Modular Properties of Non-Rational Conformal Field Theories

David Ridout

Department of Theoretical Physics & Mathematical Sciences Institute,
Australian National University

November 27, 2013

Rational CFTs

The partition function

$$Z(\tau) = \operatorname{tr} \mathrm{e}^{2\pi \mathrm{i}(L_0 - c/24)\tau} \mathrm{e}^{-2\pi \mathrm{i}\left(\overline{L}_0 - \overline{c}/24\right)\overline{\tau}}$$

of a CFT is invariant under the action of the modular group $\mathsf{SL}(2;\mathbb{Z})$:

$$Z(\tau+1) = Z(\tau), \qquad Z(-1/\tau) = Z(\tau).$$

For rational (bosonic) CFTs, the characters span a finite-dimensional representation of $\mathsf{SL}(2;\mathbb{Z})$ with

- $\mathsf{T}\colon \tau \to \tau + 1$ diagonal and unitary,
- $\mathsf{S}\colon \tau \to -1/\tau$ symmetric and unitary, and
- $\mathsf{C} = \mathsf{S}^2 = (\mathsf{ST})^3$ a permutation (called conjugation).

Example: The Ising Model $\mathbf{M}(3,4)$

This $c = \frac{1}{2}$ model is built from 3 simple modules with characters

$$\mathrm{ch}_0 = \sqrt{\frac{\vartheta_3(0,\tau)}{\eta(\tau)}}, \quad \mathrm{ch}_{1/16} = \sqrt{\frac{\vartheta_2(0,\tau)}{2\eta(\tau)}}, \quad \mathrm{ch}_{1/2} = \sqrt{\frac{\vartheta_1(0,\tau)}{\eta(\tau)}}.$$

With respect to the ordered basis $\left[\mathrm{ch}_0, \mathrm{ch}_{1/16}, \mathrm{ch}_{1/2}\right]$, we get

$$\mathsf{T} = \begin{pmatrix} \mathsf{e}^{-\mathrm{i}\pi/24} & 0 & 0 \\ 0 & \mathsf{e}^{\mathrm{i}\pi/12} & 0 \\ 0 & 0 & \mathsf{e}^{23\pi\mathrm{i}/24} \end{pmatrix},$$

$$\mathsf{S} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \qquad \mathsf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Non-Rational CFTs

Of course, we expect that a lot of the structure familiar from rational CFTs will generalise to non-rational theories.

Here, we consider separately four types of CFTs:

Case	Number of irreps	Reps are	Example
I	finite	semisimple	Ising
II	infinite	semisimple	free boson
III	finite	non-semisimple	triplet
IV	infinite	non-semisimple	singlet

Case I describes the rational CFTs whereas Cases III and IV are said to be logarithmic. Case II is what stat mech and string theorists like to study. Case IV is what they should really be studying.

A Case II Example: The Free Boson

This $c = 1$ model admits an uncountable infinity of simple modules parametrised by the momentum $p \in \mathbb{R}$. The characters are

$$\mathrm{ch}_p = \frac{\mathsf{e}^{2\pi \mathrm{i} t} \mathsf{e}^{2\pi \mathrm{i} u p} \mathsf{e}^{\mathrm{i}\pi\tau p^2}}{\eta(\tau)}.$$

With respect to the basis $[\mathrm{ch}_p]$, the transformations

$$\mathsf{T}\colon (t, u, \tau) \to (t, u, \tau + 1), \quad \mathsf{S}\colon (t, u, \tau) \to \left(t - u^2/2\tau, u/\tau, -1/\tau\right)$$

are represented by

$$\mathsf{T}_{pq} = \mathsf{e}^{\mathrm{i}\pi\left(p^2 - 1/12\right)} \delta(p - q), \quad \mathsf{S}_{pq} = \mathsf{e}^{-2\pi \mathrm{i} p q}, \quad \mathsf{C}_{pq} = \delta(p + q),$$

where, for example, $\mathrm{ch}_p(t, u, \tau + 1) = \int_{\mathbb{R}} \mathsf{T}_{pq} \, \mathrm{ch}_q(t, u, \tau) \, \mathrm{d}q$.

A Case III Example: The Triplet Model

In order to study the $c = -2$ fermionic bc ghost system, Kausch introduced the symplectic fermions algebra:

$$\chi^{\pm}(z)\,\chi^{\pm}(w) \sim 0, \qquad \chi^{\pm}(z)\,\chi^{\mp}(w) \sim \frac{\pm 1}{(z-w)^2}.$$

This CFT admits a $\mathbb{Z}_2$-orbifold called the triplet model because it is generated by three dimension 3 Virasoro primaries

$$W^{\pm} = \,:\chi^{\pm}\partial\chi^{\pm}:\,, \quad W^{0} = \,:\chi^{+}\partial\chi^{-}: - :\partial\chi^{+}\chi^{-}:\,.$$

The triplet algebra has four simple modules

$$\mathcal{A}_0, \quad \mathcal{A}_1; \qquad \mathcal{T}_{-1/8}, \quad \mathcal{T}_{3/8}.$$

Only the latter two are their own projective covers.

The simple module characters are

$$\mathrm{ch}_0 = \frac{1}{2}\left(\frac{\theta_{1,2}(0,\tau)}{\eta(\tau)} + \eta(\tau)^2\right), \qquad \mathrm{ch}_{-1/8} = \frac{\theta_{0,2}(0,\tau)}{\eta(\tau)},$$
$$\mathrm{ch}_1 = \frac{1}{2}\left(\frac{\theta_{1,2}(0,\tau)}{\eta(\tau)} - \eta(\tau)^2\right), \qquad \mathrm{ch}_{3/8} = \frac{\theta_{2,2}(0,\tau)}{\eta(\tau)}.$$

But, they do not span an SL $(2;\mathbb{Z})$-rep because of the η^2, *eg*.

$$\mathrm{ch}_0(-1/\tau) = -\frac{\mathrm{i}\tau}{2}\left(\mathrm{ch}_0(\tau) - \mathrm{ch}_1(\tau)\right) + \frac{1}{4}\left(\mathrm{ch}_{-1/8}(\tau) - \mathrm{ch}_{3/8}(\tau)\right).$$

We can get rid of the τ by noting that the characters of the projective covers of $\mathcal{A}_0$ and $\mathcal{A}_1$ coincide:

$$\mathrm{ch}_{\overline{0}} = \mathrm{ch}_{\overline{1}} = 2\left(\mathrm{ch}_0 + \mathrm{ch}_1\right) = 2\frac{\theta_{1,2}(0,\tau)}{\eta(\tau)}.$$

But, $\left\{\mathrm{ch}_{\overline{0}}, \mathrm{ch}_{-1/8}, \mathrm{ch}_{3/8}\right\}$ doesn't span an SL $(2;\mathbb{Z})$-rep either.

Torus Amplitudes

Characters are examples of torus amplitudes and the work of Zhu and Miyamoto shows that one gets an $\mathsf{SL}\,(2;\mathbb{Z})$ action on the space of *all* torus amplitudes.

For rational CFTs, this space coincides with the span of the characters. For logarithmic CFTs, it need not!

For the triplet model, with its four simple characters, the space of torus amplitudes has dimension 5. We may choose the missing generator to be

$$\mathrm{tor} = -\mathrm{i}\tau\,(\mathrm{ch}_0 - \mathrm{ch}_1).$$

With characters being formal power series in $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$, the prefactor $\tau \sim \log q$ of tor arises from an ODE whose indicial equation has repeated roots, *cf.* logs in sphere amplitudes.

Satisfaction

While generalising from characters to torus amplitudes seems to be the natural extension from Case I (rational) to Case III (dropping semisimplicity), there are still reasons to be dissatisfied:

- Partition functions are characters, so finding modular invariants requires imposing the vanishing of the coefficients of non-character torus amplitudes.
- Unitarity no longer implies the existence of canonical modular invariants, *eg*. diagonal, charge conjugation, in general.
- Does the tensor product of two copies of the torus amplitude representation of SL $(2; \mathbb{Z})$ always contain a trivial subrepresentation (a modular invariant)?
- There is no canonical basis of general torus amplitudes, which is bad news for a Verlinde formula.

Triplet Verlinde Formulae

One useful application (in Cases I and II) is to derive fusion coefficients using the Verlinde formula:

$$\mathbf{N}_{ij}{}^{k} = \int \frac{\mathsf{S}_{i\ell}\mathsf{S}_{j\ell}\mathsf{S}^{*}_{k\ell}}{\mathsf{S}_{0\ell}} \, \mathrm{d}\ell.$$

How to generalise without a canonical basis? For the triplet:

- Flohr proposed naïvely substituting the torus amplitude S-matrix into the Verlinde formula and then "truncating" the fusion. It's not clear if this works...
- FHST showed that one can modify the S-transform by an automorphy factor so as to remove all τ factors. The non-semisimple structure of the fusion algebra then gives fusion coefficients from the modified S-matrix. This works, though it's complicated, but it's not clear that it can always be done.

A Case IV Example: The Singlet Model

The symplectic fermions CFT admits $\mathbb{Z}_n$-orbifolds for all $n \in \mathbb{N}$ as well as a $\mathbb{Z}$-orbifold called the singlet model. It is generated by the dimension 3 Virasoro primary W^0.

Unlike the triplet (and the other orbifolds), the singlet algebra possesses an uncountable infinity of simple modules

$$\mathcal{A}_\lambda \quad (\lambda \in \mathbb{Z}), \qquad \mathcal{T}_\lambda \quad (\lambda \notin \mathbb{Z}).$$

The $\mathcal{T}_\lambda$ are (conjecturally) projective whereas the $\mathcal{A}_\lambda$ are not.

The characters of the $\mathcal{T}_\lambda$ are given by

$$\mathrm{ch}_\lambda = \frac{\mathsf{e}^{2\pi\mathrm{i}t}\mathsf{e}^{2\pi\mathrm{i}u(\lambda-1/2)}\mathsf{e}^{\mathrm{i}\pi\tau(\lambda-1/2)^2}}{\eta(\tau)},$$

where we include the (shifted) ghost number $\lambda - \frac{1}{2}$ in order to distinguish characters of non-isomorphic singlet modules.

Surprisingly, the $\mathcal{T}_\lambda$ carry an action of $\mathsf{SL}(2;\mathbb{Z})$:

$$\mathsf{T}_{\lambda\mu} = \mathsf{e}^{\mathsf{i}\pi(\lambda(\lambda-1)+1/6)}\delta(\lambda-\mu),$$
$$\mathsf{S}_{\lambda\mu} = \mathsf{e}^{-2\pi\mathsf{i}(\lambda-1/2)(\mu-1/2)}, \qquad \mathsf{C}_{\lambda\mu} = \delta(\lambda+\mu-1).$$

Note that T is diagonal, S is symmetric and C is a permutation. All are unitary!

Homological algebra relates the $\mathcal{A}_\lambda$ to the (non-simple) $\mathcal{T}_\lambda$:

$$\cdots \longrightarrow \mathcal{T}_{\lambda+3} \longrightarrow \mathcal{T}_{\lambda+2} \longrightarrow \mathcal{T}_{\lambda+1} \longrightarrow \mathcal{T}_{\lambda} \longrightarrow \mathcal{A}_{\lambda} \longrightarrow 0.$$

This then gives the $\mathsf{SL}(2;\mathbb{Z})$-action on the $\mathcal{A}_\lambda$, *eg*.

$$\mathsf{S}_{\underline{\lambda}\mu} = \frac{\mathsf{e}^{-2\pi\mathsf{i}\lambda(\mu-1/2)}}{2\cos[\pi(\mu-1/2)]}.$$

[The underline indicates that λ corresponds to $\mathcal{A}_\lambda$, not $\mathcal{T}_\lambda$.]

Satisfaction

This is good!

- The $\mathcal{T}_\lambda$ (simple and non-simple) provide a canonical basis of characters for which T, S and C have the expected properties.
- Unitarity implies diagonal partition function is modular invariant.
- No need to look for additional torus amplitudes.
- Substituting into the Verlinde formula (with vacuum $\mathcal{A}_0$) gives the correct (Grothendieck) fusion coefficients.
- One can also obtain the correct (Grothendieck) fusion coefficients for the triplet model by realising it as a simple current extension of the singlet.

Why does Case IV work so much better than Case III?

Because the $\mathcal{A}$-type modules correspond to a set of measure zero in Case IV, but their measure is positive for Case III.

Where to now?

While we've discussed the excellent modular behaviour of the $c = -2$ singlet model, this formalism has also been checked explicitly for other logarithmic CFTs:

- (p, q) singlet models $(p, q \in \mathbb{Z}_+)$ with $c = 1 - 6(p-q)^2/pq$
  $\Rightarrow$ effortless derivation of all triplet (Grothendieck) fusion rules.
- $\widehat{\mathfrak{sl}}(2)_k$ fractional level WZW models with $c = 3k/(k+2)$.
  $\Rightarrow$ resolution of famous negative Verlinde coefficients problem.
- $\widehat{\mathfrak{gl}}(1|1)$ WZW model with $c = 0$ and its "Takiffisation".
- Virasoro models for arbitrary c (Kac module fusion).
- Superconformal models for arbitrary c.

The (only?) known Case III examples, the triplet models, are simple current extensions of singlet models. We are therefore analysing the role of simple currents in logarithmic theories.


Thankyou
(Noosa 2002)