Nonlocal asymmetric exclusion processes [NASEP] on a ring and conformal invariance

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- 1- Stochastic Processes in continuum time
- 2- Some examples (the z=3/2 obsession!)
- 3- NASEP - Description and results
- 4- Periodic Temperley-Lieb algebra:

  a) Spin representation,

  b)Particle representation and connection with NASEP

How to produce Markovian dynamics from associative algebras?

A System with states

$$|a\rangle \quad (a = 1, 2, \ldots) \quad \rightarrow \text{ probability } \quad P_a(t)$$

$$|b\rangle \quad \rightarrow \quad |a\rangle \qquad \text{rate} \quad -H_{a,b}$$

The master equation

$$\frac{d}{dt} P_a(t) = -\sum_b H_{a,b} P_b(t),$$

H is an $N \times N$ intensity matrix (eigenvalues $\mathrm{Re}(E(k)) \geq 0$)

$$H_{a,b} \leq 0, \quad \sum_{a=1}^{N} H_{a,b} = 0$$

The stationary state $\rightarrow$ ground state $E_0 = 0$

$$\langle 0| = \sum_a 1 \langle a|, \quad |0\rangle = \sum_a P_a |a\rangle, \quad P_a = \lim_{t \to \infty} P_a(t)$$

Algebraic facets

Associative algebra with generators

$$A(1), A(2), \ldots, A(M) \quad \rightarrow \text{letters}$$

Product of generators

$$W(1), W(2), \ldots \quad \rightarrow \text{words}$$

We choose $\{W(s)\}$ independent words and associate a vector space:

Regular representation of the algebra

$$W(1) \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \quad W(2) \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \quad \cdots$$

The indendent words are chosen such that any product of them verify the relation:

$$W(r)W(s) = \sum_q p_q^{(r,s)} W(q), \quad p_q^{(r,s)} \geq 0, \quad \sum_q p_q^{(r,s)} = 1,$$

$$W(r)W(s) \quad \rightarrow \begin{pmatrix} p_1^{(r,s)} \\ p_2^{(r,s)} \\ \vdots \end{pmatrix}$$

H is an intensity matrix

$$H = \sum_r a(r)(1 - W_r) \quad a(r) \geq 0 \iff \begin{pmatrix} H_{1,1} & H_{1,2} & \cdots \\ H_{2,1} & H_{2,2} & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}$$

$$H_{i,j} \leq 0, \quad \sum_i H_{i,j} = 0$$

If the algebra has a special subset of words I_A:

I_A is a left ideal

$$a.X \in I_A \quad \text{for all } a \in A \quad \text{and} \quad X \in I_A$$

The Hamiltonian restricted to I_A defines a stochastic process

How to obtain those algebras?

Special class: semigroup algebras ($W_r \cdot W_s = W_q$)

Important example:

Temperley-Lieb algebra: generators $e_j \quad (j = 1, \ldots, L-1)$

$$e_j^2 = (q + q^{-1})e_j, \qquad e_j e_{j\pm 1} e_j = e_j, \qquad e_j e_k = e_k e_j \quad \text{for} \quad |j-k| > 1,$$

semigroup only if $q + 1/q = 1$

H is an intensity matrix

$$H = \sum_r a(r)(1 - W_r) \quad a(r) \geq 0 \quad \Longleftrightarrow \begin{pmatrix} H_{1,1} & H_{1,2} & \cdots \\ H_{2,1} & H_{2,2} & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}$$

$$H_{i,j} \leq 0, \quad \sum_i H_{i,j} = 0$$

Independent words: $e_1, e_2, \ldots, e_L, \quad e_1 e_2, e_1 e_3, \ldots, \quad e_1 e_2 e_3, \ldots$

Special choice:

$$H = \sum_{j=1}^{L-1} (1 - e_j)$$

Acting on the left ideal I_1 ($L = 2n$) ($W(s)$ any word)

$$W(s) J_0, \quad J_0 = \prod_{j=1}^{L/2} e_{2j-1},$$

Special choice

$$H = \sum_{j=1}^{L-1} (1 - e_j)$$

Acting on the left ideal I_1 ($L = 2n$) ($W(s)$ any word)

$$W(s) J_0, \quad J_0 = \prod_{j=1}^{L/2} e_{2j-1},$$

Spectrum of H

$E_0 = 0, \qquad E_i \geq 0$

GAP : $\qquad E_i - E_0 = E_i$

Critical system :

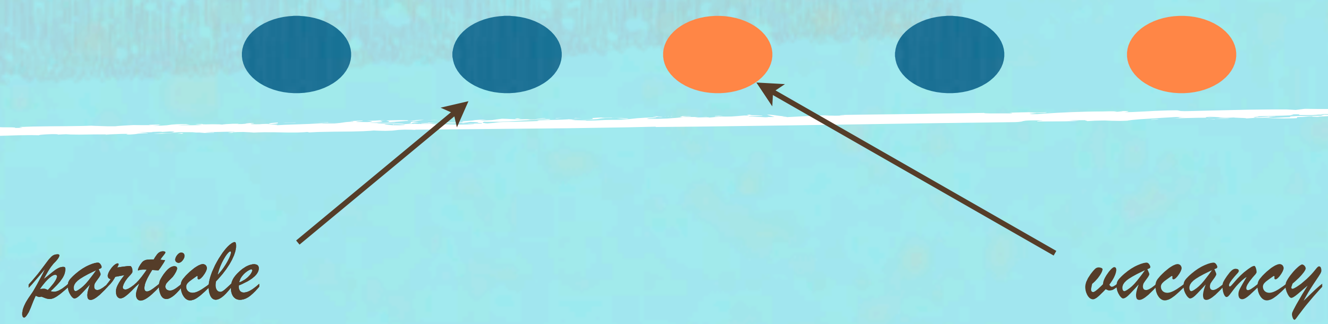
Texto

1-D system with L sites

$$E_i \sim \frac{1}{L^z}$$

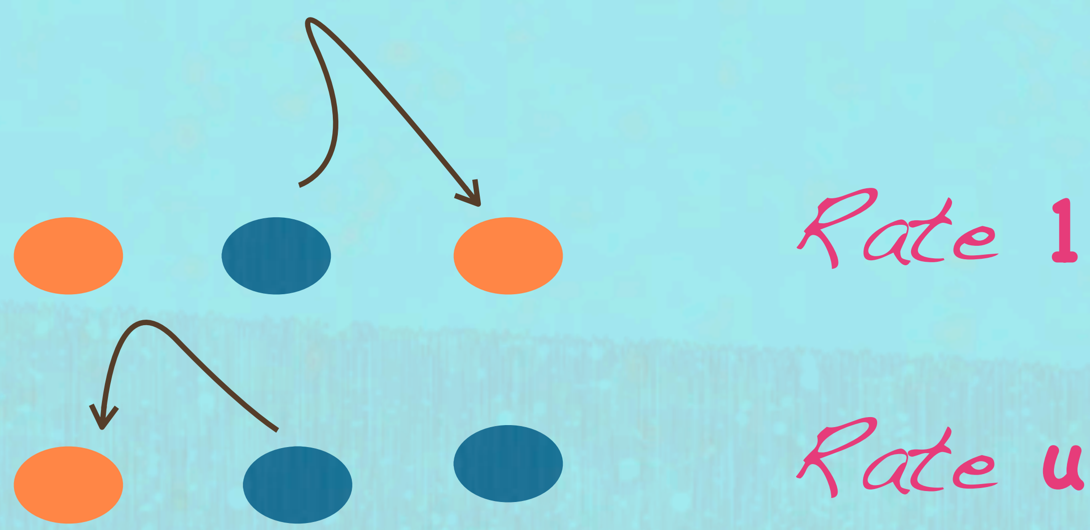
Dynamical critical exponent: z

Stochastic Processes

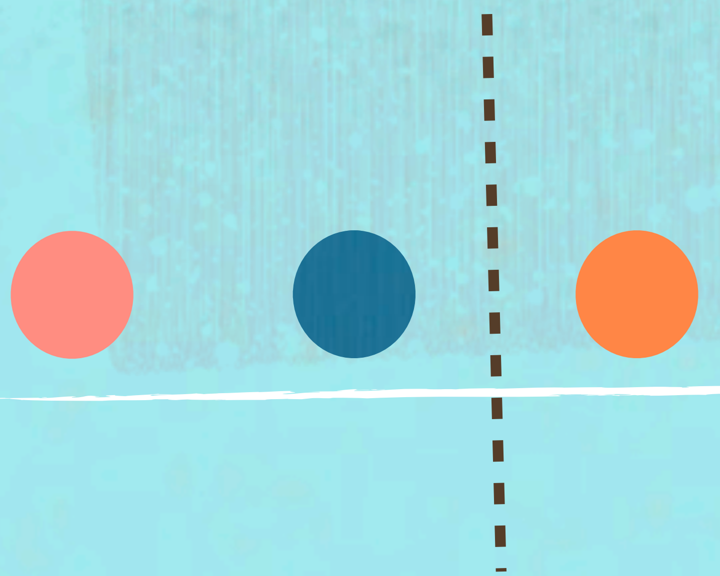


L sites: $\rho = \text{density of particles}$

ASEP on a ring



Current



Average # of particles crossing in dt

$$J = (1-u)\rho(1-\rho)$$

$$u = 1 \qquad z = 2$$

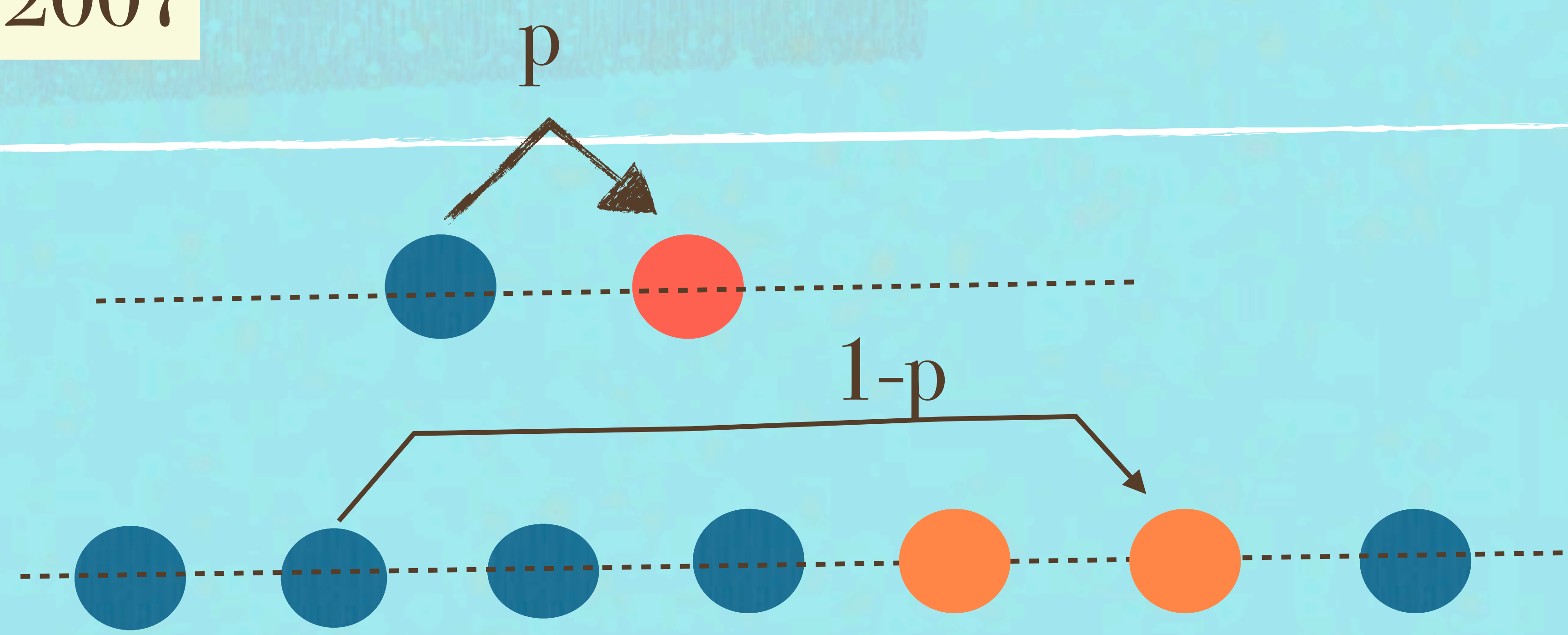
$$u \neq 1 \qquad z = \frac{3}{2} \qquad \text{KPZ}$$

Integrable, many problems anwered !!!

(Matrix models) Sasamoto, Spohn

Nonlocal Models

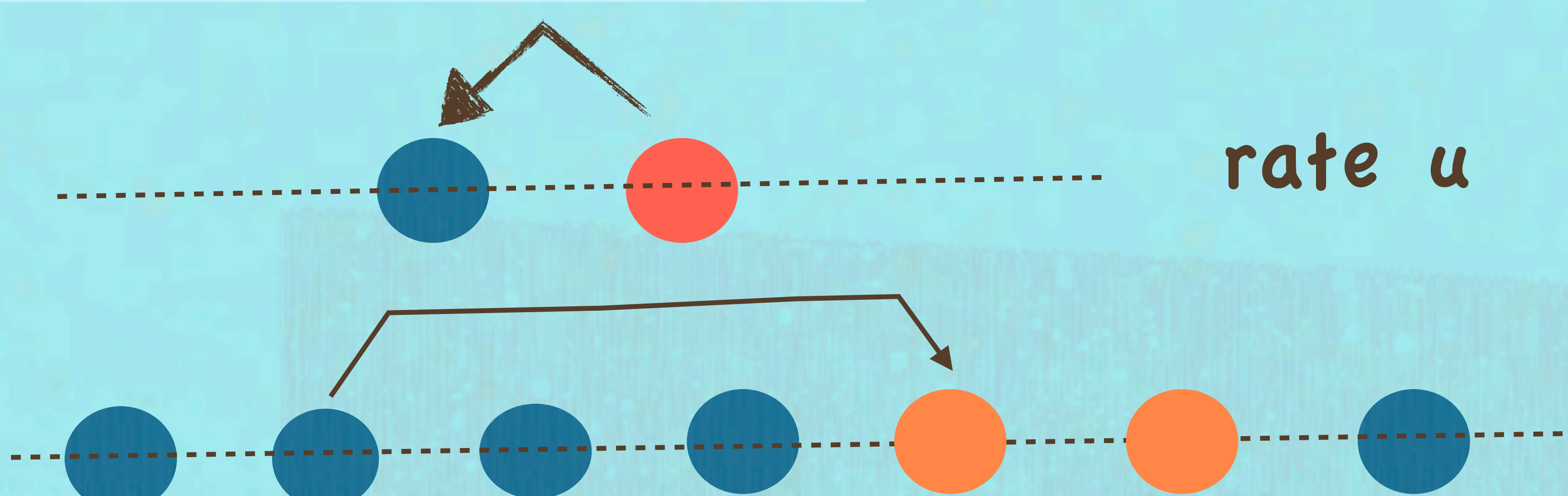
Park, den Nijs 2007



z=3/2

Hops behind the closest particle

PuahASEP (Borodin, Ferrari 2008)



rate u

z=3/2

rate 1

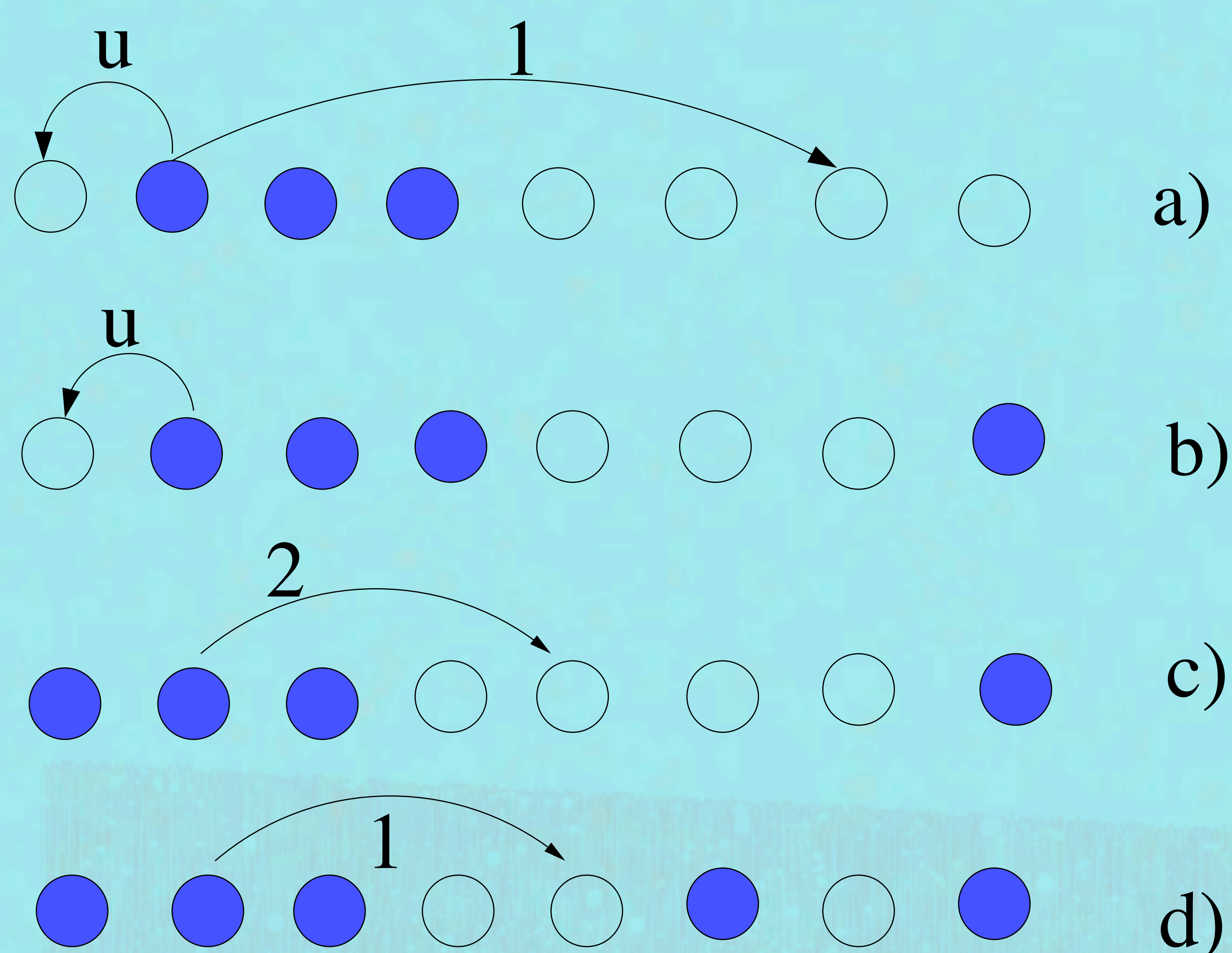
Integrable and J is a smooth function of ρ and u

Avalanches model (Priezzev et al 2001) z=3/2

Extended ASEP exclusion model (Ferreira, Alcaraz 2000) z=3/2

NASEP

no more z=3/2 !!!



Results

$$\rho = \frac{1}{2}, \qquad u = 1$$

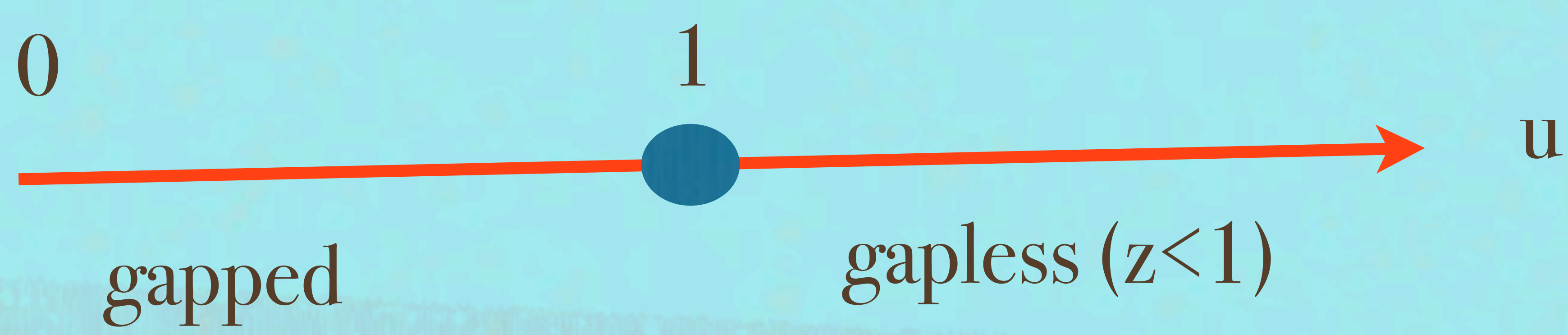
$z = 1$ conformal invariance

$$J(L) = -\frac{3L}{4(L^2-1)} \qquad (exact)$$

$$J(L)_{L\to\infty} = v_s \frac{c}{L}, \qquad v_s = \frac{3\sqrt{3}}{2}$$

C universal constant

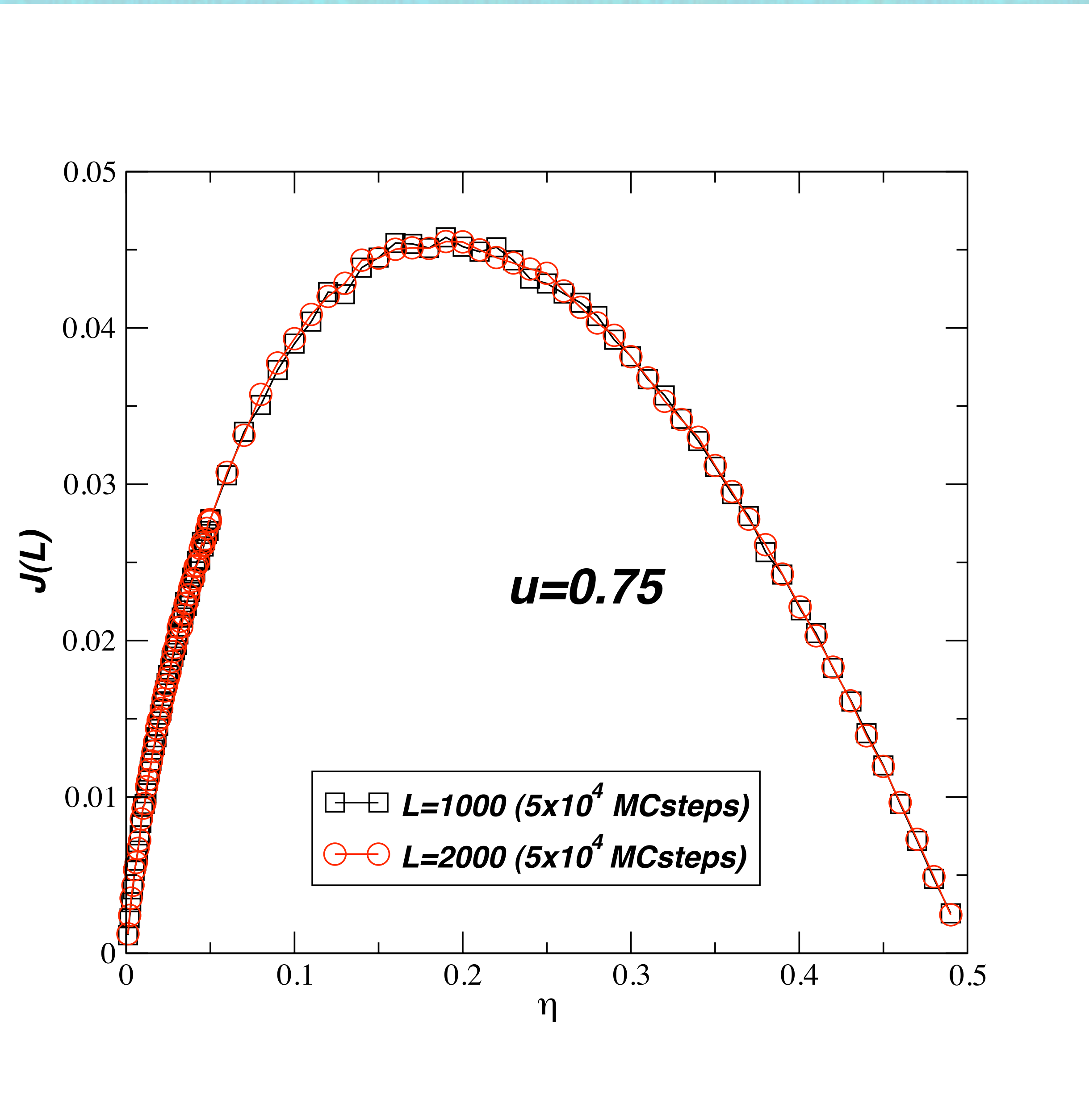
$$\rho = \frac{1}{2}$$

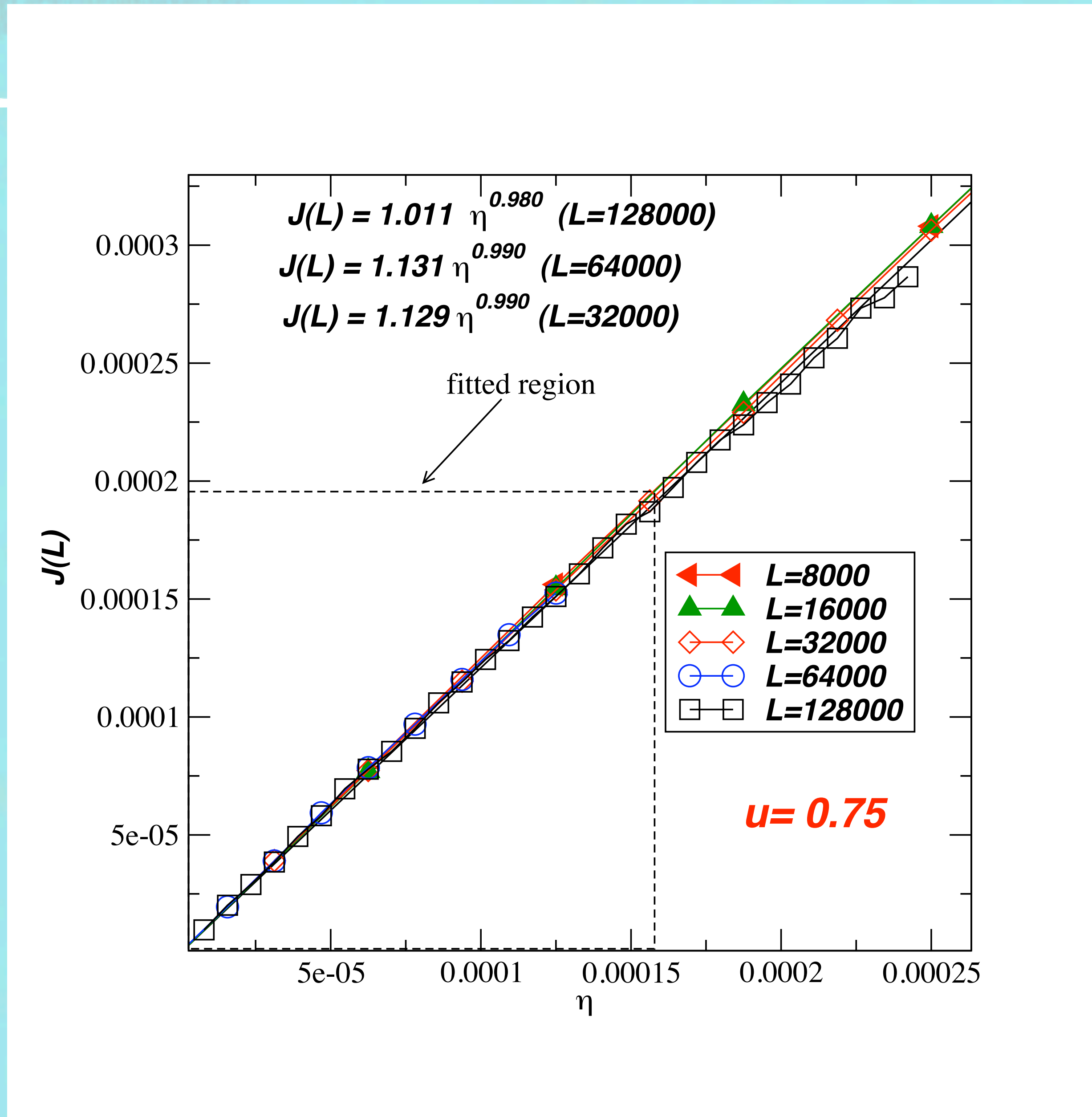


$$J(L) = e^{-\lambda L} \qquad J \sim \frac{1}{L^z}$$

z decreases with u

$$\rho \neq \frac{1}{2} \qquad \eta = \frac{1}{2} - \rho \qquad u < 1$$

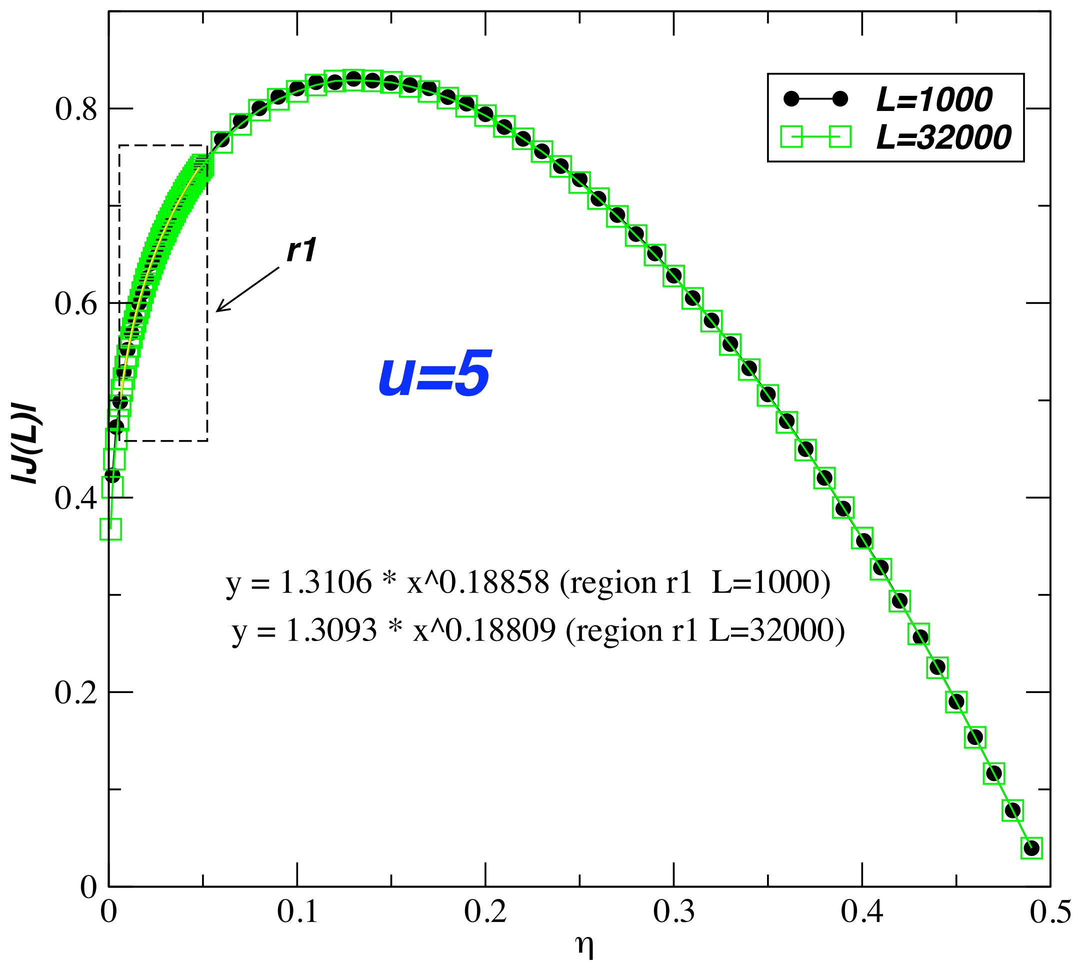


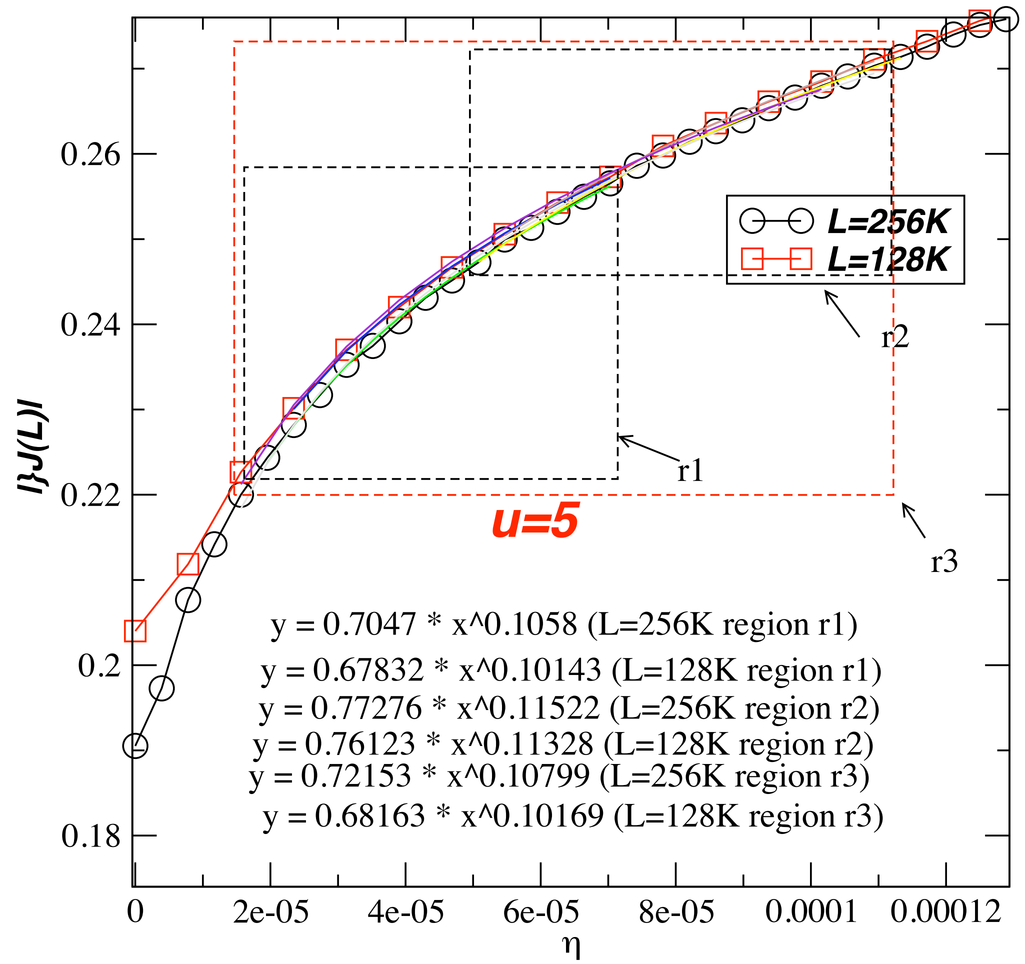


The current growth continuously

$J = A\eta^{\alpha}$ with $\alpha \approx 1$

$\rho \neq \frac{1}{2} \qquad \eta = \frac{1}{2} - \rho \qquad u > 1$





The current growth continuously

$J = A\eta^\alpha$ with $\alpha \approx 0.10$ (quite small)

α decreases with u

Spectrum of H

$$E_0 = 0, \qquad E_i \geq 0$$

GAP : $E_i - E_0 = E_i$

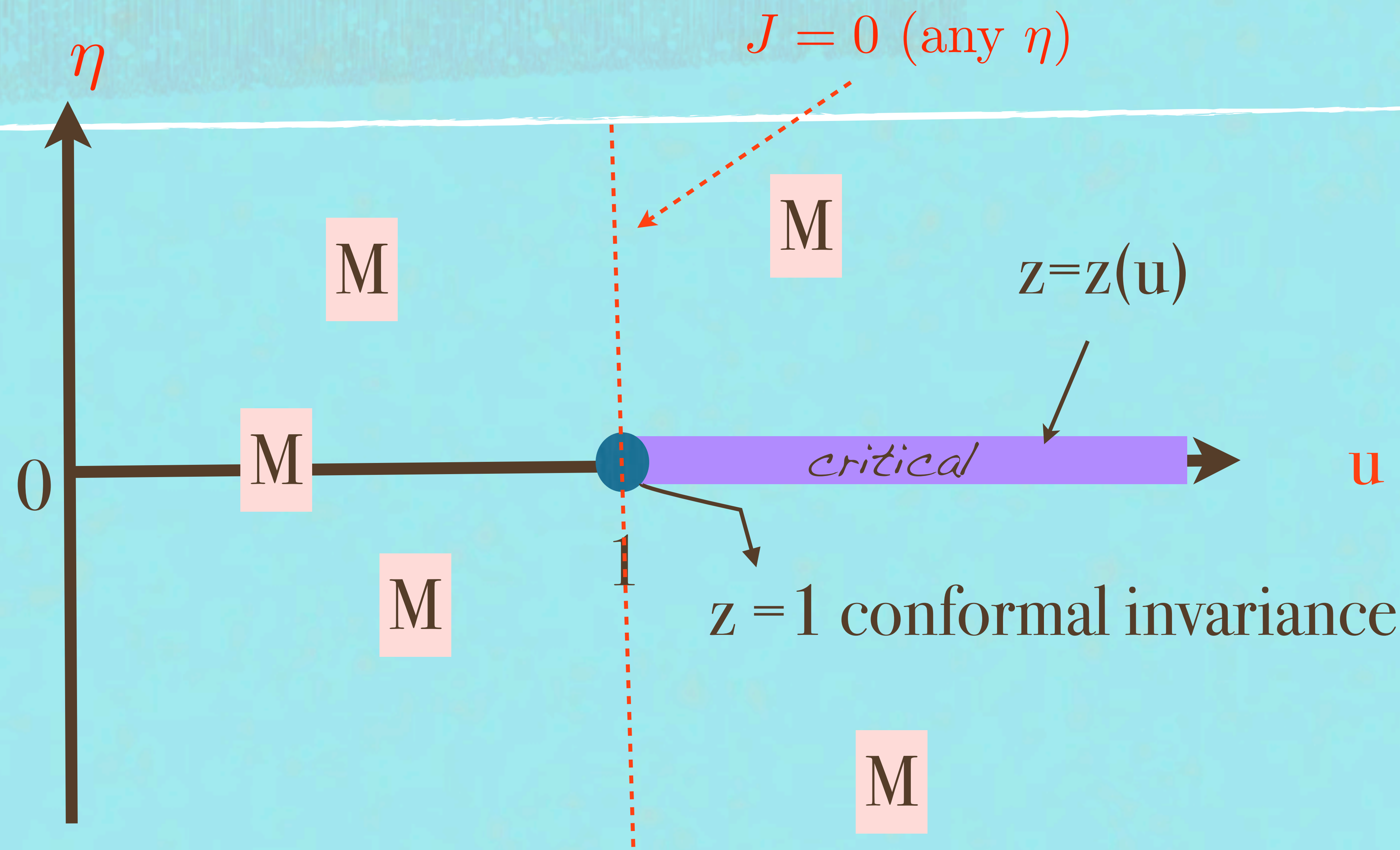
Critical system :

1-D system with L sites

$$E_i \sim \frac{1}{L^z}$$

Dynamical critical exponent: z

Phase diagram



$$J \approx A\,\eta^{\alpha}, \quad A = A(u), \;\; \alpha = \alpha(u)$$

increases with u

decreases with u

Murromboo

Periodic Temperley-Lieb algebra

$$e_{k\pm1}e_ke_{k\pm1} = e_k, \quad e_k^2 = e_k \quad (k = 1, 2, \dots, L)$$

$$[e_k, e_l] = 0 \quad (|k-l| > 1) \qquad e_{k+L} = e_k$$

The algebra is infinite dimensional

L=3

$A = e_1e_2e_3,\qquad A^N,$ for any N are independent words

$$H = \sum_{k=1}^{L}(1 - e_k)$$

2^L representations

a) Spin representation (Levi 1991)

$$e_i = \sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+ - \frac{1}{4}\sigma_i^z \sigma_{i+1}^z - i\frac{\sqrt{3}}{4}(\sigma_i^z - \sigma_{i+1}^z) + \frac{1}{4}$$

$$e_L = \sigma_L^+ \sigma_1^- e^{i\phi} + \sigma_L^- \sigma_1^+ e^{-i\phi} - \frac{1}{4}\sigma_L^z \sigma_1^z - i\frac{\sqrt{3}}{4}(\sigma_L^z - \sigma_1^z) + \frac{1}{4}$$

$$H = -\sum_{i=1}^{L-1}\left[\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+ + \frac{1}{4}\sigma_i^z \sigma_{i+1}^z - \frac{3}{4}\right]$$

$$-\sigma_L^+ \sigma_1^- e^{i\phi} - \sigma_L^- \sigma_1^+ e^{-i\phi} + \frac{1}{4}\sigma_L^z \sigma_1^z + \frac{3}{4}, \quad (L \text{ even})$$

$$\phi = -\frac{2\pi}{3}$$

Hamiltonian hermitian and conformal invariant

b) Charge particle representation, connection with NASEP

+ - + + + - ... + (L sites)

[- +] **rate u**

[+ +] + + - - - - neutral domain (**rate** 1)

[+ +] + + + + ⟶ [+ -] + + + + no neutral domain **rate v**

+ + + - - [- -] neutral domain (**rate** 1)

$u = v = 1 \quad 2^L$ Temperley-Lieb representation

+ ⟶ **particle** – ⟶ **vacancy**

Obs. u=1 half-filling NASEP = Temperley-Lieb