Rogue waves in 1+1 and 2+1 integrable models: from NLS to the KP-I equation

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Main Results

1. Multi-rogue wave solutions to the focusing NLS and Gross-Pitaevskii equation
2. General multi-rogue wave solution for n=3
3. Multi-rogue waves solutions of NLS equation and KP-I equation
4. Large parametric behavior of rank 3 solutions
5. Large parametric asymptotic of rank 4 solutions
6. Other determinant formulas for MRW solutions and P_n breathers

Abstract

The discovery of the multi-rogue waves (MRW) solutions for the focusing NLS equation made in 2010 by Philippe Dubard and myself (Eur.Phys. J,Special topics **185**, 247-258, 2010),- drastically improved the vision of the links of the rogue waves and the theory of integrable systems. The MRW solutions might be described by means of Wronskian determinant representation of a very simple structure . This structure allows to relate them with multi-rogue waves solutions of the KP-I equation via remarkable relation which we call NLS-KP-I correspondence. For the NLS case these solutions generalize both the famous Peregrine breather or P_1 breather, as we call it here ,- and its higher order versions P_n breathers or equivalently rank n Peregrine breathers. After our works of 2010-2011 it becomes clear that, starting from the rank 2,- P_n breathers are not isolated and represent the particular reduction of the MRW solutions corresponding to specific choice of the parameters.
Here we provide the formulas which (at least for small ranks) allows to show that the "extreme" rogue waves in 2+1 dimensional models occur as a result of the collision of certain number of the "simple" rogue waves.
We also describe various kinds of large parametric limits of the NLS MRW solutions.
The reported results are available at the website
http://www.kurims.kyoto-u.ac.jp/preprint/, Preprint RIMS1777 , p.1-39 , March 2013,
Published online in Nonlinearity **26** n.12 , (2013),
Additional movies describing various kinds of evolutions of the multiple rogue waves for the KP-I equation making a part of this work can be seen at stacks provided by Nonlinearity and also
http://www.kurims.kyoto-u.ac.jp/∼kirillov/MATVEEV
These movies also describe an infinite families of plots of the squared magnitude of the NLS equation.

Focusing NLS equation reads

$$iv_t + 2|v|^2 v + v_{xx} = 0, \quad x, t \in \mathbb{R}.$$

Multi rogue waves solutions of the NLS equation are quasi rational solutions:

$$v = e^{2iB^2 t} R(x,t), \quad R(x,t) = \frac{N(x,t)}{D(x,t)}, \quad B > 0,$$

Here $N(x,t), D(x,t)$ are polynomials of x and t, and deg $N(x,t)$ = deg $R(x,t) = n(n+1)$,

$$|v^2| \to B^2, \quad x^2 + t^2 \to \infty$$

The rational function $R(x,t)$ satisfies the 1D Gross-Pitaevskii equation:

$$iR_t + 2R(|R|^2 - B^2) + R_{xx} = 0, \quad |R| = |v|.$$

$$q_{2n}(k) := \prod_{j=1}^{n}\left(k^2 - \frac{\omega^{2m_j+1}+1}{\omega^{2m_j+1}-1}B^2\right), \quad \omega := \exp\left(\frac{i\pi}{2n+1}\right).$$

Numbers m_j are some positive integers satisfying the condition

$$0 \le m_j \le 2n-1, \quad m_l \ne 2n - m_j, 1 \le l, j \le n.$$

In particular, it is possible to set $m_j = j - 1$.

$$\Phi(k) := i \sum_{l=1}^{2n} \varphi_l (ik)^l, \quad \varphi_j \in \mathbf{R},$$

$$f(k,x,t) := \frac{\exp(kx + ik^2 t + \Phi(k))}{q_{2n}(k)}, \quad D_k := \frac{k^2}{k^2+B^2}\frac{\partial}{\partial k},$$

$$f_j(x,t) := D_k^{2j-1} f(k,x,t)\,|_{k=B}\ ,$$

$$f_{n+j}(x,t) := D_k^{2j-1} f(k,x,t)\,|_{k=-B}\ , \quad j = 1\ldots,n.$$

Consider two Wronskians:

$$W_1 := W(f_1, \ldots, f_{2n}) \equiv \det A, \quad A_{lj} := \partial_x^{l-1} f_j,$$

$$W_2 := W(f_1, \ldots, f_{2n}, f).$$

Multi-rogue solutions to the focusing NLS equation I

Theorem

The function $v(x,t)$ defined by the formula

$$v(x,t) = -q_{2n}(0)B^{1-2n}e^{2iB^2t}\frac{W_2}{W_1}\,|_{k=0}\ , \qquad (1)$$

represents a family of nonsingular (quasi)-rational solutions to the focusing NLS equation depending on $2n$ independent real parameters φ_j.

We choose $m_j = j - 1$ and we limit ourselves to the case $B = 1$. The whole set of solutions with any B can be obtained by applying the scaling transformation, phase transformation, and Galilean transformation

$$v(x,t) \to Bv(Bx, B^2 t), \quad v(x,t) \to v(x,t)e^{i\phi},$$

$$v(x,t) \to v(x - Vt, t)\exp\left(iVx/2 - iV^2 t/4\right),$$

preserving the NLS equation.

Peregrine solution

The case $n = 1$ is well known: we obtain the original Peregrine solution taking $\varphi_1 = 0$ and $\varphi_2 = 0$. It takes especially simple forme form if one use variables $X = 2x$, $T = 4t$:

$$v(x,t) = \left(1 - \frac{4(1+iT)}{X^2 + T^2 + 1}\right) e^{iT/2}.$$

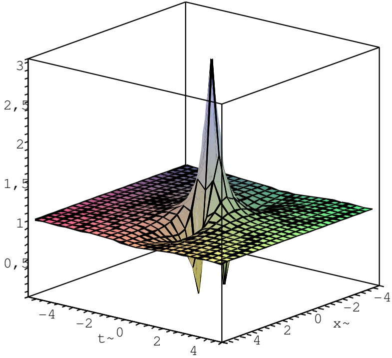


Figure: n=1 solution for $\varphi_1 = 0$ and $\varphi_2 = 0$.

For $n = 2$ the higher analog of Peregrine breather (which we call P_2 breather), found in 1995 by AEK reads:

$$P_2(x,t) = e^{iT/2}\left(1 - 12\frac{G(X,T) + iH(X,T)}{Q(X,T)}\right),$$

$$Q := (T^2 + X^2 + 1)^3 + 24(X^2 + 4T^2 - X^2T^2) + 8,$$

$$G := 5T^4 + X^4 + 6X^2T^2 + 6X^2 + 18T^2 - 3,$$

$$H := T^5 + 2T^3 + TX^4 - 15T + 2T^2 - 6TX^2.$$

It is clear that its magnitude reaches absolute maximum value 5 at the point $(0,0)$.

From $\varphi_j \to \alpha, \beta$ parametrization: $n = 2$.

Let

$$\varphi_1 = 3\varphi_3, \quad \varphi_2 = 2\varphi_4 + \frac{3+\sqrt{5}}{16} \cdot \sqrt{10 - 2\sqrt{5}},$$

$$\alpha := 2(5+\sqrt{5})\sin(\pi/5) - 48\varphi_4,$$

$$\beta := 96\varphi_3$$

. Therefore α, β are fixed by the choice of φ_3, φ_4 and the condition $\alpha = \beta = 0$ is equivalent to

$$\varphi_1 = \varphi_3 = 0, \varphi_4 = \frac{1}{24}(5+\sqrt{5})\sin(\pi/5)$$

$$\varphi_2 = \frac{1}{6}(7+2\sqrt{5})\sin(\pi/5)$$

$$v_2(x,t) = e^{iT}\left(1 - 12\frac{G_d(X,T) + iH_d(X,T)}{Q_d(X,T)}\right),$$

$$\begin{aligned}
&G_d(X,T) := G(X,T) + 2\beta\,X - 2\alpha\,T,\\
&H_d(X,T) := H(X,T) + \alpha\,X^2 + 2\beta\,TX + \alpha(1 - T^2),\\
&Q_d(X,T) :=\\
&Q(X,T) - 2\beta\,X^3 - 6\alpha\,T^3X^2 + 6\beta(T^2+1)X\\
&-2\alpha\,T^3 - 18\alpha\,T + \beta^2 + \alpha^2.
\end{aligned}$$

This new parametrization is now free of irrational factors. When $\alpha^2 + \beta^2 \to \infty$

$$v_2(x,t,\alpha,\beta) \to e^{iT/2} \equiv= e^{2it}$$

$$v_2(x,t,0,0) = P_2(x,t).$$

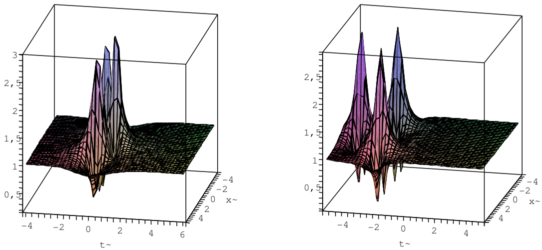
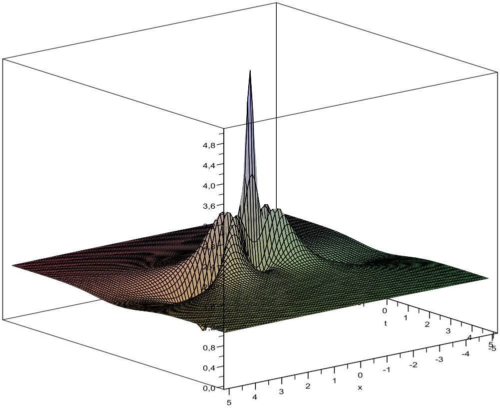


Figure: Amplitude of the solution to the NLS equation for $n = 2$ with $\varphi_2 = 1$ and $\varphi_1 = \varphi_3 = \varphi_4 = 0$ on the left, and $\varphi_4 = 1$ and $\varphi_1 = \varphi_2 = \varphi_3 = 0$ on the right.

Plot of the Magnitude of P_2 breather.



When the parameters α, β are small enough the related deformation of the higher Peregrine breather keeps its extreme rogue wave character i.e. the maximum of its magnitude is very close to 5 and a plot of the solution is quite similar to what we have when $\alpha = \beta = 0$.

$n=3$: α, β-parametrization.

$$\alpha_1 := 48(\varphi_3 - 5\varphi_5), \quad \alpha_2 = 480(\varphi_3 - 13\varphi_5),$$

$$\beta_1 := 8(12(\,4\varphi_6 - \varphi_4) + Im[\omega(1+\omega)^2])$$

$$\beta_2 := 32(60(\,8\varphi_6 - \varphi_4) + Im[\bar{\omega}(1-2\bar{\omega})^2]).$$

$$\omega := e^{-i\pi/7}.$$

$$
\begin{aligned}
\varphi_1 &= 3\varphi_3 - 5\varphi_5 \\
\varphi_2 &= 2\varphi_4 - 3\varphi_6 + \frac{\sin(\pi/7)}{4(1-\cos(\pi/7))} \\
768\varphi_3 &= 26\alpha_1 - \alpha_2 \\
1920\varphi_4 &= -40\beta_1 + \beta_2 + 96(3\sin(\pi/7) + 8\sin(2\pi/7) + 2\sin(3\pi/7)) \\
3840\varphi_5 &= 10\alpha_1 - \alpha_2 \\
7680\varphi_6 &= -20\beta_1 + \beta_2 + 32(4\sin(\pi/7) + 14\sin(2\pi/7) + \sin(3\pi/7)),
\end{aligned}
$$

$$v_3(x,t,\alpha_1,\beta_1,\alpha_2,\beta_2) = \left(1 - 24\frac{G_3(X,T) + iH_3(X,T)}{Q_3(X,T)}\right) e^{iT/2},$$

$$\begin{aligned}
G_3(X,T) &= X^{10} + 15(T^2+1)X^8 + \textstyle\sum_{n=0}^{6} g_n(T)X^n \\
H_3(X,T) &= TX^{10} + 5(T^3 - 3T + \beta_1)X^8 + \textstyle\sum_{n=0}^{6} h_n(T)X^n \\
Q_3(X,T) &= (1 + X^2 + T^2)^6 - 20\alpha_1 X^9 - 60(2T^2 - \beta_1 T - 2)X^8 + 4\textstyle\sum_{n=0}^{7} q_n(T)X^n.
\end{aligned}$$

$$
\begin{aligned}
g_6 &= 50T^4 - 60T^2 + 80\beta_1 T + 210 \\
g_5 &= 120\alpha_1 T^2 - 18\alpha_2 + 300\alpha_1 \\
g_4 &= 70T^6 - 150T^4 + 200\beta_1 T^3 + 450T^2 + 30\beta_2 T - 450 + 150\alpha_1^2 - 50\beta_1^2 \\
g_3 &= 400\alpha_1 T^4 + (3000\alpha_1 - 60\alpha_2)T^2 - 800\alpha_1\beta_1 T - 600\alpha_1 - 60\alpha_2 \\
g_2 &= 45T^8 + 420T^6 + 6750T^4 - (6000\beta_1 - 180\beta_2)T^3 - (300\alpha_1^2 - 900\beta_1^2 + 13500)T^2 \\
&\quad +(3600\beta_1 + 180\beta_2)T - 675 - 300\alpha_1^2 - 300\beta_1^2 \\
g_1 &= 280\alpha_1 T^6 + (150\alpha_2 - 2100\alpha_1)T^4 + 800\alpha_1\beta_1 T^3 - (3600\alpha_1 - 540\alpha_2)T^2 \\
&\quad +(120\beta_2\alpha_1 + 1200\alpha_1\beta_1 - 120\alpha_2\beta_1)T - 200\alpha_1\beta_1^2 - 900\alpha_1 - 90\alpha_2 - 200\alpha_1^3 \\
g_0 &= 11T^{10} + 495T^8 - 120\beta_1 T^7 + 2190T^6 - (42\beta_2 + 1200\beta_1)T^5 \\
&\quad +(350\alpha_1^2 + 150\beta_1^2 - 7650)T^4 + (6600\beta_1 - 420\beta_2)T^3 \\
&\quad -(2100\beta_1^2 + 2025 - 120\beta_2\beta_1 - 120\alpha_2\alpha_1 + 900\alpha_1^2)T^2 + (200\alpha_1^2\beta_1 + 200\beta_1^3 - 90\beta_2)T \\
&\quad +675 + 150\alpha_1^2 + 6\alpha_2^2 + 150\beta_1^2 + 6\beta_2^2.
\end{aligned}
$$

$$
\begin{aligned}
h_6 &= 10T^5 - 140T^3 + 40\beta_1 T^2 - 150T + 60\beta_1 - 5\beta_2\\
h_5 &= 40\alpha_1 T^3 + (60\alpha_1 - 18\alpha_2)T + 40\alpha_1\beta_1\\
h_4 &= 10T^7 - 210T^5 + 50\beta_1 T^4 - 450T^3 + 15\beta_2 T^2 - (50\beta_1^2 + 1350 - 150\alpha_1^2)T\\
&\quad +150\beta_1 - 15\beta_2\\
h_3 &= 80\alpha_1 T^5 + (1000\alpha_1 - 20\alpha_2)T^3 - 400\alpha_1\beta_1 T^2 - (1800\alpha_1 - 60\alpha_2)T\\
&\quad +200\alpha_1\beta_1 + 20\beta_2\alpha_1 - 20\alpha_2\beta_1\\
h_2 &= 5T^9 - 60T^7 + 1710T^5 + (45\beta_2 - 2100\beta_1)T^4 + (300\beta_1^2 - 6300 - 100\alpha_1^2)T^3\\
&\quad +(1800\beta_1 - 90\beta_2)T^2 + (4725 + 300\alpha_1^2 + 300\beta_1^2)T - 135\beta_2 - 100\beta_1^3\\
&\quad -100\alpha_1^2\beta_1 - 900\beta_1\\
h_1 &= 40\alpha_1 T^7 + (30\alpha_2 - 1140\alpha_1)T^5 + 200\alpha_1\beta_1 T^4 - (2400\alpha_1 - 60\alpha_2)T^3\\
&\quad +(60\beta_2\alpha_1 - 60\alpha_2\beta_1 + 600\alpha_1\beta_1)T^2 - (900\alpha_1 + 450\alpha_2 + 200\alpha_1^3 + 200\alpha_1\beta_1^2)T\\
&\quad +60\alpha_2\beta_1 - 60\beta_2\alpha_1\\
h_0 &= T^{11} + 25T^9 - 15\beta_1 T^8 - 870T^7 + (40\beta_1 - 7\beta_2)T^6 + (70\alpha_1^2 - 9630 + 30\beta_1^2)T^5\\
&\quad +(5850\beta_1 - 75\beta_2)T^4 + (40\beta_2\beta_1 + 40\alpha_2\alpha_1 - 2475 - 900\alpha_1^2 - 1300\beta_1^2)T^3\\
&\quad +(100\alpha_1^2\beta_1 + 495\beta_2 + 100\beta_1^3)T^2 + (6\alpha_2^2 + 4725 - 240\alpha_2\alpha_1 - 240\beta_2\beta_1\\
&\quad +750\beta_1^2 + 6\beta_2^2 + 750\alpha_1^2)T - 20\alpha_1^2\beta_2 - 675\beta_1 - 45\beta_2 - 100\alpha_1^2\beta_1 - 100\beta_1^3\\
&\quad +40\alpha_2\alpha_1\beta_1 + 20\beta_1^2\beta_2
\end{aligned}
$$

$$
\begin{aligned}
q_7 &= 3\alpha_2 - 30\alpha_1\\
q_6 &= -60T^4 + 40\beta_1 T^3 + 120T^2 - (15\beta_2 - 60\beta_1)T + 35\beta_1^2 + 15\alpha_1^2 + 580\\
q_5 &= 30\alpha_1 T^4 - (27\alpha_2 - 90\alpha_1)T^2 + 120\alpha_1\beta_1 T - 27\alpha_2 + 540\alpha_1\\
q_4 &= 30\beta_1 T^5 - 360T^4 + (15\beta_2 + 600\beta_1)T^3 + (3360 + 225\alpha_1^2 - 75\beta_1^2)T^2\\
&\quad +(135\beta_2 - 1350\beta_1)T + 225\beta_1^2 - 30\alpha_2\alpha_1 + 525\alpha_1^2 - 30\beta_2\beta_1 + 840\\
q_3 &= 40\alpha_1 T^6 + (1950\alpha_1 - 15\alpha_2)T^4 - 400\alpha_1\beta_1 T^3 + (90\alpha_2 + 4500\alpha_1)T^2\\
&\quad +(60\beta_2\alpha_1 - 1800\alpha_1\beta_1 - 60\alpha_2\beta_1)T - 450\alpha_1 + 100\alpha_1^3 + 100\alpha_1\beta_1^2 - 135\alpha_2\\
q_2 &= 60T^8 + 3360T^6 - (1620\beta_1 - 27\beta_2)T^5 + (225\beta_1^2 - 75\alpha_1^2 + 19560)T^4\\
&\quad -(16200\beta_1 - 270\beta_2)T^3 + (450\alpha_1^2 - 9120 + 4050\beta_1^2)T^2\\
&\quad +(675\beta_2 + 2700\beta_1 - 300\beta_1^3 - 300\alpha_1^2\beta_1)T + 3036 + 9\alpha_2^2 - 180\alpha_2\alpha_1\\
&\quad +225\beta_1^2 + 225\alpha_1^2 + 9\beta_2^2 - 180\beta_2\beta_1\\
q_1 &= 15\alpha_1 T^8 + (15\alpha_2 - 90\alpha_1)T^6 + 120\alpha_1\beta_1 T^5 + (405\alpha_2 - 5400\alpha_1)T^4\\
&\quad +(3000\alpha_1\beta_1 - 60\alpha_2\beta_1 + 60\beta_2\alpha_1)T^3 + (1485\alpha_2 - 300\alpha_1\beta_1^2 - 1350\alpha_1 - 300\alpha_1^3)T^2\\
&\quad +(540\beta_2\alpha_1 - 540\alpha_2\beta_1)T + 300\alpha_1^3 - 120\alpha_1\beta_1\beta_2 - 60\alpha_2\alpha_1^2 + 135\alpha_2 + 60\alpha_2\beta_1^2\\
&\quad +300\alpha_1\beta_1^2 + 2025\alpha_1\\
q_0 &= 30T^{10} - 5\beta_1 T^9 + 930T^8 - (240\beta_1 + 3\beta_2)T^7 + (15\beta_1^2 + 3820 + 35\alpha_1^2)T^6\\
&\quad +(1710\beta_1 - 153\beta_2)T^5 + (30\beta_2\beta_1 + 30\alpha_2\alpha_1 - 975\beta_1^2 + 35940 - 75\alpha_1^2)T^4\\
&\quad +(100\beta_1^3 + 100\alpha_1^2\beta_1 + 135\beta_2 - 23400\beta_1)T^3\\
&\quad +(9\beta_2^2 + 23286 + 9\alpha_2^2 - 360\beta_2\beta_1 - 360\alpha_2\alpha_1 + 4725\alpha_1^2 + 8325\beta_1^2)T^2\\
&\quad +(120\alpha_2\alpha_1\beta_1 - 60\alpha_1^2\beta_2 - 1500\alpha_1^2\beta_1 + 60\beta_1^2\beta_2 - 7425\beta_1 - 675\beta_2 - 1500\beta_1^3)T\\
&\quad +506 + 9\beta_2^2 + 100\beta_1^4 + 675\alpha_1^2 + 100\alpha_1^4 + 9\alpha_2^2 + 90\beta_2\beta_1 + 200\alpha_1^2\beta_1^2\\
&\quad +675\beta_1^2 + 90\alpha_2\alpha_1 .
\end{aligned}
$$

Solutions of the NLS equation above provide $2n$-parametric family of the smooth rational solutions to the KP-I equation:

$$\partial_x(4u_t + 6uu_x + u_{xxx}) = 3u_{yy}.$$

Replace t by y and φ_3 by t. Obviously the function

$$f(k, x, y, t) := \exp(kx + ik^2 y + k^3 t + \phi(k)),$$

where

$$\phi(k) := \Phi(k) - \varphi_3 k^3,$$

satisfies the system

$$f_t = f_{xxx}, \quad f_y = if_{xx} = 0.$$

The same is true for the functions f_j, defined above if we denote t by y and φ_3 by t. Now from (Matveev LMP 1979 p.214-216) we get following result:

Theorem

$$u(x,y,t) = 2\partial_x^2 \log\, W(f_1,\dots,f_{2n}) = 2(|v|^2 - B^2)$$

is smooth rational solution to the KP-I equation. It is obvious that

$$\int_{-\infty}^{\infty} u(x,y,t)d\,x = 0,$$

and

$$u(x,y,t) \geq -2B^2$$

CONJECTURE:

For given B and n the maximal value the solutions of KP-I equation described by the theorem above is given by the formula:

$$\max_{x,y,t\in R} u(x,y,t) = 8B^2 n(n+1).$$

For a moment this conjecture is confirmed in our works only for the small ranks but there is no doubt that it is true in general. The solutions of KP-I equation given by the previous theorem depend on $2n$ real parameters $\varphi_j, B, j \neq 3$, representing the action of the KP-I hierarchy flows. The phases φ_1, φ_2 correspond respectively to space and time translations.

This maximum value (i.e. 48) for $n = 2, B = 1$ is attended at the point $x = y = t = 0$ provided that

$$\varphi_1 = 0, \varphi_3 = t, \varphi_4 = \frac{1}{24}(5 + \sqrt{5}) \sin(\pi/5)$$

$$\varphi_2 = \frac{1}{6}(7 + 2\sqrt{5}) \sin(\pi/5)$$

.

For n=3 the related absolute maximum of u equals 96 attended at $x = y = t = 0$, with

$$\varphi_1 = \varphi_5 = 0,$$

$$\varphi_4 = 3\sin(\pi/7) + 8\sin(2\pi/7) + 2\sin(3\pi/7)/20,$$

$$\varphi_6 = (4\sin(\pi/7) + 14\sin(2\pi/7) + \sin(3\pi/7)/240,$$

$$\varphi_2 = 2\varphi_4 - 3\varphi_6 + \frac{\sin(\pi/7)}{4(1-\cos(\pi/7))}$$

.

One of the advantage of α, β parametrization of the solution is that we can analyze its limit behavior when one or several parameters tend to infinity and x and t remain bounded.

- If α_2 and β_2 remain finite and $\alpha_1^2 + \beta_1^2 \to \infty$, then, $v_3(x, t, \alpha_1, \beta_1, \alpha_2, \beta_2) \to e^{2it}$.
- If α_1 and β_1 remain finite and $\alpha_2^2 + \beta_2^2 \to \infty$, then, $v_3(x, t, \alpha_1, \beta_1, \alpha_2, \beta_2) \to P_1(x, t)$.

If

$$\alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2,\quad \to \infty$$

and

$$\beta_1 \sim b\alpha_1,\ \alpha_2 \sim c\alpha_1^r,\ \beta_2 \sim d\alpha_1^r$$

then the limit of

$$v_3(x, t, \alpha_1, \beta_1, \alpha_2, \beta_2)$$

depends on r according to the following table.

r	limit
< 2	e^{2it}
> 2	$P_1(x,t)$
2	$P_1(x-x_1, t-t_1)$

where x_1 and t_1 are defined by

$$x_1 = \frac{10(1-b^2)c+20bd}{3(c^2+d^2)}$$
$$t_1 = \frac{10(1-b^2)d-20bc}{3(c^2+d^2)}$$

So , v_3 contains all solutions of ranks 0 and 1 as appropriately chosen large parametric limits .

Here we present without details the formulas providing the 6-parametric family of multi-rogue wave solutions similar to the one of the previous section.

$$v_4(x,t,\alpha_1,\beta_1,\alpha_2,\beta_2,\alpha_3,\beta_3) = \left(1 - 40\,\frac{G_4(2x,4t) + iH_4(2x,4t)}{Q_4(2x,4t)}\right) e^{2it}$$

$$\begin{aligned}
G_4(X,T) &= X^{18} + 27(T^2+1)X^{16} - 24\alpha_1 X^{15} + \textstyle\sum_{n=0}^{14} g_n(T)X^n \\
H_4(X,T) &= TX^{18} + 9(T^3 - 3T + \beta_1)X^{16} - 24\alpha_1 TX^{15} + \textstyle\sum_{n=0}^{14} h_n(T)X^n \\
Q_4(X,T) &= (1 + X^2 + T^2)^{10} - 60\alpha_1 X^{17} - 180(2T^2 - \beta_1 T - 2)X^{16} + 4\textstyle\sum_{n=0}^{15} q_n(T)X^n.
\end{aligned}$$

Explicit formulas for the coefficients will be shown in separate pdf file.

If α_3 and β_3 remain finite and

$$\alpha_1, \beta_1, \alpha_2, \beta_2 \to \infty,$$

and

$$\beta_1 \sim b\alpha_1,\ \alpha_2 \sim c\alpha_1^r,\ \beta_2 \sim d\alpha_1^r$$

then the limit of $u_4(x, t, \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3)$ depends on r according to the following table

r	limit
$< 3/2$	$P_1(x, t)$
$> 3/2$	e^{2it}
$3/2$	$P_1(x - x_2, t - t_2)$

$$
\begin{aligned}
x_2 &= \frac{3((1-3b^2)(d^2-c^2)+2(b^2-3)bcd)}{50(b^2+1)^3} \\
t_2 &= \frac{3((3-b^2)b(d^2-c^2)+2(1-3b^2)cd)}{50(b^2+1)^3}.
\end{aligned}
$$

If α_2 and β_2 remain finite and

$$\alpha_1, \beta_1, \alpha_3, \beta_3 \to \infty,$$

and

$$\beta_1 \sim b\alpha_1,\ \alpha_3 \sim e\alpha_1^s,\ \beta_3 \sim f\alpha_1^s$$

then the limit of $v_4(x, t, \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3)$ depends on s according to the following table

s	limit
< 2	$P_1(x, t)$
> 2	e^{2it}
2	$P_1(x - x_3, t - t_3)$

where x_3 and t_3 are defined by

$$\begin{aligned} x_3 &= \frac{(2bf+(1-b^2)e)}{35(b^2+1)^2} \\ t_3 &= \frac{(2be-(1-b^2)f)}{35(b^2+1)^2}. \end{aligned}$$

If α_1 and β_1 remain finite and

$$\alpha_2, \beta_2, \alpha_3, \beta_3, \to \infty,$$

$$\beta_2 \sim d\alpha_2,\ \alpha_3 \sim e\alpha_2^p,\ \beta_2 \sim f\alpha_2^p,$$

then, the limit of $v_4(x, t, \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3)$ depending on p is described by the following table

p	limit
< 2	e^{2it}
> 2	$v_2(x, t, \alpha_1, \beta_1)$
2	$v_2(x, t, \alpha_1 - \alpha_0, \beta_1 - \beta_0)$

In the table above α_0 and β_0 are defined by

$$\begin{aligned} \alpha_0 &= \frac{21(2df+(1-d^2)e)}{10(e^2+f^2)} \\ \beta_0 &= \frac{21(2de-(1-d^2)f)}{10(e^2+f^2)}. \end{aligned}$$

If

$$\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \to \infty,$$

and

$$\beta_1 \sim b\alpha_1,\ \alpha_2 \sim c\alpha_1^r,\ \beta_2 \sim d\alpha_1^r,\ \alpha_3 \sim e\alpha_1^s,\ \beta_3 \sim f\alpha_1^s.$$

then the limit of $v_4(x, t, \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3)$ depending on r and s is given by the following table.

r	s	limit
$>3/2$	any	e^{2it}
any	>2	e^{2it}
$<3/2$	<2	$P_1(x,t)$
$3/2$	<2	$P_1(x-x_2,t-t_2)$
$<3/2$	2	$P_1(x-x_3,t-t_3)$
$3/2$	2	$P_1(x-x_2-x_3,t-t_2-t_3)$

where x_2, t_2, x_3 and t_3 are defined as above.

CONJECTURE :
It seems that in general u_n contains all solutions of rank m, $0 \leq m \leq n-2$ as appropriately chosen large parametric limits although for a moment it is proved only for the small ranks namely for $n \leq 6$
For higher rangs some more special results are available.

$$v(x,t) = \left(1 - 2\frac{\det B}{\det A}\right) e^{2it}$$

$$B = \begin{pmatrix} A & \bar{\Psi}^T \\ \Phi & 0 \end{pmatrix},$$

$$\Psi = (\psi_0, \psi_1, \dots, \psi_{N-1}), \Phi = (\phi_0, \phi_1, \dots, \phi_{N-1}),$$

$$X := e^{h(x+2it(1+f^2)+s(f))}, \quad h := f\sqrt{2+f^2},$$

$$s(f) := \sum_{i=1}^{N-1} s_i f^{2i}, \quad s_i \in C,$$

$$C_1 := h^{-1}\sqrt{1+f^2-h}, \quad C_2 := h^{-1}\sqrt{1+f^2+h},$$

$$\psi := i(XC_1 - X_1 C_2) = \sum_{i=0}^{\infty} \psi_i f^{2i},$$

$$\phi := i(XC_2 - X_1 C_1) = \sum_{i=0}^{\infty} \phi_i f^{2i}$$

$$i(\psi\bar{\psi} + \phi\bar{\phi}) = (2 + f^2 + \bar{f}^2) \sum_{i,j=0}^{\infty} A_{ij} f^{2i} \bar{f}^{2j}$$

THANK YOU FOR YOUR ATTENTION !