Functional relations in logarithmic minimal models

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Based on joint work with Paul A. Pearce and Jørgen Rasmussen

Outline

Temperley-Lieb loop models

- ↪ Planar Temperley-Lieb algebras
- ↪ Transfer tangles

Fused loop models

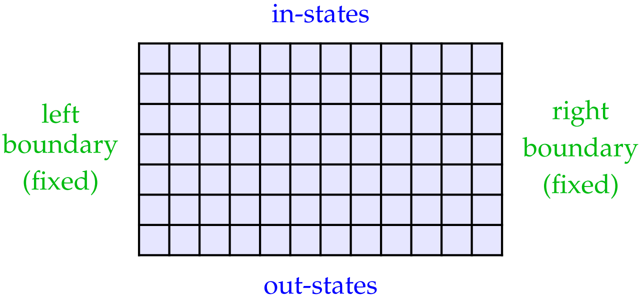
- ↪ Wenzl-Jones projectors
- ↪ Fused face operators and transfer tangles

Fusion hierarchies

- ↪ Fusion hierarchies of transfer tangles
- ↪ Closure of fusion hierarchies $\Rightarrow$ Functional relations
- ↪ Y-systems

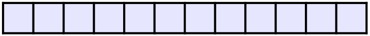
Conclusion and outlook

Transfer matrices



Transfer matrix

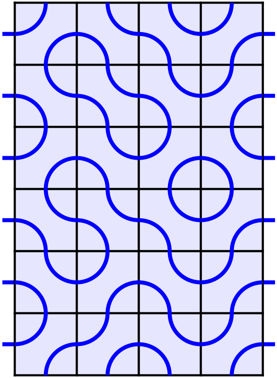
$T(u) \sim$



↪ $T(u)$ is an operator that acts on in-states and outputs the possible out-states with the correct Boltzmann weights

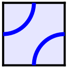
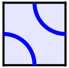
↪ The spectral parameter $u \in \mathbb{R}$ measures the lattice anisotropy

Lattice loop model



Loop configuration on the **cylinder** with **non-local** degrees of freedom

- Elementary face operator:

 $= s_1(-u)$  $+ s_0(u)$ 

- Boltzmann weights given in terms of crossing parameter λ and spectral parameter u:

$$s_k(u) = \frac{\sin(u + k\lambda)}{\sin\lambda}$$

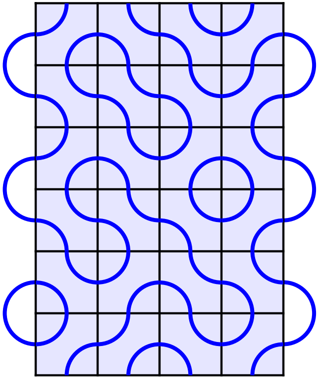
- Loop fugacities:

contractible:	$\beta = 2\cos\lambda$
non-contractible:	α

Logarithmic minimal model $\mathcal{LM}(p,p')$

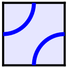
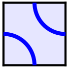
$$\lambda = \frac{(p'-p)\pi}{p'}, \qquad (p,p') = 1, \qquad p,p' \in \mathbb{N}$$

Lattice loop model



Loop configuration on the **strip** with **non-local** degrees of freedom

↳ Elementary face operator:

$= s_1(-u)$  $+ s_0(u)$ 

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↳ Loop fugacities:

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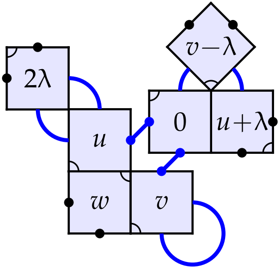
Logarithmic minimal model $\mathcal{LM}(p,p')$

$$\lambda = \frac{(p'-p)\pi}{p'}, \qquad (p,p')=1, \qquad p,p' \in \mathbb{N}$$

Planar Temperley-Lieb algebra

↪ An ***N*-tangle** is a diagram (with $2N$ free nodes) composed of face operators glued together by loop segments

4-tangle obtained as a multiplication of two 3-tangles:

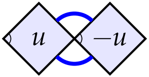


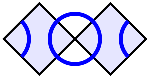
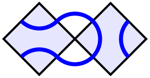
↪ Tangles also include single connectivity diagrams since

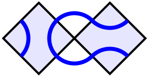
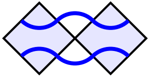
Planar TL algebra with *fixed direction of transfer*	↔	Temperley-Lieb algebra

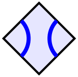
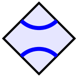
Planar identities

Local inversion relation

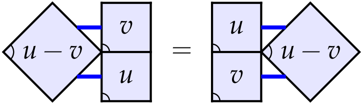


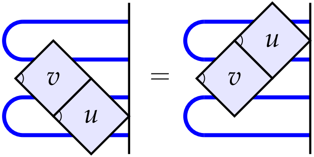
$= s_0(u)s_0(-u)$  $+ \; s_1(-u)s_0(-u)$ 

$+ \; s_0(u)s_1(u)$  $+ \; s_1(-u)s_1(u)$ 

$= \underbrace{\big(\beta s_0(u)s_0(-u) + s_1(-u)s_0(-u) + s_0(u)s_1(u)\big)}_{=0}$  $+ \; s_1(-u)s_1(u)$ 

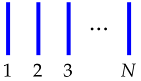
Yang-Baxter equations

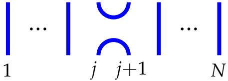




Temperley-Lieb algebra

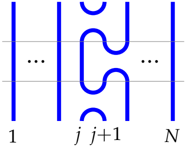
On the strip

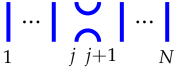
$I =$ 

$e_j =$ 

↳ Multiplication is by vertical concatenation

Example

$e_j\, e_{j+1}\, e_j =$ 

$=$ 

$= e_j$

Algebraic definition

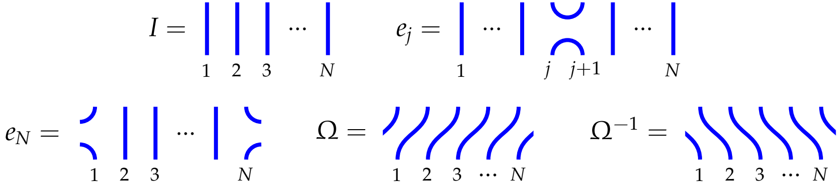
$$TL_N(\beta) = \langle I,\, e_j\,;\, j = 1, \ldots, N-1 \rangle$$

$$I\, e_j = e_j\, I = e_j$$

$$e_j^2 = \beta e_j, \qquad e_j e_{j\pm1} e_j = e_j, \qquad e_i e_j = e_j e_i, \qquad |i-j| > 1$$

Enlarged periodic Temperley-Lieb algebra

On the cylinder



Algebraic definition [subscripts interpreted mod N with $e_0 \equiv e_N$]

$$\mathcal{E}PTL_N(\alpha,\beta) = \langle I,\,\Omega,\,\Omega^{-1},\,e_j\,;\,j=1,\ldots,N\rangle$$

$$I\,e_j = e_j\,I = e_j \qquad e_j^2 = \beta e_j$$

$$e_j e_{j\pm1} e_j = e_j \qquad e_i e_j = e_j e_i, \quad |i-j|>1$$

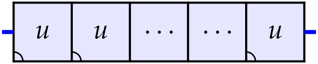
$$\Omega e_i \Omega^{-1} = e_{i-1} \qquad \Omega\Omega^{-1} = \Omega^{-1}\Omega = I$$

$$(\Omega^{\pm1}e_N)^{N-1} = \Omega^{\pm N}(\Omega^{\pm1}e_N) \qquad \Omega^{\pm N}e_N\Omega^{\mp N} = e_N$$

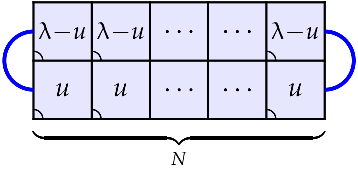
$$E\Omega^{\pm1}E = \alpha E \qquad E = e_2e_4\ldots e_{N-2}e_N \quad (N\text{ even})$$

Transfer tangles

On the cylinder

$\boldsymbol{T}(u) =$ 

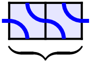
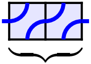
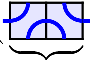
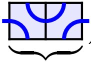
On the strip

$\boldsymbol{D}(u) =$ 

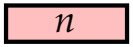
↪ Two commuting families of transfer tangles:

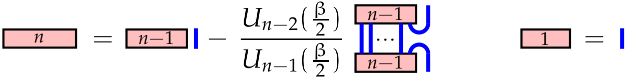
$$[\boldsymbol{T}(u), \boldsymbol{T}(v)] = 0, \qquad [\boldsymbol{D}(u), \boldsymbol{D}(v)] = 0, \qquad \forall\, u, v \in \mathbb{C}$$

↪ An example for $N = 2$:

$\boldsymbol{T}(u) = s_0^2(u)$  $= \Omega^{-1}$ $+\, s_1^2(-u)$  $= \Omega$ $+\, s_0(u)s_1(-u)\Big($  $= e_1\Omega$ $+$  $= \Omega\, e_1$ $\Big)$

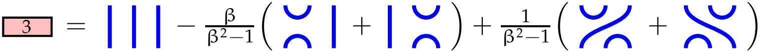
Wenzl-Jones projectors

↪ The WJ projector P_n is an n-tangle depicted as , $n \in \mathbb{N}$, and defined recursively by

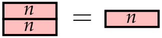


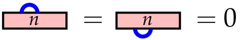
↪ Chebyshev polynomial of the second kind: $U_k(\frac{\beta}{2}) = s_{k+1}(0)$, $k \in \mathbb{N}_0$

Examples



Properties of the WJ projectors

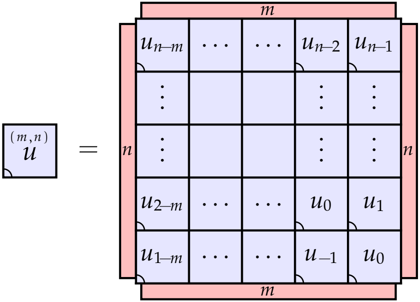
(i) P_n is a projector: 

(ii) P_n is an annihilator: 

(iii) $P_{n=p'}$ diverges for $\lambda = \frac{(p'-p)\pi}{p'}$

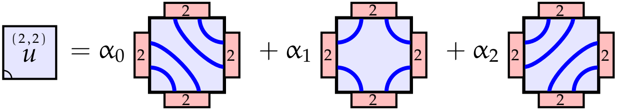
Fused faces

(m,n)-fused face operator



$$u_k = u + k\lambda$$

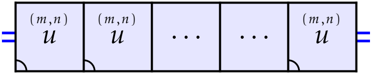
Generalised monoids [illustrated for $(m,n) = (2,2)$]



Fused transfer tangles

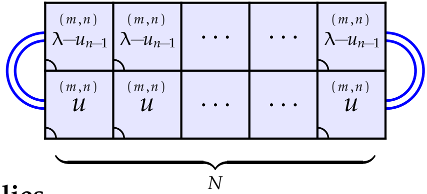
On the cylinder

$T^{m,n}(u) =$



On the strip

$D^{m,n}(u) =$

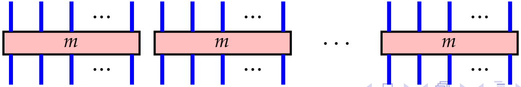


Two commuting families

$$\big[T^{m,n}(u), T^{m,n'}(v)\big] = 0 \qquad \big[D^{m,n}(u), D^{m,n'}(v)\big] = 0, \qquad \forall\, u, v \in \mathbb{C}$$

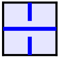
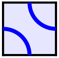
Fused identity tangle

$I^m =$

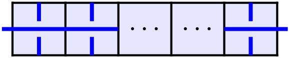


Braid tangles

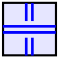
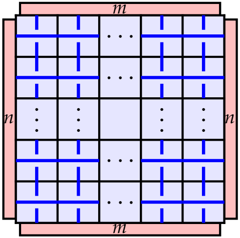
Elementary braid face operator

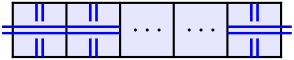
 $= e^{-\mathrm{i}\frac{\pi-\lambda}{2}}$  $+ e^{\mathrm{i}\frac{\pi-\lambda}{2}}$  $= \lim_{u\to \mathrm{i}\infty} \frac{e^{\mathrm{i}\frac{\pi-\lambda}{2}}}{s_0(u)}$ 

Braid transfer tangle on the cylinder

$\boldsymbol{F} =$  $\in Z\big[\mathcal{E}PTL_N(\alpha,\beta)\big]$

Fused braid tangles on the cylinder

 $=$ 

$\boldsymbol{F}^{m,n} =$ 

Critical dense polymers $\mathcal{LM}(1,2)$

↪ Described by $\lambda = \frac{\pi}{2}$ for which contractible loops are disallowed

$$\boxed{\lambda = \frac{\pi}{2} \quad \Rightarrow \quad \beta = 0}$$

Inversion identity

$$\boldsymbol{T}(u)\boldsymbol{T}(u+\tfrac{\pi}{2}) = \boldsymbol{I}\big(\cos^{2N}u + (-1)^N \sin^{2N}u\big) + 2\boldsymbol{J}(-\mathrm{i}\cos u \sin u)^N$$

$$\text{where} \qquad \boldsymbol{J} = \tfrac{1}{2}(\boldsymbol{F}^2 - 2\boldsymbol{I})$$

↪ Closed form for eigenvalues known for all N.

Objective

↪ Extend these results from $\mathcal{LM}(1,2)$ to general $\mathcal{LM}(p,p')$ where

$$\lambda = \frac{(p'-p)\pi}{p'}, \qquad (p,p') = 1, \qquad p,p' \in \mathbb{N}$$

Fusion hierarchies in loop models

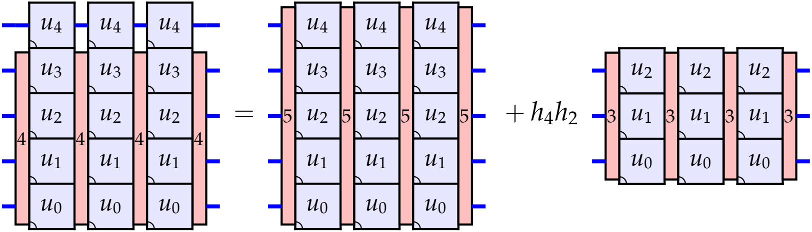
On the cylinder $[\lambda \in \mathbb{R}]$

$$\boldsymbol{T}_0^{m,n}\boldsymbol{T}_n^{m,1} = \boldsymbol{T}_0^{m,n+1} + h_n\, h_{n-2}\, \boldsymbol{T}_0^{m,n-1}$$

↪ Shorthand notations and definitions

$$\boldsymbol{T}_k^{m,n} = \boldsymbol{T}^{m,n}(u+k\lambda), \qquad \boldsymbol{T}_k^{m,0} = \boldsymbol{I}^m, \qquad h_k = \Big(\prod_{j=0}^{m-1}(-\mathrm{i})s_{k-j}(u)\Big)^N$$

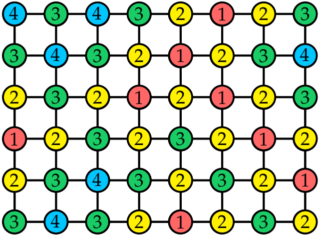
↪ Diagrammatical example $[m=1, n=4, N=3]$



Fusion hierarchies in RSOS models

- Height model with L values
- Large family of **commuting fused transfer matrices** $T^{p,q}(u,\lambda)$ (with $\lambda = \frac{\pi}{L+1}$):

$$[T^{p,q}(u,\lambda), T^{p,q'}(v,\lambda)] = 0$$



Fusion hierarchy

$$T_0^{p,q}T_q^{p,1} = f_q^p T_0^{p,q-1} + f_{q-1}^p T_0^{p,q+1} \quad \text{where} \quad T_k^{p,q} = T^{p,q}(u+k\lambda,\lambda)$$

Fusion closure

$$T_0^{p,L-1} = f_{L-1}^p R, \qquad T_0^{p,L} = 0$$

Y-system

$$t_0^{p,q}t_1^{p,q} = (1+t_1^{p,q-1})(1+t_0^{p,q+1}) \quad \text{where} \quad t_0^{p,q} = \frac{T_1^{p,q-1}T_0^{p,q+1}}{f_{-1}^p f_q^p}$$

Functional relations for loop models

Fusion closure on the cylinder $\left[\lambda = \frac{(p'-p)\pi}{p'}\right]$

$$\boldsymbol{T}_0^{m,p'} = h_0\, h_{p'-2}\, \boldsymbol{T}_1^{m,p'-2} + 2a\, \boldsymbol{J}^m, \qquad \boldsymbol{J}^m = \tfrac{1}{2}\left(\boldsymbol{F}^{m,p'} - \boldsymbol{F}^{m,p'-2}\right)$$

$$a = e^{i\theta} \prod_{j=0}^{p'-1} h_j, \qquad \theta = \tfrac{1}{2} Nm(p'-p)\pi$$

Functional relation in determinant form $\left[\lambda = \frac{(p'-p)\pi}{p'}\right]$

$$\begin{vmatrix} \boldsymbol{T}_0^{m,1} & h_{-1} & 0 & 0 & h_0 \\ h_1 & \boldsymbol{T}_1^{m,1} & h_0 & 0 & 0 \\ 0 & h_2 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \boldsymbol{T}_{p'-2}^{m,1} & h_{p'-3} \\ h_{p'-2} & 0 & 0 & h_{p'-1} & \boldsymbol{T}_{p'-1}^{m,1} \end{vmatrix} = a \begin{vmatrix} \boldsymbol{F}^{m,1} & 1 & 0 & 0 & e^{-i\theta} \\ 1 & \boldsymbol{F}^{m,1} & 1 & 0 & 0 \\ 0 & 1 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \boldsymbol{F}^{m,1} & 1 \\ e^{i\theta} & 0 & 0 & 1 & \boldsymbol{F}^{m,1} \end{vmatrix}$$

Universal Y-system

On the cylinder $[\lambda \in \mathbb{R}]$

$$\boldsymbol{t}_0^{m,n}\boldsymbol{t}_1^{m,n} = \big(\boldsymbol{I}^m + \boldsymbol{t}_0^{m,n+1}\big)\big(\boldsymbol{I}^m + \boldsymbol{t}_1^{m,n-1}\big)$$

where

$$\boldsymbol{t}_k^{m,n} = \frac{\boldsymbol{T}_{k+1}^{m,n-1}\boldsymbol{T}_k^{m,n+1}}{\mu_k}, \quad \mu_k = \prod_{j=k}^{n+k-1} h_{j+1}h_{j-1}, \quad k \in \mathbb{Z}$$

Closure of the Y-system $\left[\lambda = \frac{(p'-p)\pi}{p'}\right]$

$$\boldsymbol{t}_0^{m,\frac{3p'}{2}} = \boldsymbol{t}_{\frac{p'+2}{2}}^{m,\frac{3p'-4}{2}} + U_2(\boldsymbol{J}^m)\Big(\boldsymbol{t}_0^{m,\frac{p'}{2}} - \boldsymbol{t}_{\frac{p'+2}{2}}^{m,\frac{p'-2}{2}}\Big), \qquad p' \text{ even}$$

$$\boldsymbol{t}_0^{m,\frac{3p'-1}{2}} = \boldsymbol{t}_{\frac{p'+1}{2}}^{m,\frac{3p'-3}{2}} + U_2(\boldsymbol{J}^m)\Big(\boldsymbol{t}_0^{m,\frac{p'-1}{2}} - \boldsymbol{t}_{\frac{p'+1}{2}}^{m,\frac{p'-3}{2}}\Big), \qquad p' \text{ odd}$$

Summary and outlook

Summary

- ↪ TL loop models described by $\boldsymbol{T}(u)$ and $\boldsymbol{D}(u)$ $[\lambda \in \mathbb{R}]$
- ↪ Fusion hierarchies of fused transfer tangles $[\lambda \in \mathbb{R}]$
- ↪ Functional relations for $\boldsymbol{T}(u)$ and $\boldsymbol{D}(u)$ $\left[\lambda = \frac{(p'-p)\pi}{p'}\right]$
- ↪ Universal Y-system $[\lambda \in \mathbb{R}]$
- ↪ All computations are performed in the planar algebra. The results are therefore valid for all possible representations.

Outlook

- ↪ Examine the Y-system in the continuum scaling limit
- ↪ Explore the representation theory of fused loop models and the corresponding (logarithmic) conformal field theories
- ↪ Extend to dilute loop models
- ↪ Extend to models allowing crossings, using BMW algebras

Some references

Temperley-Lieb algebras

Temperley, Lieb (1971); Martin, Saleur (1993); Jones (1999)

Loop models

Nienhuis (1987); Blöte, Nienhuis (1989)

Fused loop models

Fendley, Read (2002); Zinn-Justin (2007)

Fusion hierarchies and Y-systems (rational CFT models)

Bazhanov, Reshetikhin (1989); Klümper, Pearce (1992);
Kuniba, Nakanishi, Suzuki (1994, 2011)

Logarithmic minimal models

Pearce, Rasmussen, Zuber (2006); Pearce, Rasmussen (2007, . . .);
AMD, Saint-Aubin (2011, 2013)

Work in progress

AMD, Pearce, Rasmussen, *Fusion hierarchies and Y-systems of logarithmic minimal models*, in preparation (2013)