green dash) and Model 2 (black) for $T = 1/2$ (top) and $T = 1$ (bottom).

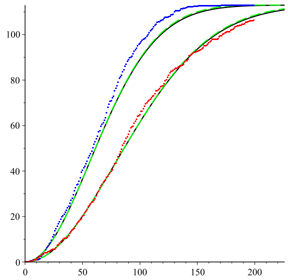

Figure: # roots along the diagonal over $\mathbb{F}_{113}$ for perturbations of Q_1 (blue) and Q_2 (red) vs predictions from Model 1 (green, dash) and Model 2 (black) for $T = (p+1)/2$ (top) and $T = p$ (bottom).

Remarks

- It appears there is a T-dependent scaling that brings all root curves to the universal curve $D(x) = 1 - exp(-x^2)$, which gives the proportion of $\mathbb{F}_p$ that appear as roots at the (scaled) distance x along the diagonal from the origin. $D(x)$ is a cumulative distribution function.
- This "integrable" model is actually a test/model for # of common factors of $x^p_{l,m}(w)$ and $z^p_{l,m}(w)$ ($T = 1$ is 1 common factor, $T = 1/2$ is 2 common factors). Non-integrable equations can produce the "integrable" signature over $\mathbb{F}_p$.
- Nevertheless, the difference in unscaled root curves can be used to test parametrised families of lattice equations over finite fields for parameter values that are possibly integrable. Helps find needle in the haystack or goldfish in the pond.

Embedding integrable equations

-We add more general terms in some integrable equations and do a factorization test over $\mathbb{F}_p[w]$.
-All the equations in the ABS list except Q_4 and equations given by Hietarinta and Viallet do not have any cubic terms.

- Write the rule in projective coordinates with free cubic coefficients.
- Impose the 'constant boundary conditions'.
- Let the free coefficient run from 0 to $p-1$ and then calculate all the points in the 3×3 square over $\mathbb{F}_p[w]$.
- Record all the values of the "free coefficient" that makes $\deg(\gcd^p_{3,3}(w)) \geq 4$ or $\gcd^p_{3,3}(w) = 0$
- Run the test with different sets of initial values.
- Intersect all the "survival" sets.
- Run with different prime numbers.
- Solve the Chinese Remainder Theorem to recover the original parameter.

Embedding integrable equations: an example

By adding cubic terms in Q_1 where $\alpha = 2, \beta = 3$, we obtain the following rule

$$
\begin{aligned}
x_{l+1,m+1} &= 6z_{l+1,m}z_{l,m}z_{l,m+1} - 2x_{l+1,m}x_{l,m}z_{l,m+1} - x_{l+1,m}x_{l,m+1}z_{l,m} + 3x_{l,m+1}xl,\ mz_{l+1,m} \\
&\quad - a\, x_{l,m}x_{l+1,m}x_{l,m+1}, \\
z_{l+1,m+1} &= x_{l,m}z_{l+1,m}z_{l,m+1} + 2x_{l,m+1}z_{l+1,m}z_{l,m} - 3x_{l+1,m}z_{l,m}z_{l,m+1} \\
&\quad + b\, x_{l,m}x_{l+1,m}z_{l,m+1} + c\, x_{l,m}x_{l,m+1}z_{l+1,m} + d,\ z_{l,m}x_{l+1,m}x_{l,m+1}.
\end{aligned}
$$

We use prime numbers $p = 7, 11, 13$ and 20 sets of initial values. For $p = 7$, the survival set is

```
 {[0, 0, 0, 0], [0, 0, 5, 5], [1, 1, 1, 1], [2, 2, 0, 0], [2, 2, 2, 2],
[3, 3, 3, 3], [4, 4, 4, 4], [5, 5, 0, 0], [5, 5, 5, 5], [6, 6, 6, 6]}
```

For $p = 11$, the survival set is

```
{[0, 0, 0, 0], [0, 0, 5, 5], [1, 1, 1, 1], [2, 2, 0, 0], [2, 2, 2, 2],
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```

For $p = 13$, the survival set is

```
{[0, 0, 0, 0], [0, 0, 8, 8], [1, 1, 1, 1], [2, 2, 2, 2], [3, 3, 3, 3],
[4, 4, 4, 4], [5, 5, 0, 0], [5, 5, 5, 5], [6, 6, 6, 6], [7, 7, 7, 7],
[8, 8, 0, 0], [8, 8, 8, 8], [9, 9, 9, 9], [10, 10, 10, 10], [11, 11, 11, 11],
[12, 12, 12, 12]}
```

They suggest that, we can take $a = b = c = d$, i.e. the new equation is just a special case of Q_V.

Embedding integrable equations: a new equation

We add 4 possible cubic terms to the following equation

$$(x_2 + x)(x_{12} + x_1) + \beta(x_1 + x_2) = 0$$

For $p = 7$, the survival set of 4 coefficients is

{[0, 0, 0, 0], [0, 1, 1, 0], [0, 2, 2, 0], [0, 3, 3, 0], [0, 4, 4, 0], [0, 5, 5, 0], [0, 6, 6, 0]}

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[0, 5, 5, 0], [0, 6, 6, 0], [0, 7, 7, 0], [0, 8, 8, 0], [0, 9, 9, 0],
[0, 10, 10, 0]}
```

They suggest that $a = d = 0, b = c = \alpha$. We obtain the following equation

$$\alpha x x_{12}(x_1 + x_2) + (x_2 + x)(x_{12} + x_1) + \beta(x_1 + x_2) = 0. \quad (17)$$

This equation has vanishing entropy and fits in our framework in the first part of the talk as one checks that $\deg(A_{l,m}) = 2\,(d_{l-1,m} + d_{l,m-1})$.

Diophantine integrability (after Halburd 2005)

If we choose initial values as rational numbers, big cancellations result in slow growth of the height of the iterates. When $k = \mathbb{Q}$, given $x \in \mathbb{Q}$ we define its height $H(x)$ as follows.

If $x = 0$**, then** $H(x) := 1$ **and if** $x = p/q$ **where** $\gcd(p, q) = 1$ **then** $H(x) := max(|p|, |q|)$
We plot $log(log(H(u_{n,n}))$ vs $log(n)$ along the diagonal ($n \geq 1$) where $\alpha = 3$, $\beta = -2$ and its perturbation $x_{12} = P(x, x_1, x_2) + 10^{-2}$.

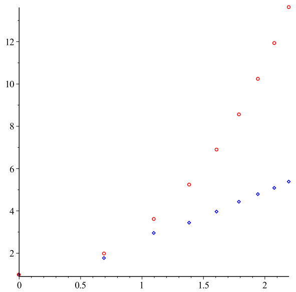

Figure: Equation (17) and its perturbation, time=450.224 seconds, size=9×9

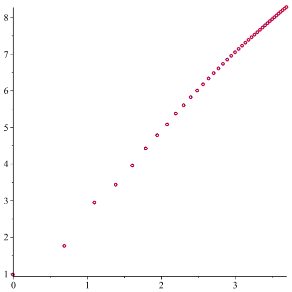

Figure: Equation (17), time=1.94 seconds, size=40×40

Thanks for your attention!

Recovery of integrable equations from parametrised families via test of spontaneous gcd

We free one of the coefficients in the integrable rule, and perform the following test to recover it.

- Write the rule in projective coordinates and free one of the coefficients.
- Impose the 'constant boundary conditions'.
- Let the free coefficient run from 0 to $p-1$ and then calculate all the points in the 3×3 square over $\mathbb{F}_p[w]$.
- Record all the values of the "free coefficient" that makes $\deg(\gcd^p_{3,3}(w)) \geq 4$ or $\gcd^p_{3,3}(w) = 0$.
- Run the test with different sets of boundary values.
- Intersect all the "survival" sets.
- Run with different prime numbers.
- Use the chinese remainder theorem to recover the original parameter

Recovery test: an example

For example, the Q_1 rule with $\alpha = 2, \beta = 3$ is

$$x_{l+1,m+1} = 6z_{l+1,m}z_{l,m}z_{l,m+1} - 2x_{l+1,m}x_{l,m}z_{l,m+1} - x_{l+1,m}x_{l,m+1}z_{l,m} + 3x_{l,m+1}x_{l,m}z_{l+1,m},$$
$$z_{l+1,m+1} = x_{l,m}z_{l+1,m}z_{l,m+1} + 2x_{l,m+1}z_{l+1,m}z_{l,m} - 3x_{l+1,m}z_{l,m}z_{l,m+1}.$$

We free the first term of $z_{l+1,m+1}$, i.e. we take

$$z_{l+1,m+1} = R\, x_{l,m}z_{l+1,m}z_{l,m+1} + 2x_{l,m+1}z_{l+1,m}z_{l,m} - 3x_{l+1,m}z_{l,m}z_{l,m+1}.$$

Prime	Initial values	Survival set	Intersection
$p = 17$	Initial values 1	$\{0, 1, 2, 6, 7, 9, 10, 12, 15, 16\}$	$\{0, 1, 10\}$
	Initial values 2	$\{0, 1, 8, 9, 10, 13\}$	
	Initial values 3	$\{0, 1, 2, 6, 7, 10, 13\}$	
$p = 19$	Initial values 1	$\{1, 4, 5, 6, 8, 12, 15, 17, 18\}$	$\{1, 4\}$
	Initial values 2	$\{0, 1, 2, 4, 6, 7, 10, 11, 12, 13, 17, 18\}$	
	Initial values 3	$\{0, 1, 3, 4, 5, 8, 10, 13, 16\}$	
$p = 13$	Initial values 1	$\{0, 1, 2, 3, 8, 10, 11, 17, 21, 22\}$	$\{1, 2\}$
	Initial values 2	$\{1, 2, 5, 7, 9, 12, 16, 20\}$	
	Initial values 3	$\{1, 2, 6, 7, 9, 11, 13, 16\}$	

Using the Chinese remainder theorem to solve the systems $R \equiv a_i \pmod{p_i}$ gives 12 values Only $R = 1$ will stabilize we use more prime numbers. It confirms that $R = 1$ in the original equation.