The Toda lattice and a quantum curve.

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Joint with Dunin-Barkowski, Mulase, Popolitov, Shadrin.

Outline: a story of a wave function Ψ.

- Toda lattice: discrete and continuous ... $\mathcal{L}\Psi = 0$.
- Rational behaviour of wave function.
- WKB method for producing wave function.
- Quantum curve for Gromov-Witten invariants of $\mathbb{P}^1$:

$$\hat{A}(\hat{x},\hat{y})\Psi = 0 \longleftrightarrow A(x,y) = 0, \quad \hat{x} = x, \hat{y} = \hbar\frac{d}{dx}$$

- Gukov-Sułkowski conjecture.

Toda lattice: $\ddot{q}_n = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}}$

- $u_n = q_{n-1} - q_n,\ v_n = -\dot{q}_n$
- $\tilde{\mathcal{L}} = \Lambda + v_n + e^{u_n}\Lambda^{-1}, \quad \Lambda f_n = f_{n+1}$
- $\dfrac{d\tilde{\mathcal{L}}}{dt} = [\Lambda + v_n, \tilde{\mathcal{L}}]$

$$
\begin{aligned}
\Leftrightarrow \quad \dot{v}_n + \dot{u}_n e^{u_n}\Lambda^{-1} &= [\Lambda + v_n, \Lambda + v_n + e^{u_n}\Lambda^{-1}] \\
&= (\Lambda + v_n)e^{u_n}\Lambda^{-1} - e^{u_n}\Lambda^{-1}(\Lambda + v_n) \\
&= e^{u_{n+1}} + v_n e^{u_n}\Lambda^{-1} - e^{u_n} - e^{u_n} v_{n-1}\Lambda^{-1} \\
&= e^{u_{n+1}} - e^{u_n} + (v_n - v_{n-1})e^{u_n}\Lambda^{-1}
\end{aligned}
$$

$$\Leftrightarrow \quad \dot{v}_n = e^{u_{n+1}} - e^{u_n}, \quad \dot{u}_n = v_n - v_{n-1}$$

Continuous Toda

- Interpolation: $u_n = u(\hbar n), \quad v_n = v(\hbar n)$

$$\mathcal{L} = \Lambda + v(x) + e^{u(x)}\Lambda^{-1}, \ \Lambda = e^{\hbar \frac{\partial}{\partial x}}$$

- $\dfrac{\partial \mathcal{L}}{\partial t} = [\Lambda + v(x), \mathcal{L}]$

$$\Leftrightarrow \quad \frac{\partial}{\partial t} v(x) = e^{u(x+\hbar)} - e^{u(x)}, \quad \frac{\partial}{\partial t} u(x) = v(x) - v(x-\hbar)$$

- Wave function: (for solution: $u = -\hbar t, \ v = -x$)

$$\mathcal{L} = \Lambda - x + q\Lambda^{-1}, \quad \mathcal{L}\Psi = 0, \quad \Psi = \Psi(x, \hbar, q)$$

- Aganagic-Dijkgraaf-Klemm-Mariño-Vafa

$$H = e^{\hbar \frac{\partial}{\partial x}} + x + e^{-\hbar \frac{\partial}{\partial x}}$$

$H\psi = 0$: insertion of D-brane at fixed x.

Rational behaviour.

$$\text{Consider } \Psi = \sum_{d\geq 0} q^d \Psi_d = \Psi_0 \left(1 + q\frac{\Psi_1}{\Psi_0} + q^2\frac{\Psi_2}{\Psi_0} + \dots\right)$$

$\mathcal{L}\Psi = 0 \Rightarrow \frac{\Psi_d}{\Psi_0}$ is

- rational in x, with d simple poles

$$\frac{\Psi_d}{\Psi_0} = \sum_{\lambda\vdash d} \frac{1}{H_\lambda^2} \prod_{m=1}^{d} \frac{x - (m - \frac{1}{2} - \lambda_m)\hbar}{x - (m - \frac{1}{2})\hbar}, \; H_\lambda := \prod_{ij} h_{ij}$$

$$= a_{0,d} + \frac{a_{1,d}\hbar}{x - \frac{\hbar}{2}} + \frac{a_{2,d}\hbar}{x - \frac{3\hbar}{2}} + \dots + \frac{a_{d,d}\hbar}{x - (d - \frac{1}{2})\hbar}$$

- $a_{m,d} = \frac{1}{d!}\frac{1}{(m-1)!} L_{d-m}^{(m)}(1)$
- *Laguerre polynomials* $L_n^{(\alpha)}(z) = \sum_{i=0}^{n} (-1)^i \binom{n+a}{n-i} \frac{z^i}{i!}$.

WKB method

- Assume the wave function has the form

$$\log \Psi(x) = \hbar^{-1} S_0(x) + \hbar^0 S_1(x) + \hbar S_2(x) + \ldots$$

- $\mathcal{L}\Psi = 0 \Rightarrow$ can recursively solve for $S_k(x)$.
- $\Lambda = e^{\hbar \frac{\partial}{\partial x}} = \sum \frac{\hbar^n}{n!} \left(\frac{\partial}{\partial x}\right)^n$ acts on Laurent series in $\hbar$
- Ψ is not Laurent in $\hbar$ ... make sense of $\mathcal{L}\Psi$ by conjugation:

$$\begin{aligned}
\mathcal{L}_1 &= e^{-\frac{1}{\hbar} S_0} \mathcal{L} e^{\frac{1}{\hbar} S_0} \\
&= e^{-\frac{1}{\hbar} S_0} \Lambda e^{\frac{1}{\hbar} S_0} + q e^{-\frac{1}{\hbar} S_0} \Lambda^{-1} e^{\frac{1}{\hbar} S_0} - x \\
&= e^{\frac{1}{\hbar}(S_0(x+\hbar) - S_0(x))} \Lambda + q e^{\frac{1}{\hbar}(S_0(x-\hbar) - S_0(x))} \Lambda^{-1} - x
\end{aligned}$$

now acts on $e^{-\frac{1}{\hbar} S_0} \Psi$ which is a Laurent series in $\hbar$.

Spectral curve

- Semi-classical limit:

$$\lim_{\hbar\to 0}\mathcal{L}_1 = \lim_{\hbar\to 0}\{e^{\frac{1}{\hbar}(S_0(x+\hbar)-S_0(x))}\Lambda + qe^{\frac{1}{\hbar}(S_0(x-\hbar)-S_0(x))}\Lambda^{-1} - x\}$$
$$= e^{S_0'(x)} + qe^{-S_0'(x)} - x$$

which is a multiplication operator and hence must vanish.

- so it defines an algebraic curve

$$z + qz^{-1} - x = 0, \quad z = e^{S_0'(x)}.$$

- we say that $\mathcal{L}$ is a quantisation of the curve

$$x = z + \frac{q}{z}, \quad y = \log z, \quad (x, y) \mapsto (x, \hbar\frac{d}{dx}).$$

Exact solution to the WKB method

- Summary:
  - $\mathcal{L} = \Lambda - x + q\Lambda^{-1}, \quad \mathcal{L}\Psi = 0$
  - $\log \Psi(x) = \hbar^{-1} S_0(x) + \hbar^0 S_1(x) + \hbar S_2(x) + \dots$
  - $\hbar \to 0 \quad x = z + \frac{q}{z}, \quad y = \log z, \quad (x, y) \mapsto (x, \hbar \frac{d}{dx})$

- Gukov-Sułkowski: conjectured an exact expression for $S_k(x)$
- $S_k(x)$ are meromorphic functions on the curve $x = z + \frac{q}{z}$
- Moreover they conjectured an explicit relation between the $S_k(x)$ for $k > 1$ and CEO invariants:

$$\log \Psi(x) = \hbar^{-1} S_0(x) + \hbar^0 S_1(x) + \sum_{2g-2+n>0} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}(x, \dots, x)$$

- $d_1 \cdots d_n F_{g,n}(x_1, \dots, x_n) = W_{g,n}(x_1, \dots, x_n)$ multidifferentials on $x = z + \frac{q}{z}$ produced by CEO recursion

Results

- Theorem (N., Scott, $g = 0, 1$, 2011; DOSS, all g, 2012)
  The CEO recursion applied to $x = z + \frac{q}{z}$, $y = \log z$ *produces stationary Gromov-Witten invariants of* $\mathbb{P}^1$*, i.e. an expansion of the multidifferentials* $W_{g,n}(x_1, \dots, x_n)$ *are generating functions for stationary Gromov-Witten invariants of* $\mathbb{P}^1$*.*
- Theorem (DuninBarkowski-Mulase-N-Popolitov-Shadrin 2013)
  The Gukov-Sułkowski conjecture holds for $x = z + \frac{q}{z}$, $y = \log z$*.*

  - Ψ is non-perturbative—it collects different genus
  - Place target stationary points at a single point p.
    $\Psi(x) = \sum N_k x^{-k}$, $N_k = \#$ covers with ramification k over p.