Off-critical parafermionic observables and the winding angle distribution of the O(n) loop model

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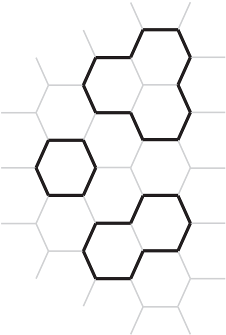
Outline of the talk

- The $O(n)$ loop model on the honeycomb lattice
- Smirnov's parafermionic observable away from criticality
- The winding angle distribution and critical exponents

$O(n)$ loop model

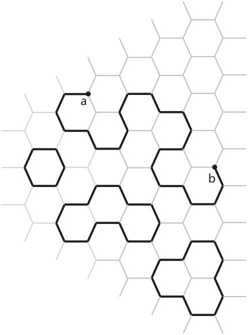
- Closed non-intersecting loops on the honeycomb lattice. Each loop contributes a weight $n = 2\cos\phi$, $\phi \in [-2, 2]$ and each loop segment x.
- *Partition function* given by the sum over all configurations, $Z = \sum_{g \in G} n^{c(g)} x^{l(g)}$.
- Some much-studied models at particular values of n: $n \to 0$ (SAW), $n \to 1$ (Ising), $n = 2$ (classical XY model).
- Conjectured critical points and critical exponents (Nienhuis '82).



Winding angle distribution of the $O(n)$ model

Duplantier and Saleur ('88) calculated the winding angle distribution of the walk component in the $O(n)$ loop model, where $n = -2\cos(\frac{4\pi}{\kappa})$.

$$P(\theta) = (2\pi\nu\kappa \log \ell)^{-1/2} \exp(-\frac{\theta^2}{2\nu\kappa \log \ell}), \qquad \nu = \frac{1}{4-\kappa}.$$

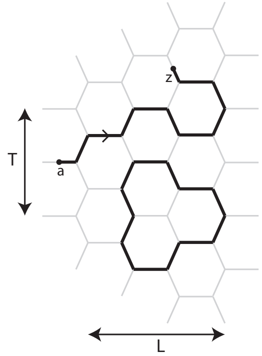


Parafermionic observable

Smirnov ('08) defined the following function at the *mid-edges* of the lattice.

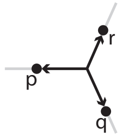
$$F_\sigma(z) = \sum_\gamma P(\gamma) e^{-i\sigma W(\gamma)}.$$

- σ is the *parafermionic spin* (related to the central charge in CFT and κ in SLE).
- $W(\gamma)$ is the winding angle of the loop segment from a to z.
- $P(\gamma)$ is the total weight of the configuration γ.
- Ω is the set of all mid-edges and $\partial\Omega$ is the set of all boundary mid-edges.



Discrete holomorphicity

We say that $F(z)$ is (partially) *discrete holomorphic* if around a given vertex v it satisfies the following linear condition.

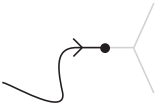


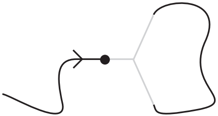
$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0.$$

- Discrete analogue of $\oint f(z)dz = 0$. Not sufficient to determine $F_\sigma(z)$ given its boundary values.
- Square-lattice Ising *fermionic* observable satisfies a much stronger condition. Used to prove various conformal invariance conjectures about the Ising model (Chelkak, Hongler, Smirnov)
- If the scaling limit of $F_\sigma(z)/\delta^\sigma$ is a holomorphic function conformal invariance of corresponding lattice model follows. (Smirnov '07).

Discrete holomorphicity

For a loop segment entering the vertex via a given mid-edge there are two sets of configurations.



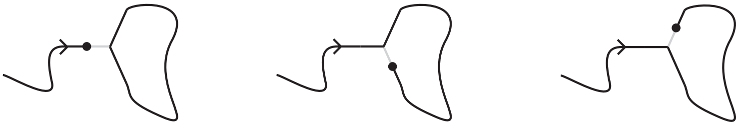


For each set we calculate:

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0.$$

Discrete holomorphicity

Define $\lambda = \mathrm{e}^{-\pi\sigma\mathrm{i}/3}$ as the weight of a left turn and $j = \mathrm{e}^{\pi\mathrm{i}/3}$. From the first set of configurations,



we have

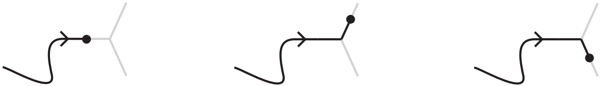
$$P(\gamma_1)(-n + \bar{j}\bar{\lambda}^4 + j\lambda^4) = 0.$$

Recalling that $n = 2\cos\phi$ and solving for σ we find two sets of solutions:

$$\sigma = \frac{\pi - 3\phi}{4\pi}, \qquad \text{or} \qquad \sigma = \frac{\pi + 3\phi}{4\pi}.$$

Discrete holomorphicity

Second set of configurations:



We obtain the following equation

$$P(\gamma_2)(-1 + j\lambda + \bar{j}\bar{\lambda}) = 0.$$

Solving for x we find

$$x^{-1} = x_c^{-1} = 2\cos(\frac{\pi + \phi}{4}), \qquad x^{-1} = x_c^{-1} = 2\cos(\frac{\pi - \phi}{4}),$$

corresponding to dense and dilute phases respectively.

- These are the critical values of the $O(n)$ model predicted by Nienhuis. ('82)
- The $n = 0$ (SAW) case was rigorously proven only recently (Duminil-Copin and Smirnov '10),

$$x_c^{-1} = \sqrt{2 + \sqrt{2}}.$$

Off-critical discrete holomorphicity

Relax the discrete holomorphicity condition: σ fixed fixed but $x < x_c$. This leads to

$$(p-v)F(p) + (q-v)F(q) + (r-v)F(r) = (x - x_c)G(v),$$

The *vertex observable* $G(v)$ is defined by

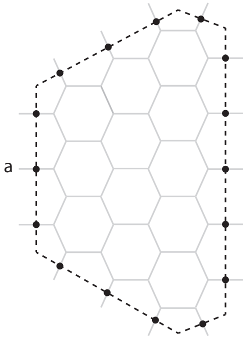
$$G(v) = (p-v)F(p;v) + (q-v)F(q;v) + (r-v)F(r;v),$$

where $F(p;v)$ consists of walks terminating at the mid-edge p before the vertex v and where there is no loop connected to the remaining mid-edges. A similar off-critical observable for the Ising model was considered by Beffara and Duminil-Copin ('12)

Identity between boundary and bulk walks

By summing the discrete holomorphicity condition over all vertices of the domain, Duminil-Copin and Smirnov ('10) derived the following identity between walks terminating on the boundary and the interior

$$\sum_{\gamma:a\to z\in\partial\Omega\setminus\{a\}} \mathrm{e}^{\mathrm{i}(1-\sigma)W(\gamma)} x^{|\gamma|} n^{c(\gamma)} - \sum_{\gamma:a\to a} x^{|\gamma|} n^{c(\gamma)} = 0.$$



Off-critical identity

In the off-critical case, the same summation leads to

$$\sum_{\gamma:a\to z\in\partial\Omega\setminus\{a\}} \mathrm{e}^{\mathrm{i}(1-\sigma)W(\gamma)}x^{|\gamma|}n^{c(\gamma)} \quad + \quad (1-x/x_c)\sum_{\gamma:a\to z\in\Omega\setminus\partial\Omega} \mathrm{e}^{\mathrm{i}(1-\sigma)W(\gamma)}x^{|\gamma|}n^{c(\gamma)}$$
$$= \sum_{\gamma:a\to a} x^{|\gamma|}n^{c(\gamma)},$$

which we write more concisely as

$$\underbrace{H_\Omega(x)}_{\text{Boundary}} + (1-\frac{x}{x_c})\sum_\theta \mathrm{e}^{\mathrm{i}(1-\sigma)\theta}\underbrace{G_{\Omega,\theta}(x)}_{\text{Interior}} = \underbrace{C_\Omega(x)}_{\text{Loops}},$$

where

$$G_{\Omega,\theta}(x) = \sum_{\gamma:a\to z\in\Omega\setminus\partial\Omega,\, W(\gamma)=\theta} x^{|\gamma|}n^{c(\gamma)}, \quad H_\Omega(x) = \sum_{\gamma:a\to z\in\partial\Omega\setminus\{a\}} \mathrm{e}^{\mathrm{i}(1-\sigma)W(\gamma)}x^{|\gamma|}n^{c(\gamma)}.$$

Off-critical identity: critical exponents

$$H_\Omega(x) + (1 - \frac{x}{x_c}) \sum_\theta \mathrm{e}^{\mathrm{i}(1-\sigma)\theta} G_{\Omega,\theta}(x) = C_\Omega(x).$$

Surprisingly this simple off-critical deformation allows us to relate critical exponents. Dividing through by $C_\Omega(x)$ and taking the width and length of the domain to ∞ (the domain then becomes a half-plane) we find

$$H^*(x) + (1 - \frac{x}{x_c}) \sum_\theta \mathrm{e}^{\mathrm{i}(1-\sigma)\theta} G^*_\theta(x) = 1.$$

Assuming the following (standard) asymptotic form of $H^*(x)$

$$H^*(x) \sim 1 + \mathrm{const} \times (1 - x/x_c)^{-\gamma_{11}}.$$

and therefore

$$\sum_\theta \mathrm{e}^{\mathrm{i}(1-\sigma)\theta} G^*_\theta(x) = (1 - H^*(x)) \sim \mathrm{const} \times (1 - x/x_c)^{-\gamma_{11}-1}.$$

Critical exponents

We denote by $a_\theta(j)$ the number of walks of length j with winding angle θ. We then write

$$G^*_\theta(x) = \sum_{j=0}^{\infty} a_\theta(j) x^j.$$

$G^*_\theta(x)$ has the asymptotic form

$$\sum_\theta G^*_\theta(x) \sim \mathrm{const} \times (1 - x/x_c)^{-\gamma_1}.$$

This gives the asymptotics of the coefficients $a_\theta(j)$:

$$\sum_\theta a_\theta(j) \sim \mathrm{const} \times x_c^{-j} j^{\gamma_1 - 1}, \qquad \sum_\theta \mathrm{e}^{\mathrm{i}(1-\sigma)\theta} a_\theta(j) \sim \mathrm{const} \times x_c^{-j} j^{\gamma_{11}}.$$

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Winding angle exponent

For a walk of length j the winding angle distribution is defined as

$$P(\theta, j) = \frac{a_\theta(j)}{\sum_\theta a_\theta(j)},$$

and the Fourier transform is given by

$$\sum_\theta \mathrm{e}^{\mathrm{i}\bar{\sigma}\theta} P(\theta, j) = \frac{\sum_\theta \mathrm{e}^{\mathrm{i}\bar{\sigma}\theta} a_\theta(j)}{\sum_\theta a_\theta(j)}.$$

Using the result from before:

$$\sum_\theta a_\theta(j) \sim \mathrm{const} \times x_c^{-j} j^{\gamma_1 - 1}, \qquad \sum_\theta \mathrm{e}^{\mathrm{i}(1-\sigma)\theta} a_\theta(j) \sim \mathrm{const} \times x_c^{-j} j^{\gamma_{11}},$$

we arrive at (with $\bar{\sigma} = 1 - \sigma = 5/8$ for SAW)

$$\sum_\theta \mathrm{e}^{\mathrm{i}(1-\sigma)\theta} P(\theta, j) \sim \mathrm{const} \times j^{\gamma_{11} - \gamma_1 + 1}.$$

Winding angle distribution of the $O(n)$ model

Recall the predicted winding angle distribution of the $O(n)$ model has the asymptotic form

$$P(\theta) \propto \exp(-\frac{\theta^2}{2\kappa\nu \log \ell}), \qquad \ell \to \infty,$$

Consider the Fourier transform:

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(1-\sigma)\theta} P(\theta, \ell) \propto \ell^{-\omega},$$

We find the winding angle exponent in terms of the bulk and boundary critical exponents:

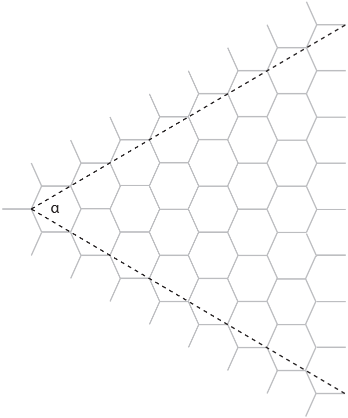
$$\gamma_1 - \gamma_{11} - 1 = \omega.$$

Wedge exponents

The argument can be extended to wedge shaped domains with opening angle α. In this case we have wedge exponents $\gamma_2(\alpha), \gamma_{21}(\alpha)$ which satisfy the relation

$$\gamma_{21}(\alpha) - \gamma_2(\alpha) + 1 = \omega,$$

where ω is the winding angle exponent from before. Setting $\alpha = \pi$ gives the previous results.



Summary

- Off-critical observables lead to an off-critical generating function identity
- Gives relation between critical exponents and winding angle exponent
- Nothing other than a simple linear condition satisfied by $F_\sigma(z)$ is required.

Further work:

- Can $F_\sigma(z)$ be calculated explicitly?
- Understand the relations satisfied by the vertex observable $G(v)$.
- For the square lattice Ising model, $G(v)$ is known. In this case can the elliptic integrable weights be determined from discrete complex analysis?