On the Yang-Baxter equation for the six-vertex model

Vladimir Mangazeev

The Australian National University

Second ANZAMP meeting,
Mooloolaba , 26-29 November 2013

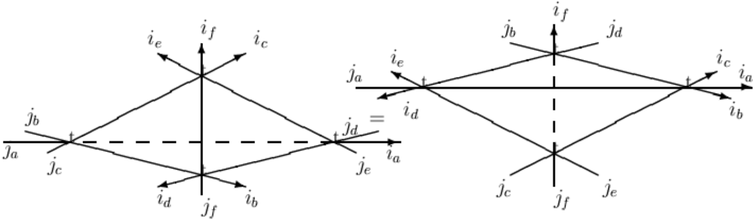
Collaboration:
V.Bazhanov (ANU) and S.Sergeev (UC, Canberra)

Outline

1. The Tetrahedron Equation
2. Dressed 3D R-matrix
3. The 2-layer case and higher-spin R-matrices
4. The main result
5. Q-operator
6. Non-compact case
7. Conclusion

Tetrahedron Equation

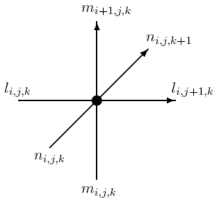
The tetrahedron equation is a 3D generalization of the YBE equation



$$R_{abc}R_{ade}R_{bdf}R_{cef} = R_{cef}R_{bdf}R_{ade}R_{abc}$$

Consider a 3D cubic lattice $M \times L \times N$ with periodic BC. We label each vertex by a triple (i, j, k).

$$R^{m_{i+1,j,k},\, l_{i,j+1,k},\, n_{i,j,k+1}}_{m_{i,j,k},\, l_{i,j,k},\, n_{i,j,k}} =$$



Solution of the TE

We consider a model where spin indices are assigned to the edges and can take any positive integer values $0, 1, 2, \ldots, \infty$.
Such a solution based on the Fock representation of the q-oscillator algebra has been proposed by (Bazhanov, Sergeev,2006).
We use a solution based on the dual Fock representation (see M, Bazhanov, Sergeev, arXiv:1308.4773, J. Phys. A:**46**, 465206)

$$R_{n_1,n_2,n_3}^{k_1,k_2,k_3} = \delta_{n_1+n_2,k_1+k_2}\, \delta_{n_2+n_3,k_2+k_3}\, q^{n_2(n_2+1)-(n_2-k_1)(n_2-k_3)} \times$$
$$\times \sum_{r=0}^{n_2} \frac{(q^{-2k_1};q^2)_{n_2-r}}{(q^2;q^2)_{n_2-r}} \frac{(q^{2+2n_1};q^2)_r}{(q^2;q^2)_r}\, q^{-2r(n_3+k_1+1)}$$

It is easy to rewrite this formula in the form of a terminated hypergeometric series ${}_2\phi_1$.
For $0 < q < 1$ all nonzero matrix elements of R are positive.

All R-matrices in the TE are the same ! How to construct a commutative family ??

Let us introduce a set of parameters (λ_i, μ_i, c_i) associated with the spaces $\mathcal{F}_q^{(i)}$ (Fock spaces) $i = 1, \ldots, 6$.

$$R_{124} R_{135} R_{236} R_{456} = R_{456} R_{236} R_{135} R_{124}$$

It is easy to show that for a solution R_{123} of the TE given above we can define a “dressed” R-matrix

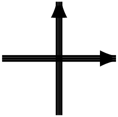
$$\mathrm{R}_{123} \;=\; \left(\frac{\mu_3}{\lambda_1}\right)^{\mathbf{N}_2} R_{123} \left(\frac{\lambda_2}{\lambda_3}\right)^{\mathbf{N}_1} \left(\frac{\mu_1}{\mu_2}\right)^{\mathbf{N}_3} \;,$$

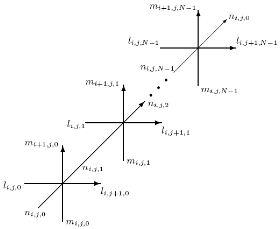
$$R'_{123} = c_1^{\mathbf{N}_1} c_2^{\mathbf{N}_2} c_3^{\mathbf{N}_3}\, R_{123}\, c_1^{-\mathbf{N}_1} c_2^{-\mathbf{N}_2} c_3^{-\mathbf{N}_3}.$$

$$R_{n_1,n_2,n_3}^{k_1,k_2,k_3}(\phi) = \phi_1^{n_1}\phi_2^{n_2}\phi_3^{n_3}\phi_4^{k_2}\, R_{n_1,n_2,n_3}^{k_1,k_2,k_3}.$$

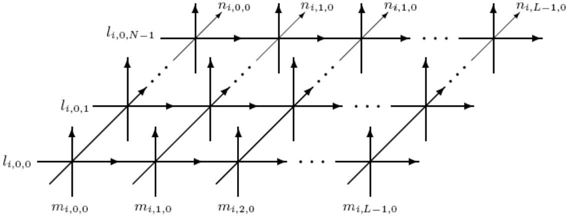
The “layer-layer” TM will commute for two different sets of fields ϕ_i and ϕ'_i.

Commuting transfer-matrices

$$\mathbb{R}_{\mathbf{l},\mathbf{m}}^{\mathbf{l}'.\mathbf{m}'} = \sum_{\mathbf{n}} \prod R_{l,m,n}^{l',m'.n'} =$$ 

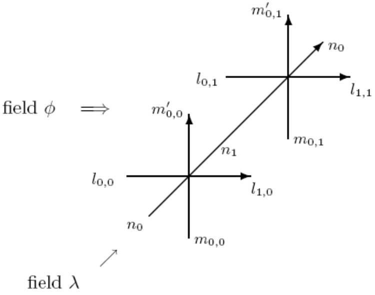


$$T(\Phi)_{\mathbf{M}}^{\mathbf{M}'} =$$



$$[T(\Phi), T(\Phi')] = 0$$

n in the 3rd direction corresponds to the $sl(n)$ algebra. We consider $n = 2$.



Conservation laws

$$l_{0,0} + l_{0,1} = l_{1,0} + l_{1,1} = \mathbf{l_1},$$
$$m_{0,0} + m_{0,1} = m'_{0,0} + m'_{0,1} = \mathbf{l_2}$$

The sum over n_0 from 0 to ∞ is the geometric progression with converges provided $|\lambda| < q^{2(\mathbf{l_1}+\mathbf{l_2})}$.

This construction gives the $U_q(sl(2))$ R-matrix $R^{(\mathbf{l_1},\mathbf{l_2})}(\lambda)$ in the horizontal field ϕ acting in the tensor product of representations with spins $\mathbf{l_1}/2$ and $\mathbf{l_2}/2$.

The higher-spin R-matrix

The matrix elements of the $(l_1+1)\times(l_2+1)$ R-matrix are given by

$$[R^{(l_1,l_2)}(\lambda)]_{i_1,i_2}^{j_1,j_2} = \delta_{i_1+i_2,j_1+j_2}\,\phi^{2i_1-l_1}\,\frac{q^{i_2^2+(l_2-i_2)(l_1-j_1)-j_2(j_2-i_1)+2l_2+\frac{1}{2}l_1l_2-\frac{1}{2}m(l_1,l_2)}}{(q^2;q^2)_{i_2}(q^2;q^2)_{l_2-i_2}}$$

$$\times\lambda^{i_2-j_2-m(l_1,l_2)}(\lambda^2 q^{-l_1-l_2};q^2)_{m(l_1,l_2)+1}\sum_{k=0}^{i_2}\sum_{l=0}^{l_2-i_2}\frac{(-1)^{k+l}q^{2k(j_2-i_1)-2l(l_1-l_2-i_1+i_2)}}{(1-\lambda^2 q^{l_2-l_1-2k-2l})q^{k(k+1)+l(l+1)}}$$

$$\times\frac{(q^{-2i_2},q^{2+2i_1};q^2)_k(q^{-2j_1};q^2)_{i_2-k}}{(q^2;q^2)_k}\,\frac{(q^{-2(l_2-i_2)},q^{2(1+l_1-i_1)};q^2)_l(q^{-2(l_1-j_1)};q^2)_{l_2-i_2-l}}{(q^2;q^2)_l}$$

where

$$m(i,j)=\min(i,j).$$

and $i_1,j_1=0,\ldots,l_1,\ i_2,j_2=0,\ldots,l_2.$
This is the simplest formula for $R^{(l_1,l_2)}(\lambda)$ we know. It satisfies the Yang-Baxter equation

$$R_{12}^{(l_1,l_2)}(\lambda)R_{13}^{(l_1,l_3)}(\lambda\mu)R_{23}^{(l_2,l_3)}(\mu)=R_{23}^{(l_2,l_3)}(\mu)R_{13}^{(l_1,l_3)}(\lambda\mu)R_{12}^{(l_1,l_2)}(\lambda)$$

Fusion procedure or spectral decomposition lead to more complicated formulas.

Symmetries of the R-matrix

$$\mathcal{P}_{12} R^{(l_1,l_2)}(\lambda)\mathcal{P}_{12} = R^{(l_2,l_1)}(\lambda) \quad \text{at} \quad \phi = 1$$

$$R^{(l_1,l_2)}(\lambda;\phi)^{j_1,j_2}_{i_1,i_2} = R^{(l_1,l_2)}(\lambda;\phi^{-1})^{l_1-j_1,l_2-j_2}_{l_1-i_1,l_2-i_2}.$$

Proof:

$$[R^{(l_1,l_2)}(\lambda)]^{j_1,j_2}_{i_1,i_2} = \delta_{i_1+i_2,j_1+j_2}\, q^{i_2^2+(l_2-i_2)(l_1-j_1)-j_2(j_2-i_1)+2l_2+\frac{1}{2}l_1 l_2-\frac{1}{2}m(l_1,l_2)} \times$$

$$\phi^{2i_1-l_1}\lambda^{i_2-j_2-m(l_1,l_2)}(\lambda^2 q^{-l_1-l_2};q^2)_{m(l_1,l_2)+1}\frac{(q^{-2j_1};q^2)_{i_2}(q^{-2(l_1-j_1)};q^2)_{l_2-i_2}}{(q^2;q^2)_{i_2}(q^2;q^2)_{l_2-i_2}} \times$$

$$\sum_{s=0}^{l_2} q^{2j_2 s}\frac{(q^{-2l_2},q^{2(1+l_1-i_1-i_2)};q^2)_s\; {}_4\phi_3\left(\begin{matrix} q^{-2s},q^{-2i_2},q^{-2j_2},q^{2(1+l_1-l_2+s)} \\ q^{-2l_2},q^{2(1+i_1-j_2)},q^{2(1+l_1-i_1-i_2)}\end{matrix}\;\middle|\; q^2,q^2\right)}{(1-\lambda^2 q^{l_2-l_1-2s})(q^2,q^{2(1-i_1+j_2+l_1-l_2)};q^2)_s}$$

It follows by applying a sequence of Sears' transformations for terminated balanced ${}_4\phi_3$ series.

There is the explicit expression for the R-matrix which in not restricted to the case of integer I and J, i.e. works for Verma modules. It is polynomial in λ and can be expressed in terms of continuous q-Hahn polynomials.

$$[R^{I,J}(\lambda;\phi)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'}\, c_{i,j}^{i',j'}(\lambda)\ {}_4\overline{\phi}_3 \left(\begin{array}{c} q^{-2i}; q^{-2i'}, \lambda^{-2}q^{J-I}, \lambda^2 q^{2+J-I} \\ q^{-2I}, q^{2(1+j'-i)}, q^{2(1+J-i-j)} \end{array} \middle| q^2, q^2 \right)$$

where $c_{i,j}^{i',j'}(\lambda)$ ia some simple product of q-Pochhammer symbols and ${}_4\overline{\phi}_3$ is a regularized hypergeometric function.

All symmetry properties are now a simple consequence of Sears' transformations for terminating balanced ${}_4\phi_3$ series.

$$\mathcal{P}_{12} R^{(I_1,I_2)}(\lambda) \mathcal{P}_{12} = R^{(I_2,I_1)}(\lambda) \quad \text{at} \quad \phi = 1$$

$$R^{(I_1,I_2)}(\lambda;\phi)_{i_1,i_2}^{j_1,j_2} = R^{(I_1,I_2)}(\lambda;\phi^{-1})_{I_1-i_1,I_2-i_2}^{I_1-j_1,I_2-j_2}.$$

Example

Consider $l_1 = 1$ case. Then we get a 2×2 Lax operator in the $l_2 + 1$-dim representation.

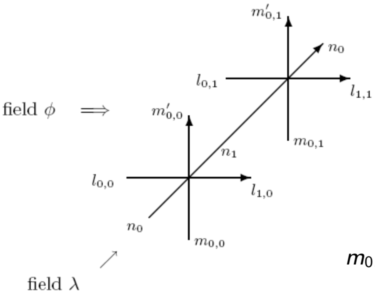
$$[R^{(1,l_2)}(\lambda)]_{i_1,i_2}^{j_1,j_2} = \begin{pmatrix} \delta_{i_2,j_2}\phi^{-1}[\lambda q^{\frac{1+l_2}{2}-j_2}] & \delta_{i_2,j_2+1}\phi^{-1}[q^{l_2-j_2}]q^{j_2-\frac{l_2-1}{2}} \\ \delta_{i_2+1,j_2}\phi\,[q^{j_2}]q^{\frac{l_2+1}{2}-j_2} & \delta_{i_2,j_2}\phi\,[\lambda q^{\frac{1-l_2}{2}+j_2}] \end{pmatrix}_{i_1+1,j_1+1}$$

where $[x] = x - x^{-1}$.
Conjugating by $\sigma_x^{(1)}$ we get the standard $U_q(sl(2))$ L-operator $L(\mu)$ in the field ϕ

$$L(\mu) = \sigma_x^{(1)}\mathbf{R}^{(1,l_2)}(\lambda)\sigma_x^{(1)} = \begin{pmatrix} \phi^{-1}[\mu K^{1/2}] & \phi^{-1}[q]F \\ \phi[q]E & \phi\,[\mu K^{-1/2}] \end{pmatrix}$$

where $\mu = \lambda q^{1/2}$ and

$$K v_j = q^{j-l_2/2} v_j, \quad F v_j = \phi q^{\frac{1+l_2}{2}-j}\frac{[q^j]}{[q]} v_{j-1}, \quad E v_j = \phi^{-1} q^{\frac{1-l_2}{2}+j}\frac{[q^{l_2-j}]}{[q]} v_{j+1}.$$



$$l_{0,0} + l_{0,1} = l_{1,0} + l_{1,1} = \mathbf{l}$$

$$m_{0,0} + m_{0,1} = m'_{0,0} + m'_{0,1} \to \infty$$

$m_{0,0} = n_1,\ m'_{0,0} = n_2,\ l_{0,0} = k_1,\ l_{1,0} = k_2$ are fixed,

In the 3D picture we take the limit $l_2 \to \infty$ keeping n_1 and n_2 finite. Then the second 3D R-matrix degenerates into the product of q-Pochhammer symbols.

In this way we arrive to the local Baxter's Q-operator. The Q-operator of the spin l (the trace is taken over the q-oscillator representation labeled by $n_1, n_2, \ldots$)

$$Q_+^{(l)}(\lambda) = \mathrm{Tr}_n\,[Q_+^{(l))}]_{1n}(\lambda)\ldots[Q_+^{(l)}]_{Mn}(\lambda)$$

Taking the limit $l_2 \to \infty$ in our general expression for the R-matrix we immediately derive the local $Q - operator$ acting in the tensor product of the spin l representation and q-oscillator representation

$$Q_{+}^{(l)}(\lambda)_{k_1,n_1}^{k_2,n_2} = \delta_{k_1+n_1,k_2+n_2} \frac{q^{3k_1(k_2-n_1)+k_2 n_1 - k_1^2 - k_2^2 + l(k_1+n_1)}}{\phi^{2n_1}(q\lambda)^{\frac{k_1+k_2}{2}}(q^2;q^2)_{k_1}} \, {}_3\overline{\phi}_2 \left(\begin{array}{c} q^{-2k_1}; q^{-2k_2}, \lambda^2 q^{1-l} \\ q^{-2l}, q^{2(1+n_1-k_2)} \end{array} \middle| q^2, q^2 \right)$$

The second Q-operator is obtained by applying the transformation

$$Q_{-}^{(l)}(\lambda)_{k_1,n_1}^{k_2,n_2} = \delta_{n_1-k_1,n_2-k_2} \; Q_{+}^{(l)}(\lambda)_{l-k_1,n_1}^{l-k_2,n_2}\Big|_{\phi\to\phi^{-1}}$$

One can derive the expression for $Q_{-}^{(l)}$ valid for general l

$$Q_{-}^{(l)}(\lambda)_{k_1,n_1}^{k_2,n_2} = \delta_{n_1-k_1,n_2-k_2} \phi^{2n_1}(-1)^{k_1+k_2} q^{k_2^2+k_1 n_1 - 3k_2 n_2 + l(n_1+k_2)+k_1-k_2}(q\lambda)^{\frac{k_1+k_2}{2}-l}$$
$$\times \frac{(q^2;q^2)_{n_2}(\lambda^2 q^{1-l+2(k_1-n_1)};q^2)_{l-k_1-k_2}}{(q^2;q^2)_{n_1}(q^2;q^2)_{k_1}} \, {}_3\overline{\phi}_2 \left(\begin{array}{c} q^{-2k_1}; q^{-2k_2}, \lambda^2 q^{1-l} \\ q^{-2l}, q^{2(1+n_1-k_1)} \end{array} \middle| q^2, q^2 \right)$$

TQ-relation

For the case $l = 1$ (spin $= 1/2$) this Q-operator reproduces the formula by Bazhanov, Lukyanov and Zamolodchikov (1996).

$$Q_-^{(l)}(\lambda)_{k_1,n_1}^{k_2,n_2} = \delta_{n_1-k_1,n_2-k_2}\; Q_+^{(l)}(\lambda)_{l-k_1,n_1}^{l-k_2,n_2}\Big|_{\phi\to\phi^{-1}} \tag{1}$$

Since

$$R^{(l_1,l_2)}(\lambda;\phi)_{i_1,i_2}^{j_1,j_2} = R^{(l_1,l_2)}(\lambda;\phi^{-1})_{l_1-i_1,l_2-i_2}^{l_1-j_1,l_2-j_2}.$$

we have the following TQ-relations for the spin l:

$$T_1^{(l)}(\lambda)Q_\pm^{(l)}(\lambda) = (-\phi)^{\pm M}\,[\lambda q^{\frac{1-l}{2}}]^M Q_\pm^{(l)}(q\lambda) + (-\phi)^{\mp M}\,[\lambda q^{\frac{1+l}{2}}]^M Q_\pm^{(l)}(\lambda/q),$$

Wronskian relation

$$\phi^M Q_+^{(l)}(\lambda q^{1/2})Q_-^{(l)}(\lambda q^{-1/2}) - \phi^{-M} Q_-^{(l)}(\lambda q^{1/2})Q_+^{(l)}(\lambda q^{-1/2}) = \prod_{k=1}^{l}[\lambda q^{k-l/2}]^M\,\mathrm{Wr}[\phi]$$

where

$$\mathrm{Wr}[\phi] = \frac{(-1)^{Ml}}{q^{Ml+n}(q^{Ml-2n}\phi^{-M} - \phi^M)},$$

where n is the sum over all M outgoing spins.

Functional relations

There are two expressions for the transfer-matrices in terms of the Q-operators for compact spins

$$T_i^{(j)}(\lambda) = Q_{\pm}^{(j)}(\lambda q^{-\frac{i+1}{2}}) Q_{\pm}^{(j)}(\lambda q^{\frac{i+1}{2}}) \sum_{l=0}^{i} \frac{(-\phi)^{\pm M(i-2l)} (F_{ij;l}(\lambda q^{l-\frac{i}{2}}))^M}{Q_{\pm}^{(j)}(\lambda q^{l-\frac{i+1}{2}}) Q_{\pm}^{(j)}(\lambda q^{l-\frac{i-1}{2}})}$$

where

$$F_{ij;l}(\lambda) = (-1)^{i+j} \frac{\prod_{k=1}^{j} [\lambda q^{k-\frac{j}{2}}]}{\prod_{k=0}^{j-i-1} [\lambda^{-1} q^{l+k-\frac{j}{2}}]}$$

or

$$T_i^{(j)}(\lambda) = \frac{\phi^{(i+1)M} Q_+^{(j)}(\lambda q^{\frac{i+1}{2}}) Q_-^{(j)}(\lambda q^{-\frac{i+1}{2}}) - \phi^{-(i+1)M} Q_-^{(j)}(\lambda q^{\frac{i+1}{2}}) Q_+^{(j)}(\lambda q^{-\frac{i+1}{2}})}{\mathrm{Wr}[\phi]\, h_{ij}(\lambda)^M}$$

where

$$h_{ij}(\lambda) = (-1)^j \prod_{k=0}^{j-i-1} [\lambda^{-1} q^{k+\frac{i-j}{2}}]$$

The expression for the local Q-matrix $Q_+^{(l)}(\lambda)$

$$Q_+^{(l)}(\lambda)_{k_1,n_1}^{k_2,n_2} = \delta_{k_1+n_1,k_2+n_2} q^{k_1^2+(k_1-k_2)(k_2-n_1)+l(n_1-k_1)-(k_1+k_2)/2} \phi^{-2n_1} \lambda^{-(k_1+k_2)/2} \times$$
$$\times \frac{(q^{-2n_2};q^2)_{k_1}(q^2;q^2)_l}{(q^2;q^2)_{k_1}(q^2;q^2)_{l-k_1}} \, {}_3\phi_2\left(\begin{array}{ccc} \lambda^2 q^{1-l}, & q^{-2k_1}, & q^{-2k_2} \\ q^{-2l}, & q^{2+2n_1-2k_2} & \end{array} \middle| \, q^2, q^2 \right)$$

$k_1, k_2 = 0, \ldots, l$, $l \in \mathbb{Z}_{\geq 1}$.
When l is non-integer, there is no pole and $k_1, k_2 = 0, \ldots, \infty$.
Therefore, the action of this Q-operator can be generalized to non-compact spins and in the limit $l \to 1, 2, \ldots$ this action is non-singular and gives the correct $(l+1) \times (l+1)$-dim irreducible block.

There is an alternative "saw" construction of the Q-operator. It is based on the factorization property of the $U_q(sl(2))$ L-operator (Korepin, Tarasov).
The idea goes back to Bazhanov and Stroganov (1987) (XXZ, $q^N = 1$). For the non-compact case XXX case it has been done by Kuznetsov, Sklyanin; Derkachov (1999). The deformed case was studied recently by Chicherin, Derkachov, et al (2011). It works well for the non-compact case. However, in the compact spin limit the action of the "saw"-like Q-operator becomes singular and it is not clear how to extract a finite-dimensional block.

Conclusions/prospects

- We constructed the higher spin R-matrix for the XXZ chain which works for both compact and non-compact representations.
- This leads to explicit expressions for two Q-operators for any representation with arbtrary spin I.
- There is no analog of the q-oscillator and “field” for the elliptic case. However, the “saw” type Q-operator can still be constructed.
- Are there elliptic versions of the higher spin R-matrix and ${}_3\phi_2$ Q-matrix representations ?