Quantum systems involving fourth Painlevé transcendent, rational solutions and new ladder operators

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I will present a brief review of quantum systems related with Painlevé transcendents. I will present results on a quantum system involving the fourth Painlevé transcendent and how this Hamiltonian is connected with supersymmetric quantum mechanics and also superintegrability. I will explain how this system in the reducible case contains families of systems related to Hermite exceptionnal orthogonal polynomials. I will show how we can construct new ladder operators in such case and how this is important in regard of applications in context of superintegrable systems and algebraic derivation of their energy spectrum.

Outline

- Painlevé transcendents, Painlevé transcendents in quantum mechanics

- Systems with fourth the Painlevé transcendent P_4
  - SUSYQM
  - Superintegrability
  - Rational solutions and generalized Hermite polynomials
- 1-step and 2-step extensions of Harmonic oscilaltor
  - EOP X_m,standard operator with 1 singlet and new ladder operators for 1-step with only infinite sequence
  - EOP X_{m_1,m_2} standard operators with 2 singlet, new ladder operators with doublet or infinite sequence
- Application to superintegrable systems and systems with fourth Painlevé transcendent and algebraic calculation of the energy spectrum
- Concluding remarks

The Painlevé transcendents

- The Painlevé transcendents arise in the study of ordinary differential equations.
- Painlevé found 50 equations whose only movable singularities are poles. ($\frac{d^2w}{dz^2} = F(z, w, \frac{dw}{dz})$)
- The most interesting of the fifty types are those which are irreducible and serve to define new transcendents (Painlevé transcendents)
- The other 44 can be integrated in terms of classical functions and transcendents or transformed into the remaining six equations.
- Only the first three were found by Painlevé. The last three were subsequently added by Gambier and Fuchs.

- Gromak, Laine and Shimomura Painlevé differential equations in the complex plane (2002)

$$P_1''(z) = 6P_1^2(z) + z$$
$$P_2''(z) = 2P_2(z)^3 + zP_2(z) + \alpha$$
$$P_3(z)'' = \frac{P_3'(z)^2}{P_3(z)} - \frac{P_3'(z)}{z} + \frac{\alpha P_3^2(z)+\beta}{z} + \gamma P_3^3(z) + \frac{\delta}{P_3(z)}$$
$$P_4''(z) = \frac{P_4'^2(z)}{2P_4(z)} + \frac{3}{2}P_4^3(z) + 4zP_4^2(z) + 2(z^2-\alpha)P_4(z) + \frac{\beta}{P_4(z)}$$
$$P_5''(z) = \left(\frac{1}{2P_5(z)} + \frac{1}{P_5(z)-1}\right)P_5'(z)^2 - \frac{1}{z}P_5'(z) + \frac{(P_5(z)-1)^2}{z^2}\left(\frac{aP_5^2(z)+b}{P_5(z)}\right) + \frac{cP_5(z)}{z} + \frac{dP_5(z)(P_5(z)+1)}{P_5(z)-1}$$
$$P_6''(z) = \frac{1}{2}\left(\frac{1}{P_6(z)} + \frac{1}{P_6(z)-1} + \frac{1}{P_6(z)-z}\right)P_6'(z)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{P_6(z)-z}\right)P_6'(z) + \frac{P_6(z)(P_6(z)-1)(P_6(z)-z)}{z^2(z-1)^2}\left(\gamma_1 + \frac{\gamma_2 z}{P_6(z)^2} + \frac{\gamma_3(z-1)}{(P_6(z)-1)^2} + \frac{\gamma_4 z(z-1)}{(P_6(z)-z)^2}\right)$$

- Statistical mechanics, quantum field theory, relativity, symmetry reduction of various equations (Kdv, Boussineq, Sine-Gordon, Kadomstev-Petviashvile, nonlinear Schrödinger).

Painlevé transcendents in quantum mechanics

- Dressing chains method : (Veselov and Shabat 1993 and 2001) : P_4 , Willox and Hietarinta (2003) : P_3 , P_4 and P_5
- Higher symmetries : (Fushchych and Nikitin, 1997) : P_1 , P_2 and P_4
- Superintegrability : Gravel and Winternitz (2004), Marquette and Winternitz (2008), Marquette (2009,2010,2011), Tremblay and Winternitz (2010) : P_1 , P_2 , P_4, P_5 and P_6 .
- Supersymmetric quantum mechanics : Cannata, Ioffe, Junker and Nishnianidze (1999) : P_2
  Andrianov, Cannata, Ioffe and Nishnianidze (2000) : P_4
  Carballo, Fernandez, Negro and Nieto (2004) : P_5

2D Superintegrable systems involving Painlevé transcendents

$$V_a(x,y)=\hbar^2(\omega_1^2 P_1(\omega_1 x)+\omega_2^2 P_1(\omega_2 y))$$
$$V_b(x,y)=ay+\hbar^2\omega_1^2 P_1(\omega_1 x)$$

$$V_c(x,y)=bx+ay+(2\hbar b)^{\frac{2}{3}}P_2^2((\tfrac{2b}{\hbar^2})^{\frac{1}{3}}x,0)$$
$$V_d(x,y)=ay+(2\hbar^2 b^2)^{\frac{1}{3}}(P_2'((\tfrac{-4b}{\hbar^2})^{\frac{1}{3}}x,\alpha)+P_2^2((\tfrac{-4b}{\hbar^2})^{\frac{1}{3}}x),\alpha)$$

$$V_e(x,y)=\frac{\omega^2}{2}(x^2+y^2)+\frac{\hbar^2}{2}P_4^2(\sqrt{\tfrac{\omega}{\hbar}}x,\alpha,\beta)+2\omega\sqrt{\omega\hbar}P_4(\sqrt{\tfrac{\omega}{\hbar}}x,\alpha,\beta)$$
$$+\frac{\epsilon\hbar\omega}{2}P_4'(\sqrt{\tfrac{\omega}{\hbar}}x,\alpha,\beta)+\frac{\hbar\omega}{3}(\epsilon-\alpha)$$

$$V_f(r,\theta)=\frac{1}{r^2}(\hbar^2 W'(sin^2(\frac{\theta}{2}))-\frac{\pm 8\hbar^2 cos(\theta)W(sin^2(\frac{\theta}{2}))+4\beta_1+\hbar^2}{4sin^2(\theta)})$$

$$W'=\frac{x(1-x)}{4P_6(P_6-1)}(P_6'-\sqrt{2\gamma_1}\frac{P_6(P_6-1)}{x(x-1)})^2-(\frac{\gamma_2}{2(x-1)P_6}+\frac{\gamma_3}{2x(P_6-1)})(P_6-x)$$

Fourth Painleve transcendent systems

$$H_1 L_1^\dagger = L_1^\dagger (H_2 + \hbar\omega), \quad H_1 L_2^\dagger = L_2^\dagger H_2$$

- L_1 order 1 and L_2 order 2 : SUSYQM Factorization, interwining, ...
- The solution is given in term of P_4 : reducible/irreducible

- We can construct $A^\dagger = L_1^\dagger L_2$ (with $[H_1, A^\dagger] = \hbar\omega A^\dagger$).
- $[a, a^\dagger] = P^{(+)}(H^{(+)} + \lambda) - P^{(+)}(H^{(+)})$

- The zero modes can be written ($A\psi_k^{(0)} = 0$ and $A^\dagger \phi_k^{(0)} = 0$) as $(F_1(P_4, P_4')e^{\int^x F_2(P_4, P_4')dx'})$
- E_i are written in terms of α and β

- 1,2,3 infinite sequence of levels, singlet , doublet
- $(A^\dagger, A)$ allow to construct superintegrable systems

N dimensional Euclidean space

Marquette (2011)

$$V=\sum_i^N \frac{\omega_i^2}{2}(x_i^2)+\frac{\hbar^2}{2}P_4^2+2\omega_i\sqrt{\omega_i\hbar}P_4+\frac{\epsilon\hbar\omega}{2}P_4'+\frac{\hbar\omega_i}{3}(\epsilon_i-\alpha_i)$$

$$V=\sum_i^N \frac{\omega_i^2}{8}\left(1+\frac{4(P_5+\epsilon_i P_5')^2-P_5^2}{(P_5-1)^2P_5}\right)x_i^2+\frac{\hbar^2}{x_i^2}\left(a_i-b_i-\frac{1}{8}+\frac{b_i-a_iP_5^2}{P_5}\right)$$

$$-\hbar\omega_i\left(1+\frac{(1+2c_iP_5)}{2(P_5-1)}\right)$$

with

$$P_4=P_4\left(\sqrt{\frac{\omega}{\hbar}}x_i,\alpha_i,\beta_i\right),\qquad P_5=P_5\left(\frac{\omega}{\hbar}x_i^2,a_i,b_i,c_i,-\frac{1}{8}\right)$$

- Do we have algebraic structures that explain the degenerate energy spectrum ?

Application superintegrability

A 2D system with separation of variables in Cartesian coordinates :

$$H = H_x + H_y = -\frac{d^2}{dx^2} - \frac{d^2}{dy^2} + V_x(x) + V_y(y)$$

with ladder operators that satisfy PHA

$$[H_x, a_x^\dagger] = \lambda_x a_x^\dagger, \quad [H_x, a_x] = -\lambda_x a_x$$

$$a_x a_x^\dagger = Q(H_x + \lambda_x), \quad a_x^\dagger a_x = Q(H_x)$$

$$[H_y, a_y^\dagger] = \lambda_y a_y^\dagger, \quad [H_y, a_y] = -\lambda_y a_y$$

$$a_y a_y^\dagger = S(H_y + \lambda_y), \quad a_y^\dagger a_y = S(H_y)$$

- λ_x and λ_y, $Q(x)$ and $S(y)$ are polynomials
- integrals of motion ($k_1 n_1 + k_2 n_2$) for $n_1\lambda_x = n_2\lambda_y = \lambda$, $n_1, n_2 \in \mathbb{Z}^*$
- related recurrence approach Kalnins, Kress, Miller (2011,2012)

$$K = \frac{1}{2\lambda}(H_x - H_y), \quad I_- = a_x^{n_1} a_y^{\dagger n_2}, \quad I_+ = a_x^{\dagger n_1} a_y^{n_2}.$$

- the method allows to generate a polynomial algebra of order $k_1 n_1 + k_2 n_2 - 1$

$$[K, I_\pm] = \pm I_\pm, \quad [I_-, I_+] = F_{n_1,n_2}(K+1, H) - F_{n_1,n_2}(K, H),$$

$$F = \prod_{i=1}^{n_1} Q\left(\frac{H}{2} + \lambda K - (n_1 - i)\lambda_x\right) \prod_{j=1}^{n_2} S\left(\frac{H}{2} - \lambda K + j\lambda_y\right)$$

- a generalised deformed oscillator algebra (Daskaloyannis (1991,2001))
- $b^\dagger = I_+$, $b = I_-$, $N = K - u$ and $\Phi(H, u, N) = F_{n_1,n_2}(K, H)$
- Problem in the case of singlet and doublet (not all spectrum and degeneracies)

1-step,2-step and fourth Painleve

- Gromak (2002) : P_4 has families of rational solutions (related to reducible case)

P_4 has rational solution if and only if

$$\alpha = m, \quad \beta = 2(1+2n-m)^2$$

$$\alpha = m, \quad \beta = -\frac{2}{9}(1+6n-3m)^3, m, n \in \mathbb{Z}$$

There are three families of rational solution of the form

$$w_1(z,\alpha_1,\beta_1) = P_{1,n-1}/Q_{1,n}$$

$$w_2(z,\alpha_2,\beta_2) = -2z + P_{2,n-1}/Q_{2,n}$$

$$w_3(z,\alpha_3,\beta_3) = -\frac{2}{3}z + P_{3,n-1}/Q_{3,n}$$

Generalized Hermite polynomials

- $P_{j,n-1}$ $Q_{j,n}$ are polynomial of degree n
- associated with a different set of α and β

- They can be rewritten in other form
- In the case of $-2z$ and $-\frac{1}{z}$ hierarchies involve $H_{m,n}$
- $H_{m,n}$ are generalized Hermite polynomials of Noumi and Yamada $H_{m,n}$
- also in the form of determinant $\tau_{m,n} = c_{m,n} H_{m,n}$

- $H_{m,n}$ they satisfy the recurrence relation

$$2mH_{m+1,n}H_{m-1,n} - H_{m,n}H''_{m,n} + (H'_{m,n})^2 + 2mH^2_{m,n}$$

with $H_{0,0} = H_{1,0} = H_{0,1} = 1, H_{1,1} = 2z$

- $\mathcal{H}_m(x)$ is a pseudo or twisted Hermite polynomial $((-i)^m H_m(ix))$
- Clarkson (2003)

$$w^{I}_{m,n} = w(z, \alpha^{I}_{m,n}, \beta^{I}_{m,n}) = -\frac{d}{dz}(ln(\frac{H_{m,n+1}}{H_{m,n}}))$$

$$w^{II}_{m,n} = w(z, \alpha^{II}_{m,n}, \beta^{II}_{m,n}) = -\frac{d}{dz}(ln(\frac{H_{m,n+1}}{H_{m,n}}))$$

- We can use formula on Wronskian from (Odake and Sasaki (2013))

For m=1

$$H_{1,n} \propto W(H_1, H_2, ..., H_n) \propto \mathcal{H}_n$$

For $m = 2$

$$H_{2,n} \propto W(H_2, H_3, ..., H_{n+1}) \propto W(\mathcal{H}_n, \mathcal{H}_{n+1})$$

SUSYQM, 1-step, EOP

Marquette and Quesne(JMP 2013, JPA 2013)

$$H^{(+)} = -\frac{d^2}{dx^2} + x^2 + 2m + 1 = A^\dagger A,$$

$$H^{(-)} = -\frac{d^2}{dx^2} + x^2 - 2\left(\frac{\mathcal{H}''_m}{\mathcal{H}_m} - \left(\frac{\mathcal{H}'_m}{\mathcal{H}_m}\right)\right) + 2m - 1 = AA^\dagger,$$

$$A = \frac{d}{dx} + q_0(x) = \frac{d}{dx} - x - \frac{\mathcal{H}'_m}{\mathcal{H}_m}$$

The interwining and factorisation relation of SUSYQM are

$$AH^{(+)} = H^{(-)}A, \quad A^\dagger H^{(-)} = H^{(+)}A^\dagger$$

$$H^{(+)} - E = A^\dagger A, \quad H^{(-)} - E = AA^\dagger, E = -(2m+1)$$

The corresponding boud state energies are

$$E_\nu^{(+)} = 2(\nu + m + 1), \quad \nu = 0, 1, 2, 3, \ldots$$

$$E_\nu^{(-)} = 2(\nu + m + 1), \quad \nu = -m-1, 0, 1, 2, 3, \ldots$$

- The SUSYQM is constructed using seed solution ($\phi_m(x)$)

$$\phi_m(x) = \mathcal{H}_m(x)e^{\frac{1}{2}x^2}, \quad q_0(x) = -\frac{\phi_m'}{\phi_m}$$

- The seed solution is nodeless on the real line if we take $m = 0, 2, 4, 6, \ldots$, its inverse $\phi_m^{-1}(x)$ is an acceptable physical wavefunction of the superpartner potential

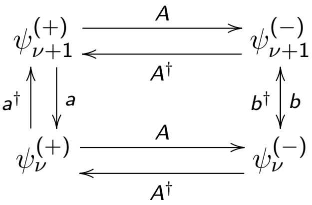
$$\psi_\nu^{(-)} = N_\nu^{(-)} \frac{e^{-\frac{1}{2}x^2}}{\mathcal{H}_m} y_{\nu+m+1}^{(m)}(x), \quad \nu = -m-1, 0, 1, 2, \ldots$$

$$y_0^{(m)}(x) = 1 \quad y_{\nu+m+1}^{(m)}(x) = -\mathcal{H}_m H_{\nu+1} - 2m\mathcal{H}_{m-1} H_\nu$$

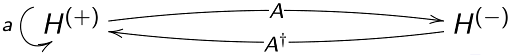
- Hermite EOP $y_n(x)$ (with $n = m + \nu + 1$)

- The supercharges relate the wavefunction and give the isospectral property and relate ladder operators
- The Hamiltonian has ladder operators of the form

$$b = AaA^{\dagger}, b^{\dagger} = Aa^{\dagger}A^{\dagger}$$



- We can also use the following diagram
- we come back by the same path (also for $a^{\dagger}$)



- We can relate $H^{(+)}$ to $H^{(-)}$ by a chain of m first order SUSYQM transformations with supercharges

$$\hat{A}_i = \frac{d}{dx} + \hat{W}_i(x), \quad \hat{A}_i^\dagger = -\frac{d}{dx} + \hat{W}_i(x), \quad \hat{W}_i = x + \frac{\mathcal{H}'_{i-1}}{\mathcal{H}_{i-1}} - \frac{\mathcal{H}'_i}{\mathcal{H}_i}$$

$i =, 1, 2, ..., m.$

$$\hat{H}_i = -\frac{d^2}{dx^2} + x^2 - 2(\frac{\mathcal{H}''_{i-1}}{\mathcal{H}_{i-1}} - (\frac{\mathcal{H}'_{i-1}}{\mathcal{H}_{i-1}})^2) - 3, \quad i = 1, 2, ..., m+1$$

- We have

$$\hat{A}_i^\dagger \hat{A}_i = \hat{H}_i, \quad \hat{A}_i \hat{A}_i^\dagger = \hat{H}_{i+1} + 2, \quad \hat{A}_i \hat{H}_i = (\hat{H}_{i+1} + 2)\hat{A}_i$$

$$H^{(+)} = \hat{H}_1 + 2m + 4, \quad H^{(-)} = \hat{H}_{m+1} + 2m + 2,$$

$$\hat{A}_m\hat{A}_{m-1}...\hat{A}_1 H^{(+)} = (H^{(-)} + 2m + 2)\hat{A}_m\hat{A}_{m-1}...\hat{A}_1$$
$$H^{(+)}\hat{A}_1^\dagger...\hat{A}_{m-1}^\dagger\hat{A}_m^\dagger = \hat{A}_1^\dagger...\hat{A}_{m-1}^\dagger\hat{A}_m^\dagger(H^{(-)} + 2m + 2)$$

$$\hat{H}_1 \xrightarrow{\hat{A}_1} \hat{H}_2 + 2 \xrightarrow{\hat{A}_2} \hat{H}_3 + 4 \xrightarrow{\hat{A}_3} \cdots \xrightarrow{\hat{A}_{m-1}} \hat{H}_m + 2m - 2 \xrightarrow{\hat{A}_m} \hat{H}_{m+1} + 2m$$
$$H^{(+)} \xrightarrow{\hat{A}_m\cdots\hat{A}_2\hat{A}_1} H^{(-)} + 2m$$

- some of the intermediate $\hat{H}_i$ are singular
- The ladder operators for $H^{(-)}$ can be obtained by combining 2 types of supercharges

$$c = \hat{A}_m...\hat{A}_2\hat{A}_1 A^\dagger, \quad c^\dagger = A\hat{A}_1^\dagger\hat{A}_2^\dagger...\hat{A}_m^\dagger$$

$$H^{(-)} \xrightarrow{A^{\dagger}} H^{(+)} \xrightarrow{\hat{A}_m \cdots \hat{A}_2 \hat{A}_1} H^{(-)} + 2m + 2$$

$$H^{(-)} \xrightarrow{c} H^{(-)} + 2m + 2$$

The operator $H^{(-)}$,c and $c^{\dagger}$ satisfy a PHA of order m

$$[H^{(-)}, c^{\dagger}] = (2m+2)c^{\dagger}, \quad [H^{(+)}, c] = -(2m+2)c$$
$$[c, c^{\dagger}] = Q(H^{(-)} + 2m + 2) - Q(H^{(-)}),$$
$$Q(H^{(-)}) = H^{(-)} \prod_{i}^{m} (H^{(-)} - 2m - 2 - 2i)$$

- action of the raising operator $c^{\dagger}$ on $\psi_{\nu}^{(-)}(x)$

$$c^{\dagger}\psi_{-m-1}^{(-)} \propto \psi_0^{(-)}, c^{\dagger}\psi_{\nu}^{(-)} \propto \psi_{\nu+m+1}^{(-)}, \qquad \nu = 0, 1, 2, \ldots .$$
$$c\psi_{\nu}^{(-)} = 0, \qquad \nu = -m-1, 1, 2, \ldots, m,$$

- the PHA generated by $H^{(-)}$, $c^{\dagger}$, and c has $m+1$ infinite-dimensional unirreps

2-step systems and new ladder

- $H^{(1)}$ and $H^{(2)}$ related by second order supercharges $\mathcal{A}^\dagger$ and $\mathcal{A}$
- reducible : $\mathcal{A}^\dagger = A^{(1)\dagger} A^{(2)\dagger}$

with

$$A^{(i)\dagger} = -d/dx + W^{(i)}(x), A^{(i)} = d/dx + W^{(i)}(x), i = 1, 2,$$

- seed eigenfunctions $\phi^{(1)}(x)$ and $\phi^{(2)}(x)$ as $W^{(i)}(x) = -(\phi^{(i)}(x))'$, $i = 1{,}2$
- $\phi^{(1)}(x) = \phi_1(x)$ and $\phi^{(2)}(x) = A^{(1)}\phi_2(x) = \mathcal{W}(\phi_1, \phi_2)/\phi_1$, where $\mathcal{W}(\phi_1, \phi_2)$
- factorization energy $E_1 = -2m_1 - 1$ being less than the ground-state of starting system
- with $E_2 = -2m_2 - 1$ and $\phi_2(x) = \phi_{m_2}(x) = \mathcal{H}_{m_2}(x)e^{x^2/2}$ with m_2 odd ($m_2 > m_1$)

$$W^{(1)}(x) = -x - \frac{\mathcal{H}'_{m_1}}{\mathcal{H}_{m_1}}, \qquad W^{(2)}(x) = -x + \frac{\mathcal{H}'_{m_1}}{\mathcal{H}_{m_1}} - \frac{g'_\mu}{g_\mu},$$

The potentials

$$V^{(1)}(x) = x^2 + m_1 + m_2 + 1,$$
$$V(x) = x^2 - 2\left[\frac{\mathcal{H}''_{m_1}}{\mathcal{H}_{m_1}} - \left(\frac{\mathcal{H}'_{m_1}}{\mathcal{H}_{m_1}}\right)^2\right] + m_1 + m_2 - 1,$$
$$V^{(2)}(x) = x^2 - 2\left[\frac{g''_\mu}{g_\mu} - \left(\frac{g'_\mu}{g_\mu}\right)^2\right] + m_1 + m_2 - 3,$$

The Wronskian therefore becomes (μth-degree polynomial)

$$g_\mu(x) = \mathcal{W}(\mathcal{H}_{m_1}, \mathcal{H}_{m_2}), \quad \mu = m_1 + m_2 - 1$$

From SUSYQM, we directly get the energy spectra of $H^{(1)}$, H and $H^{(2)}$ in the form

$$E_\nu^{(1)} = 2\nu + m_1 + m_2 + 2, \qquad \nu = 0, 1, 2, \dots,$$
$$E_\nu = 2\nu + m_1 + m_2 + 2, \qquad \nu = -m_1 - 1, 0, 1, 2, \dots,$$
$$E_\nu^{(2)} = 2\nu + m_1 + m_2 + 2, \qquad \nu = -m_2 - 1, -m_1 - 1, 0, 1, 2, \dots.$$

wavefunctions given by ($n = \nu + \mu + 2$)

$$\psi_\nu^{(2)}(x) = \mathcal{N}_\nu^{(2)} \frac{e^{-\frac{1}{2}x^2}}{g_\mu(x)} y_n^{(\mu)}(x), \quad \nu = -m_2 - 1, -m_1 - 1, 0, 1, 2, \dots,$$

$y_n^{(\mu)}(x)$ is an nth-degree polynomial (X_{m_1,m_2})

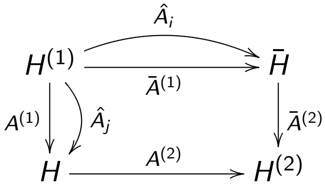
$$y_{m_1}^{(\mu)}(x) = \mathcal{H}_{m_1},$$
$$y_{m_2}^{(\mu)}(x) = \mathcal{H}_{m_2},$$
$$y_{m_1+m_2+\nu+1}^{(\mu)}(x) = (m_2 - m_1)\mathcal{H}_{m_1}\mathcal{H}_{m_2}H_{\nu+1} + \dots H_\nu, \qquad \nu = 0, 1, 2, \dots,$$

$$\left[\frac{d^2}{dx^2} - 2\left(x + \frac{g'_\mu}{g_\mu}\right)\frac{d}{dx} + 2n + 2\frac{\bar{g}_{\mu-2}}{g_\mu}\right] y_n^{(\mu)}(x) = 0,$$
$$\bar{g}_{\mu-2}(x) = \mathcal{W}(\mathcal{H}'_{m_1}, \mathcal{H}'_{m_2}).$$

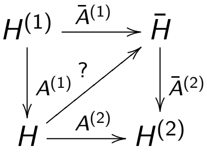
- The order of ϕ_1 and ϕ_2 may be changed without affecting the final results
- $\bar{\phi}_1(x) = \phi_2(x)$, $\bar{\phi}_2(x) = \phi_1(x)$, $\bar{A}^{(i)}$

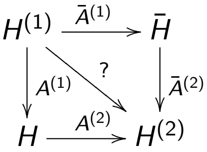
$$\bar{V}^{(1)}(x) = V^{(1)}(x) = x^2 + m_1 + m_2 + 1,$$
$$\bar{V}(x) = x^2 - 2\left[\frac{\mathcal{H}''_{m_2}}{\mathcal{H}_{m_2}} - \left(\frac{\mathcal{H}'_{m_2}}{\mathcal{H}_{m_2}}\right)^2\right] + m_1 + m_2 - 1,$$
$$\bar{V}^{(2)}(x) = V^{(2)}(x) = x^2 - 2\left[\frac{g''_\mu}{g_\mu} - \left(\frac{g'_\mu}{g_\mu}\right)^2\right] + m_1 + m_2 - 3,$$

- factorizations can be summarized in the following commutative diagram
- $\bar{H}$ can be useful to construct ladder operators for $H^{(2)}$



- Maybe we need introduce new path ?





- alternative ladder operators for $H^{(2)}$ from possibility of going from H to $\bar{H}$ (up to some additive constant) by a chain of $\ell = m_2 - m_1$ with supercharges

$$\hat{A}_i^\dagger = -\frac{d}{dx} + \hat{W}_i(x), \qquad \hat{A}_i = \frac{d}{dx} + \hat{W}_i(x),$$

$$\hat{W}_i(x) = x + \frac{\mathcal{H}'_{m_1+i-1}}{\mathcal{H}_{m_1+i-1}} - \frac{\mathcal{H}'_{m_1+i}}{\mathcal{H}_{m_1+i}}, \qquad i = 1, 2, \ldots, \ell.$$

$$\hat{H}_i = -\frac{d^2}{dx^2} + x^2 - 2\left[\frac{\mathcal{H}''_{m_1+i-1}}{\mathcal{H}_{m_1+i-1}} - \left(\frac{\mathcal{H}'_{m_1+i-1}}{\mathcal{H}_{m_1+i-1}}\right)^2\right] - 3,$$

$$i = 1, 2, \ldots, \ell + 1,$$

$$\hat{A}_i^\dagger \hat{A}_i = \hat{H}_i, \qquad \hat{A}_i \hat{A}_i^\dagger = \hat{H}_{i+1} + 2, \qquad i = 1, 2, \ldots, \ell,$$

- as $\hat{A}_i \hat{H}_i = (\hat{H}_{i+1} + 2)\hat{A}_i$ for $i = 1, 2, \ldots, \ell$

$$H = \hat{H}_1 + m_1 + m_2 + 2, \qquad \bar{H} = \hat{H}_{\ell+1} + m_1 + m_2 + 2,$$
$$\hat{A}_\ell \cdots \hat{A}_2 \hat{A}_1 H = (\bar{H} + 2\ell)\hat{A}_\ell \cdots \hat{A}_2 \hat{A}_1,$$
$$H \hat{A}_1^\dagger \hat{A}_2^\dagger \cdots \hat{A}_\ell^\dagger = \hat{A}_1^\dagger \hat{A}_2^\dagger \cdots \hat{A}_\ell^\dagger (\bar{H} + 2\ell),$$

- In diagrammatic form

$$\hat{H}_1 \xrightarrow{\hat{A}_1} \hat{H}_2 + 2 \xrightarrow{\hat{A}_2} \hat{H}_3 + 4 \xrightarrow{\hat{A}_3} \cdots \xrightarrow{\hat{A}_{\ell-1}} \hat{H}_\ell + 2\ell - 2 \xrightarrow{\hat{A}_\ell} \hat{H}_{\ell+1} + 2\ell$$
$$H \xrightarrow{\hat{A}_\ell \cdots \hat{A}_2 \hat{A}_1} \bar{H} + 2\ell$$

conbine ℓth-order SUSYQM from H to $\bar{H} + 2\ell$ with other going from $H^{(2)}$ to H or from $\bar{H}$ to $H^{(2)}$ $(cH^{(2)} = (H^{(2)} + 2\ell)c)$

$$c^\dagger = A^{(2)} \hat{A}_1^\dagger \hat{A}_2^\dagger \cdots \hat{A}_\ell^\dagger \bar{A}^{(2)\dagger}, \qquad c = \bar{A}^{(2)} \hat{A}_\ell \cdots \hat{A}_2 \hat{A}_1 A^{(2)\dagger},$$

- In diagrammatic form

$$H^{(2)} \xrightarrow{A^{(2)\dagger}} H \xrightarrow{\hat{A}_\ell \cdots \hat{A}_2 \hat{A}_1} \bar{H} + 2\ell \xrightarrow{\bar{A}^{(2)}} H^{(2)} + 2\ell, \qquad H^{(2)} \xrightarrow{c} H^{(2)} + 2\ell$$

The operators $H^{(2)}$, $c^\dagger$ and c satisfy the commutation relations

$$[H^{(2)}, c^\dagger] = 2\ell c^\dagger, \quad [H^{(2)}, c] = -2\ell c, \quad [c, c^\dagger] = Q(H^{(2)}+2\ell) - Q(H^{(2)}),$$

$$Q(H^{(2)}) = (H^{(2)} - 3\ell)\left[\prod_{i=1}^{\ell}(H^{(2)} - 2m_1 - \ell - 2i)\right](H^{(2)} + \ell)$$

- a PHA of $(\ell + 1)$th order

- PHA of $(m_2 - m_1 + 1)$th order and zero modes of c and $c^\dagger$ can be obtained
- one 2-dime unirreps spanned by doublet $\{\psi^{(2)}_{-m_2-1}, \psi^{(2)}_{-m_1-1}\}$
- ℓ infinite-dimensional ones spanned by $\{\psi^{(2)}_{i+\ell j} \mid j = 0, 1, 2, \ldots\}$ with $i = 0,\ 1,\ \ldots,\ \ell - 1$

$$c\psi^{(2)}_{-m_2-1} = c\psi^{(2)}_0 = c\psi^{(2)}_1 = \cdots = c\psi^{(2)}_{m_2-m_1-1} = 0,$$

$$c\psi^{(2)}_{-m_1-1} = (m_2 - m_1)\left(\frac{2^{m_2-m_1+2} m_2!}{m_1!}\right)^{1/2} \psi^{(2)}_{-m_2-1},$$

$$c\psi^{(2)}_{\nu} = \left(\frac{2^{m_2-m_1+2}\nu!(\nu + 2m_1 - m_2 + 1)(\nu + m_2 + 1)}{(\nu + m_1 - m_2)!}\right)^{1/2} \psi^{(2)}_{\nu+m_1-m_2},$$

$$\nu = m_2 - m_1, m_2 - m_1 + 1, \ldots.$$

- Ladder operators without singlet ou doublet can be found and written in term of Wronskian
- The action is the following

$$c\psi^{(2)}_{-m_2-1} = c\psi^{(2)}_{-m_1-1} = c\psi^{(2)}_{1} = \cdots = \ldots = c\psi^{(2)}_{m_2-m_1-1} = 0,$$
$$= c\psi^{(2)}_{m_2-m_1-1} = c\psi^{(2)}_{m_2-m_1+1} = \ldots = c\psi^{(2)}_{m_2} = 0,$$
$$c\psi^{(2)}_{0} = \left(2^{m_2+1}(m_2+1)!\frac{m_1+1}{m_2-m_1}\right)^{1/2}\psi^{(2)}_{-m_2-1},$$
$$c\psi^{(2)}_{m_2-m_1} \propto \psi^{(2)}_{-m_1-1},$$
$$c\psi^{(2)}_{\nu} \propto \psi^{(2)}_{\nu-m_2-1},$$
$$\nu = m_2+1, m_2+2, \ldots .$$

- A family of 2D superintegrable systems H
- We use method discuss earlier based on ladder operators

$$H = H_x + H_y$$

$$H_x = -\frac{d^2}{dx^2} + x^2 - 2\left[\frac{\mathcal{H}_m''}{\mathcal{H}_m} - \left(\frac{\mathcal{H}_m'}{\mathcal{H}_m}\right)^2 + 1\right], \quad m \text{ even},$$

$$H_y = -\frac{d^2}{dy^2} + y^2.$$

- This family is a particcvular case of the superintegrable systems involving the fourth Painleve transcendent.
- The integrals will be product of ladder operators and a polynomial algebra can be constructed from PHA

Integrals from standard ladder

- One important properties of superintegrable systems is the symmetry algebra that "explain" the degenerate energy spectrum
- Unirreps from deformed oscillator algebra realization of the polynomial algebra

	u	p	Energy E	Structure function Φ	Physical states
1	u_1	$\mathbb{N}$	$2(p+1)$	$16x(p+1-x)(x+m)(x+1+m)$	$\nu_x = 0, 1, 2, \ldots,$ $\nu_y = 0, 1, 2, \ldots$
2	u_3	0	$2(p-m)$	$16x(p+1-x)(x-1-m)(x-1)$	$\nu_x = -m-1$ $\nu_y = 0$

- we obtain the ground state but not all the states
- do not recover all degeneracies

$$E = E_x + E_y = 2(\nu_x + \nu_y + 1), \quad \nu_x = -m-1, 0, 1, 2, \ldots, \quad \nu_y = 0, 1, 2, \ldots$$

Integrals from new ladder

The finite-dimensional unirreps

$$E_1 = 2[(m+1)p + 1 - k]$$

$$E_2 = 2[(m+1)(p+1) + l - k + 1],$$

$$\Phi_1 = 2^{2(m+1)}(m+1)x \prod_{i=1}^{m}[(m+1)x - m - 1 - i]$$

$$\times \prod_{j=1}^{m+1}[(m+1)(p+1-x) - m + j - k]$$

$$\Phi_2 = 2^{2(m+1)}[(m+1)x + m + 1 + l] \prod_{i=1}^{m}[(m+1)x + l - i]$$

$$\times \prod_{j=1}^{m+1}[(m+1)(p+1-x) + j - k]$$

Direct approach

- an analysis from the two solutions E_1 and E_2 allow to recover the degeneracies and all the levels

$$E = E_x + E_y = 2(\nu_x + \nu_y + 1),$$

$$\nu_x = -m-1, 0, 1, 2, \ldots, \quad \nu_y = 0, 1, 2, \ldots.$$

- we found the number of unirreps per level, the corresponding set of p values with their number of occurrences and the total degeneracy

$$(E_N) = \begin{cases} 1 & \text{if } N = -m, -m+1, \ldots, -1, \\ N+1 & \text{if } N = 0, 1, 2, \ldots. \end{cases}$$

- is the algebraic structure generated by integrals explains the degenerate energy spectrum ?

unirreps and energy spectrum

- setting $\nu_x = (m+1)n_x + a_1$, $\nu_y = (m+1)n_y + a_2$, with $n_x, n_y \in \mathbb{N}$, $a_1 \in \{-m-1, 1, 2, \ldots, m\}$, and $a_2 \in \{0, 1, \ldots, m\}$

- E can be rewritten as $E = 2[(m+1)(n_x+n_y) + a_1 + a_2 + 1]$.

- E_1 and E_2, correspond to E with $n_x + n_y = p \in \mathbb{N}$, $a_2 = m + 1 - k \in \{0, 1, \ldots, m\}$, and $a_1 = -m-1$ or $a_1 = l \in \{1, 2, \ldots, m\}$

- $m = 2$: nine unirreps for each $p \in \mathbb{N}$, associated with the energies $6p-4$, $6p-2$, $6p$, $6p+4$, $(6p+6)^2$, $(6p+8)^2$, and $6p+10$

- the sequence of energy levels with their degeneracy is -4, -2, 0, 2^2, 4^3, 6^4, 8^5, 10^6, $\ldots$.

unirreps and degeneracies

TABLE : p occurrences, number $\mathcal{N}$ of unirreps per level, total level degeneracy $(N = (m+1)\lambda + \mu)$

λ	μ	p	$\mathcal{N}$	(E_N)
-1	$1, 2, \ldots, m$	0	1	1
0	0	0	1	1
0	$1, 2, \ldots, m$	1	μ	$N+1$
		$0^{\mu-1}$		
$1, 2, \ldots$	0	λ	$m+1$	$N+1$
		$(\lambda-1)^m$		
$1, 2, \ldots$	$1, 2, \ldots, m$	$\lambda+1$	$m+1$	$N+1$
		$\lambda^{\mu-1}$		

Conclusion

- New ladder are created using two differents SUSYQM path
- Wronskian , Darboux-Crum and Krein-Adler equivalence,
- new ladder for k-step extension of harmonic oscillator and singular oscillator related to type III Laguerre EOP, P_5 ?
- Moreover, we need also to create new integrals from ladder operators with only infinite sequence (i.e. no multiplet)
- We obtain all levels and degeneracies but for a given level the degeneracies is given by taking into account union of unirreps
- relations between integrals from $(b,\ b^\dagger)$ and $(c,\ c^\dagger)$?