Symmetries of curved superspace

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Based on:
SMK, arXiv:1212.6179

Background and motivation

Exact results (partition functions, Wilson loops etc.) in rigid supersymmetric field theories on curved backgrounds (e.g., S^3, S^4 , $S^3 \times S^1$ etc.) using localization techniques

V. Pestun (2007, 2009)
A. Kapustin, B. Willett & I. Yaakov (2010)
D. Jafferis (2010)
. . . .

Necessary technical ingredients:

- Curved space $\mathcal{M}$ should admit some unbroken rigid supersymmetry (supersymmetric background);
- Rigid supersymmetric field theory on $\mathcal{M}$ should be off-shell.

These developments have inspired much interest in the construction and classification of supersymmetric backgrounds that correspond to off-shell supergravity formulations.

Classification of supersymmetric backgrounds in off-shell supergravity

Component approaches

G. Festuccia and N. Seiberg (2011)
B. Jia and E. Sharpe (2011)
H. Samtleben and D. Tsimpis (2012)
C. Klare, A. Tomasiello and A. Zaffaroni (2012)
T. Dumitrescu, G. Festuccia and N. Seiberg (2012)
D. Cassani, C. Klare, D. Martelli, A. Tomasiello and A. Zaffaroni (2012)
T. Dumitrescu and G. Festuccia (2012)
A. Kehagias and J. Russo (2012)
. . . .

Formalism to determine (conformal) isometries of curved superspaces
I. Buchbinder & SMK, *Ideas and Methods of Supersymmetry and Supergravity or a Walk Through Superspace*, IOP, Bristol, 1995 (1998)

Superspace formalism is universal, for it is geometric and may be generalized to supersymmetric backgrounds associated with any supergravity theory formulated in superspace.

Applications of the formalism:

- Rigid supersymmetric field theories in 5D $\mathcal{N} = 1$ AdS superspace
  SMK & G. Tartaglino-Mazzucchelli (2007)
- Rigid supersymmetric field theories in 4D $\mathcal{N} = 2$ AdS superspace
  SMK & G. Tartaglino-Mazzucchelli (2008)
  D. Butter & SMK (2011)
  D. Butter, SMK, U. Lindström & G. Tartaglino-Mazzucchelli (2012)
- Rigid supersymmetric field theories in 3D (p, q) AdS superspaces
  SMK, & G. Tartaglino-Mazzucchelli (2012)
  SMK, U. Lindström & G. Tartaglino-Mazzucchelli (2012)
  D. Butter, SMK & G. Tartaglino-Mazzucchelli (2012)

Three formulations for gravity

Metric formulation

Gauge field: metric $g_{mn}(x)$

Gauge transformation: $\delta g_{mn} = \nabla_m \xi_n + \nabla_n \xi_m$

with $\xi = \xi^m(x)\partial_m$ a vector field generating an infinitesimal diffeomorphism.

Vielbein formulation

Gauge field: vielbein $e_m{}^a(x)$, $e := \det(e_m{}^a) \neq 0$

The metric is a composite field $g_{mn} = e_m{}^a e_n{}^b \eta_{ab}$

Gauge transformation: $\delta\nabla_a = [\xi^b\nabla_b + \frac{1}{2}K^{bc}M_{bc}, \nabla_a]$

Gauge parameters: $\xi^a = \xi^m e_m{}^a(x)$ and $K^{ab}(x) = -K^{ba}(x)$

Covariant derivatives (M_{bc} the Lorentz generators)

$$\nabla_a = e_a{}^m\partial_m + \frac{1}{2}\omega_a{}^{bc}M_{bc} , \qquad [\nabla_a, \nabla_b] = \frac{1}{2}R_{ab}{}^{cd}M_{cd}$$

$e_a{}^m$ the inverse vielbein, $e_a{}^m e_m{}^b = \delta_a{}^b$; $\omega_a{}^{bc}[e]$ the Lorentz connection

Three formulations for gravity

Weyl transformations

The torsion-free constraint

$$T_{ab}{}^{c} = 0 \quad \Longleftrightarrow \quad [\nabla_a, \nabla_b] \equiv T_{ab}{}^{c}\nabla_c + \frac{1}{2}R_{ab}{}^{cd}M_{cd} = \frac{1}{2}R_{ab}{}^{cd}M_{cd}$$

is invariant under Weyl (local scale) transformations

$$\nabla_a \to \nabla'_a = \mathrm{e}^{\sigma}\Big(\nabla_a + (\nabla^b\sigma)M_{ba}\Big) ,$$

with the parameter $\sigma(x)$ being completely arbitrary.

$$e_a{}^m \to \mathrm{e}^{\sigma} e_a{}^m , \qquad e_m{}^a \to \mathrm{e}^{-\sigma} e_m{}^a , \qquad g_{mn} \to \mathrm{e}^{-2\sigma} g_{mn}$$

Three formulations for gravity

Conformal formulation

Gauge fields: vielbein $e_m{}^a(x)$, $e := \det(e_m{}^a) \neq 0$

& conformal compensator $\varphi(x)$, $\varphi \neq 0$

Gauge transformations ($\mathcal{K} := \xi^b \nabla_b + \frac{1}{2} K^{bc} M_{bc}$)

$$\delta \nabla_a = [\xi^b \nabla_b + \frac{1}{2} K^{bc} M_{bc}, \nabla_a] + \sigma \nabla_a + (\nabla^b \sigma) M_{ba} \equiv (\delta_{\mathcal{K}} + \delta_\sigma) \nabla_a \ ,$$

$$\delta \varphi = \xi^b \nabla_b \varphi + \sigma \varphi \equiv (\delta_{\mathcal{K}} + \delta_\sigma) \varphi$$

Gauge-invariant gravity action (in four spacetime dimensions)

$$S = \frac{1}{2} \int \mathrm{d}^4 x\, e \left(\nabla^a \varphi \nabla_a \varphi + \frac{1}{6} R \varphi^2 + \lambda \varphi^4 \right)$$

Weyl gauge condition: $\varphi = \sqrt{6/\kappa} = \mathrm{const.}$

Action turns into the Einstein-Hilbert action with a cosmological term.

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^4 x\, e\, R - \frac{\Lambda}{\kappa^2} \int \mathrm{d}^4 x\, e$$

Conformal isometries

Conformal Killing vector fields

A vector field $\xi = \xi^m \partial_m = \xi^a e_a$, with $e_a := e_a{}^m \partial_m$, is conformal Killing if there exist local Lorentz, $K^{bc}[\xi]$, and Weyl, $\sigma[\xi]$, parameters such that

$$\Big[\xi^b \nabla_b + \frac{1}{2} K^{bc}[\xi] M_{bc}, \nabla_a\Big] + \sigma[\xi] \nabla_a + (\nabla^b \sigma[\xi]) M_{ba} = 0$$

A short calculation gives

$$K^{bc}[\xi] = \frac{1}{2}\big(\nabla^b \xi^c - \nabla^c \xi^b\big) , \qquad \sigma[\xi] = \frac{1}{4} \nabla_b \xi^b$$

Conformal Killing equation

$$\nabla^a \xi^b + \nabla^b \xi^a = 2\eta^{ab} \sigma[\xi]$$

Equivalent spinor form ($\nabla_a \to \nabla_{\alpha\dot\alpha}$ and $\xi_a \to \xi_{\alpha\dot\alpha}$)

$$\nabla_{(\alpha}{}^{(\dot\alpha} \xi_{\beta)}{}^{\dot\beta)} = 0$$

Conformal isometries

- Lie algebra of conformal Killing vector fields
- Conformally related spacetimes (∇_a, φ) and $(\widetilde{\nabla}_a, \widetilde{\varphi})$

$$\widetilde{\nabla}_a = \mathrm{e}^{\rho}\Big(\nabla_a + (\nabla^b \rho) M_{ba}\Big) , \qquad \widetilde{\varphi} = \mathrm{e}^{\rho}\varphi$$

have the same conformal Killing vector fields $\xi = \xi^a e_a = \tilde{\xi}^a \tilde{e}_a$.

The parameters $K^{cd}[\tilde{\xi}]$ and $\sigma[\tilde{\xi}]$ are related to $K^{cd}[\xi]$ and $\sigma[\xi]$ as follows:

$$\mathcal{K}[\tilde{\xi}] := \tilde{\xi}^b \widetilde{\nabla}_b + \frac{1}{2} K^{cd}[\tilde{\xi}] M_{cd} = \mathcal{K}[\xi] ,$$
$$\sigma[\tilde{\xi}] = \sigma[\xi] - \xi\rho$$

- Conformal field theories

Isometries

Killing vector fields

A vector field $\xi = \xi^m \partial_m = \xi^a e_a$, with $e_a := e_a{}^m \partial_m$, is Killing if there exist local Lorentz $K^{bc}[\xi]$ and Weyl $\sigma[\xi]$ parameters such that

$$\Big[\xi^b \nabla_b + \frac{1}{2} K^{bc}[\xi] M_{bc}, \nabla_a\Big] + \sigma[\xi] \nabla_a + (\nabla^b \sigma[\xi]) M_{ba} = 0 ,$$
$$\xi \varphi + \sigma[\xi] \varphi = 0$$

These Killing equations are Weyl invariant in the following sense:
Given a conformally related spacetime $(\widetilde{\nabla}_a, \widetilde{\varphi})$

$$\widetilde{\nabla}_a = \mathrm{e}^\rho \Big(\nabla_a + (\nabla^b \rho) M_{ba}\Big) , \qquad \widetilde{\varphi} = \mathrm{e}^\rho \varphi ,$$

the above Killing equations have the same functional form when rewritten in terms of $(\widetilde{\nabla}_a, \widetilde{\varphi})$, in particular

$$\xi \widetilde{\varphi} + \sigma[\tilde{\xi}] \widetilde{\varphi} = 0$$

Isometries

Because of Weyl invariance, we can work with a conformally related spacetime such that

$$\varphi = 1$$

Then the Killing equations turn into

$$\Big[\xi^b \nabla_b + \frac{1}{2} K^{bc}[\xi] M_{bc}, \nabla_a\Big] = 0 \;, \qquad \sigma[\xi] = 0$$

Standard Killing equation:

$$\nabla^a \xi^b + \nabla^b \xi^a = 0$$

- Lie algebra of Killing vector fields
- Rigid symmetric field theories in curved space

The Wess-Zumino superspace geometry

4D $\mathcal{N}=1$ curved superspace $\mathcal{M}^{4|4}$ parametrized by $z^M = (x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}})$.

The superspace geometry is described by covariant derivatives of the form

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^{\dot{\alpha}}) = E_A + \Omega_A \ .$$

E_A is the inverse vielbein, $E_A = E_A{}^M \partial_M$
Ω_A is the Lorentz connection,

$$\Omega_A = \frac{1}{2}\,\Omega_A{}^{bc} M_{bc} = \Omega_A{}^{\beta\gamma} M_{\beta\gamma} + \Omega_A{}^{\dot{\beta}\dot{\gamma}} \bar{M}_{\dot{\beta}\dot{\gamma}} \ ,$$

with $M_{bc} \Leftrightarrow (M_{\beta\gamma}, \bar{M}_{\dot{\beta}\dot{\gamma}})$ the Lorentz generators.
Supergravity gauge transformation (U matter tensor superfield)

$$\delta_{\mathcal{K}} \mathcal{D}_A = [\mathcal{K}, \mathcal{D}_A] \ , \qquad \delta_{\mathcal{K}} U = \mathcal{K} U$$

The gauge parameter $\mathcal{K}$ is

$$\mathcal{K} = \xi^B \mathcal{D}_B + \frac{1}{2} K^{bc} M_{bc}$$

The Wess-Zumino superspace geometry

To describe supergravity, the superspace torsion tensor must obey nontrivial constraints that generalize the torsion-free condition, $T_{ab}{}^c = 0$, in ordinary gravity.

The covariant derivatives obey the following anti-commutation relations:

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\} = -2\mathrm{i}\mathcal{D}_{\alpha\dot\alpha} ,$$

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = -4\bar{R} M_{\alpha\beta} , \qquad \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = 4R\bar{M}_{\dot\alpha\dot\beta} ,$$

$$\Big[\bar{\mathcal{D}}_{\dot\alpha}, \mathcal{D}_{\beta\dot\beta}\Big] = -\mathrm{i}\varepsilon_{\dot\alpha\dot\beta}\Big(R\,\mathcal{D}_\beta + G_\beta{}^{\dot\gamma}\bar{\mathcal{D}}_{\dot\gamma} - (\bar{\mathcal{D}}^{\dot\gamma} G_\beta{}^{\dot\delta})\bar{M}_{\dot\gamma\dot\delta} + 2W_\beta{}^{\gamma\delta} M_{\gamma\delta}\Big) \\ -\mathrm{i}(\mathcal{D}_\beta R)\bar{M}_{\dot\alpha\dot\beta} ,$$

Torsion tensors R, $G_a = \bar{G}_a$ and $W_{\alpha\beta\gamma} = W_{(\alpha\beta\gamma)}$ satisfy the Bianchi identities

$$\bar{\mathcal{D}}_{\dot\alpha} R = 0 , \qquad \bar{\mathcal{D}}_{\dot\alpha} W_{\alpha\beta\gamma} = 0 ,$$

$$\bar{\mathcal{D}}^{\dot\gamma} G_{\alpha\dot\gamma} = \mathcal{D}_\alpha R ,$$

$$\mathcal{D}^\gamma W_{\alpha\beta\gamma} = \mathrm{i}\,\mathcal{D}_{(\alpha}{}^{\dot\gamma} G_{\beta)\dot\gamma} .$$

The Wess-Zumino superspace geometry

Super-Weyl transformations

$$\delta_\sigma \mathcal{D}_\alpha = (\frac{1}{2}\sigma - \bar{\sigma})\mathcal{D}_\alpha - (\mathcal{D}^\beta \sigma)\, M_{\alpha\beta} \ ,$$
$$\delta_\sigma \bar{\mathcal{D}}_{\dot\alpha} = (\frac{1}{2}\bar{\sigma} - \sigma)\bar{\mathcal{D}}_{\dot\alpha} - (\bar{\mathcal{D}}^{\dot\beta}\bar{\sigma})\bar{M}_{\dot\alpha\dot\beta} \ ,$$
$$\delta_\sigma \mathcal{D}_{\alpha\dot\alpha} = -\frac{1}{2}(\sigma + \bar{\sigma})\mathcal{D}_{\alpha\dot\alpha} - \frac{\mathrm{i}}{2}(\bar{\mathcal{D}}_{\dot\alpha}\bar{\sigma})\mathcal{D}_\alpha - \frac{\mathrm{i}}{2}(\mathcal{D}_\alpha\sigma)\bar{\mathcal{D}}_{\dot\alpha} - (\mathcal{D}^\beta{}_{\dot\alpha}\sigma)M_{\alpha\beta} - (\mathcal{D}_\alpha{}^{\dot\beta}\bar{\sigma})\bar{M}_{\dot\alpha\dot\beta} \ ,$$

with the scalar parameter σ being covariantly chiral,

$$\bar{\mathcal{D}}_{\dot\alpha}\sigma = 0$$

Conformal isometries

A supervector field $\xi = \xi^B E_B$ on $\mathcal{M}^{4|4}$ is called conformal Killing if there exists a bivector $K^{bc} = -K^{cb}$ and a covariantly chiral scalar σ such that

$$(\delta_{\mathcal{K}} + \delta_\sigma)\mathcal{D}_A = 0 \ .$$

Implications (I):

$$\begin{aligned}
\delta_\alpha{}^\beta \bar\xi^{\dot\beta} &= \frac{\mathrm{i}}{4}\mathcal{D}_\alpha \xi^{\beta\dot\beta} \quad \Rightarrow \quad \bar\xi^{\dot\alpha} = \frac{\mathrm{i}}{8}\mathcal{D}_\alpha \xi^{\alpha\dot\alpha} \ , \\
K_{\alpha\beta} &= \mathcal{D}_{(\alpha}\xi_{\beta)} - \frac{\mathrm{i}}{2}\xi_{(\alpha}{}^{\dot\beta} G_{\beta)\dot\beta} \ , \\
\sigma &= \frac{1}{3}(\mathcal{D}^\alpha \xi_\alpha + 2\bar{\mathcal{D}}_{\dot\alpha}\bar\xi^{\dot\alpha} - \mathrm{i}\xi^a G_a) \ ,
\end{aligned}$$

Implications (II):

$$\begin{aligned}
\mathcal{D}_\alpha \bar\xi_{\dot\alpha} &= -\frac{\mathrm{i}}{2}\xi_{\alpha\dot\alpha}\bar R \ , \\
\bar{\mathcal{D}}_{\dot\alpha} K^{\beta\gamma} &= \mathrm{i}\xi_{\alpha\dot\alpha} W^{\alpha\beta\gamma} \ , \\
\mathcal{D}_\alpha K^{\beta\gamma} &= -\delta_\alpha{}^{(\beta}\mathcal{D}^{\gamma)}\sigma - 4\delta_\alpha{}^{(\beta}\xi^{\gamma)}\bar R + \frac{\mathrm{i}}{2}\delta_\alpha{}^{(\beta}\xi^{\gamma)\dot\gamma}\bar{\mathcal{D}}_{\dot\gamma}\bar R + \frac{\mathrm{i}}{2}\xi_{\alpha\dot\alpha}\mathcal{D}^{(\beta}G^{\gamma)\dot\alpha} \ .
\end{aligned}$$

The real vector ξ^a is the only independent parameter. It obeys the equation

$$\mathcal{D}_{(\alpha}\xi_{\beta)\dot\beta} = 0 \qquad (*)$$

which in fact contains all information about the conformal Killing supervector field. In particular, it implies the ordinary conformal Killing equation

$$\mathcal{D}_a\xi_b + \mathcal{D}_b\xi_a = \frac{1}{2}\eta_{ab}\mathcal{D}^c\xi_c$$

Alternative definition of the conformal Killing supervector field:
It is a real supervector field

$$\xi = \xi^A E_A~, \qquad \xi^A = \Big(\xi^a, -\frac{\mathrm{i}}{8}\bar{\mathcal{D}}_\beta\xi^{\alpha\dot\beta}, -\frac{\mathrm{i}}{8}\mathcal{D}^\beta\xi_{\beta\dot\alpha}\Big)$$

obeying the equation $(*)$.

Isometries

Each off-shell formulation for $\mathcal{N} = 1$ supergravity can be realized as a super-Weyl invariant coupling of the old minimal supergravity, discussed above, to a scalar compensator Ψ and its conjugate $\bar{\Psi}$ (if the compensator is complex) with a super-Weyl transformation of the form

$$\delta_\sigma \Psi = -(p\sigma + q\bar{\sigma})\Psi \ ,$$

where p and q are fixed parameters which are determined by the off-shell structure of Ψ. The compensator is assumed to be nowhere vanishing.

Killing supervector field

A supervector field $\xi = \xi^B E_B$ on $\mathcal{M}^{4|4}$ is called Killing if there exists a bivector $K^{bc} = -K^{cb}$ and a covariantly chiral scalar σ such that

$$(\delta_\mathcal{K} + \delta_\sigma)\mathcal{D}_A = 0 \ ,$$
$$(\delta_\mathcal{K} + \delta_\sigma)\Psi = 0$$

Supersymmetric backgrounds

Look for those curved backgrounds which admit some unbroken (conformal) supersymmetries. By definition, such a superspace possesses a (conformal) Killing supervector field ξ^A with the property

$$\xi^a| = 0 \ , \qquad \epsilon^\alpha := \xi^\alpha| \neq 0 \ ,$$

where $U|$ denotes the $\theta, \bar{\theta}$ independent part of a tensor superfield $U(z) = U(x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}})$,

$$U| := U|_{\theta^\mu = \bar{\theta}_{\dot{\mu}} = 0}$$

By definition, supersymmetric backgrounds are defined to have no covariant fermionic fields

$$\mathcal{D}_\alpha R| = 0 \ , \qquad \mathcal{D}_\alpha G_{\beta\dot{\beta}}| = 0 \ , \qquad W_{\alpha\beta\gamma}| = 0 \ .$$

As a result, the gravitino can be gauged away.
The remaining supergravity fields are ${e_a}^m$ and

$$R| = -\frac{1}{3} M \ , \qquad G_a| = -\frac{2}{3} b_a \ .$$

Conformal supersymmetry

$$2\nabla_a\epsilon_\beta - \frac{\mathrm{i}}{3}\Big\{(\sigma_a\bar{\zeta})_\beta + (\sigma_a\bar{\epsilon})_\beta M - 2(\sigma_{ac}\epsilon)_\beta b^c + 2b_a\epsilon_\beta\Big\} = 0$$

ϵ_β & $\bar{\epsilon}_{\dot\beta}$ Q-supersymmetry parameters
ζ_β & $\bar{\zeta}_{\dot\beta}$ S-supersymmetry parameters ($\zeta_\beta := \mathcal{D}_\beta\sigma|$).
Introducing a gauge-covariant derivative

$$\mathfrak{D}_a\epsilon_\beta := (\nabla_a - \frac{\mathrm{i}}{2}b_a)\epsilon_\beta$$

and expressing ζ_β & $\bar{\zeta}_{\dot\beta}$ in terms of ϵ_β & $\bar{\epsilon}_{\dot\beta}$ gives

$$\mathfrak{D}_a\epsilon_\beta + \frac{1}{4}(\sigma_a\tilde{\sigma}^c\mathfrak{D}_c\epsilon)_\beta = 0 \quad \Longleftrightarrow \quad \mathfrak{D}_{\alpha\dot\gamma}\epsilon_\beta + \mathfrak{D}_{\beta\dot\gamma}\epsilon_\alpha = 0$$

ϵ_β is a charged conformal Killing spinor

Given a commuting conformal Killing spinor ϵ_β,
the *null* vector $\mathcal{V}_{\beta\dot\beta} := \epsilon_\beta\bar{\epsilon}_{\dot\beta}$ is a conformal Killing vector field

$$\nabla_{(\alpha}{}^{(\dot\alpha}\mathcal{V}_{\beta)}{}^{\dot\beta)} = 0$$

Rigid supersymmetry

In the non-conformal case, $\bar{\zeta}^{\dot{\alpha}} = 0$ and the equation for unbroken rigid supersymmetry

$$2\nabla_a \epsilon_\beta - \frac{\mathrm{i}}{3}\Big\{(\sigma_a \bar{\epsilon})_\beta M - 2(\sigma_{ac}\epsilon)_\beta b^c + 2b_a \epsilon_\beta\Big\} = 0$$

Nontrivial consistency conditions on the backgrounds fields M & b_c

Rigid supersymmetry

Spacetimes admitting four supercharges

$$
\begin{aligned}
\nabla_a M &= 0 \ , \\
M \cdot b_c &= 0 \ , \\
\nabla_a b_c &= 0 \ , \\
C_{abcd} &= 0 \ ,
\end{aligned}
$$

with C_{abcd} the Weyl tensor. The Riemann tensor proves to be

$$
R_{abcd} = \frac{1}{9}\Big\{ b_c(b_a\eta_{bd} - b_b\eta_{ad}) \ - \ b_d(b_a\eta_{bc} - b_b\eta_{ac}) - b^2(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}) \Big\} \\
- \ \frac{1}{9} M\bar{M}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc})
$$

This gives the Ricci tensor

$$
R_{ab} = \frac{2}{9}(b_a b_b - b^2\eta_{ab}) - \frac{1}{3} M\bar{M}\eta_{ab}
$$

G. Festuccia and N. Seiberg (2011)