Integrable aspects of Yang-Baxter maps

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Introduction to Yang-Baxter maps

Let $\mathcal{X}$ be any set. A map $R : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$, $R : (x, y) \mapsto (u(x, y), v(x, y))$, that satisfies the *Yang-Baxter equation* :

$$R_{23} \circ R_{13} \circ R_{12} = R_{12} \circ R_{13} \circ R_{23}$$

is called *Yang-Baxter Map* . (Yang 67, Baxter 72, Drinfel'd 92)

$$\begin{aligned} R_{12}(x, y, z) &= (u(x, y), v(x, y), z), \\ R_{13}(x, y, z) &= (u(x, z), y, v(x, z)), \\ R_{23}(x, y, z) &= (x, u(y, z), v(y, z)), \end{aligned}$$

for $x, \ y, \ z \in \mathcal{X}$. The YB map R is called *quadrirational* if the maps

$$u(\cdot, y) : \mathcal{X} \to \mathcal{X} \text{ and } v(x, \cdot) : \mathcal{X} \to \mathcal{X}$$

are bijective rational maps.

A *parametric YB map* is a YB map:

$$R : ((x, \alpha), (y, \beta)) \mapsto ((u(x, \alpha, y, \beta), \alpha), (v(x, \alpha, y, \beta), \beta))$$

where $x,\ y \in \mathcal{X}$ and the parameters $\alpha, \beta \in \mathbb{C}^n$. We usually denote $R(x, \alpha, y, \beta)$ by $R_{\alpha,\beta}(x, y)$

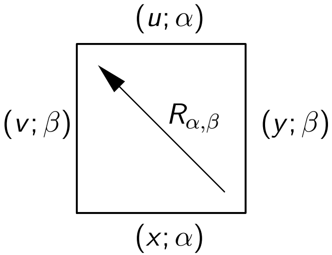
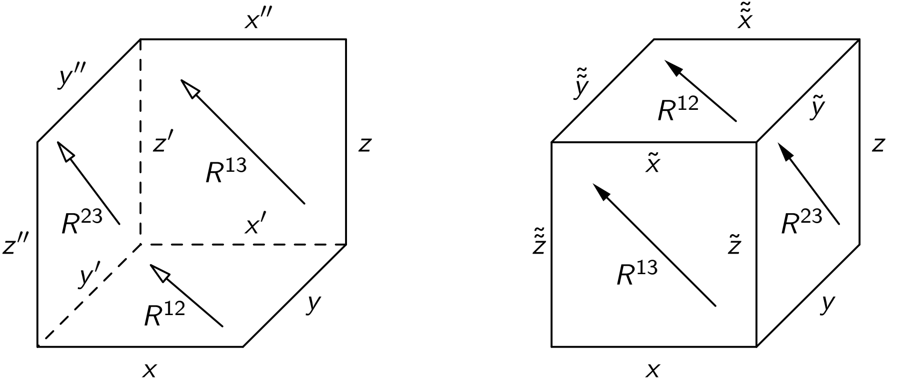


Figure : A map assigned to the edges of a quadrilateral



(i) $R_{23}R_{13}R_{12}(x, y, z) = (x'', y'', z'')$ (ii) $R_{12}R_{13}R_{23}(x, y, z) = (\tilde{\tilde{x}}, \tilde{\tilde{y}}, \tilde{\tilde{z}})$

Figure : Cubic representation of the Yang–Baxter property

The YB equation is equivalent to $x'' = \tilde{\tilde{x}}$, $y'' = \tilde{\tilde{y}}$ and $z'' = \tilde{\tilde{z}}$.

Lax matrices for Yang-Baxter maps

A Lax Matrix of a parametric YB map $R_{\alpha,\beta}(x,y) \mapsto (u,v)$ is a map $L : \mathcal{X} \times \mathbb{K} \to Mat_{n\times n}$ ($\mathbb{K} \subset \mathbb{C}$ or $\mathbb{R}$), such that

$$L(u,\alpha,\zeta)L(v,\beta,\zeta) = L(y,\beta,\zeta)L(x,\alpha,\zeta) \qquad (1)$$

Furthermore, if equation (1) is equivalent to $(u,v) = R_{\alpha,\beta}(x,y)$ then we will call $L(x,\alpha)$ *strong Lax matrix*.

Proposition

If $u = u_{\alpha,\beta}(x,y), v = v_{\alpha,\beta}(x,y)$ satisfy (1), for a matrix L and the equation

$$L(\hat{x},\alpha)L(\hat{y},\beta)L(\hat{z},\gamma) = L(x,\alpha)L(y,\beta)L(z,\gamma)$$

implies that $\hat{x} = x$, $\hat{y} = y$ and $\hat{z} = z$, for every $x, y, z \in \mathcal{X}$, then $R_{\alpha,\beta}(x,y) \mapsto (u,v)$ is a parametric YB map with Lax matrix L.

Adler's Map

The equation $L(u;\alpha)L(v;\beta) = L(y;\beta)L(x;\alpha)$, with

$$L(x;\alpha) = \begin{pmatrix} x & x^2+\alpha-\zeta \\ 1 & x \end{pmatrix},$$

admits the unique solution $u = y - \frac{\alpha-\beta}{x+y}$, $v = x - \frac{\beta-\alpha}{x+y}$. Furthermore,

$$L(\hat{x};\alpha)L(\hat{y};\beta)L(\hat{z};\gamma) = L(x;\alpha)L(y;\beta)L(z;\gamma)$$

implies $(\hat{x},\hat{y},\hat{z}) = (x,y,z)$. So the map

$$R_{\alpha,\beta}(x,y) = (y - \frac{\alpha-\beta}{x+y}, x - \frac{\beta-\alpha}{x+y})$$

is a YB map with (strong) Lax matrix $L(x;\alpha)$.

The standard periodic staircase initial value problem

$R_{\alpha,\beta}:(x,y)\mapsto(x',y')$ a YB map with Lax matrix $L(x_i;\alpha)$.

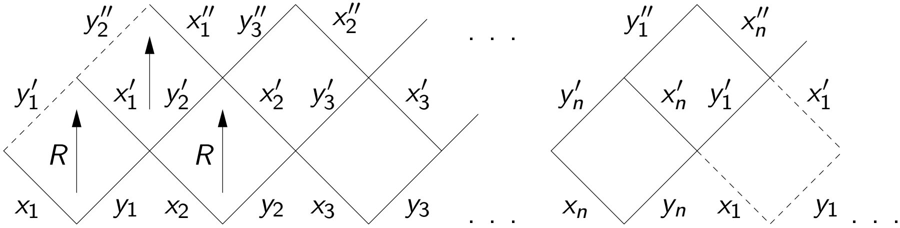


Figure : n-period mapping

Transfer map $T_n:(x_1,\ldots,x_n,y_1,\ldots,y_n)\mapsto(x_1',\ldots,x_n',y_2',y_3'\ldots,y_n',y_1')$

The k-transfer map: $T_n^k=\underbrace{T_n\circ\ldots\circ T_n}_{k}$.

$(T_n^n(x_1,\ldots,x_n,y_1,\ldots,y_n)=(x_1^{(n)},\ldots,x_n^{(n)},y_1^{(n)},\ldots,y_n^{(n)}))$

For any n-periodic initial value problem we define the *monodromy matrix* :

$$M_n(x_1, ..., x_n, y_1, ..., y_n) = \prod_{i=1}^{\overset{\curvearrowleft}{n}} L(y_i; \beta) L(x_i; \alpha)$$

Proposition

The transfer map preserves the spectrum of the monodromy matrix.

$$M_n(T_n(x_1, ..., x_n, y_1, ..., y_n)) L(y_1'; \beta_1) = L(y_1'; \beta_1) M_n(x_1, ..., x_n, y_1, ..., y_n).$$

We consider Adler's map

$$R_{\alpha,\beta}(x,y) = (y + \frac{\alpha-\beta}{x+y}, x - \frac{\alpha-\beta}{x+y})$$

with Lax matrix

$$L(x;\alpha) = \begin{pmatrix} x & x^2-\alpha-\zeta \\ 1 & x \end{pmatrix}.$$

$R_{\alpha,\beta} \circ R_{\alpha,\beta} = Id.$

• 2-periodic initial value problem

The transfer map :

$T_2(x_1, x_2, y_1, y_2) = (y_1 + \frac{\alpha_1 - \beta_1}{x_1 + y_1}, y_2 + \frac{\alpha_2 - \beta_2}{x_2 + y_2}, x_2 - \frac{\alpha_2 - \beta_2}{x_2 + y_2}, x_1 - \frac{\alpha_1 - \beta_1}{x_1 + y_1})$
Monodromy matrix :

$$M_2(x_1, x_2, y_1, y_2) = L(y_2; \beta_2)L(x_2; \alpha_2)L(y_1; \beta_1)L(x_2; \alpha_1),$$

From the trace of $M_2(x_1, x_2, y_1, y_2)$ we derive the integrals

$$\begin{aligned} J_1 &= -\alpha_2(x_1 + y_1)(x_1 + y_2) - \beta_1(x_1 + y_2)(x_2 + y_2) \\ &\quad +(x_2 + y_1)(x_2 + y_2)(x_1 + y_1)(x_1 + y_2) \\ &\quad -\alpha_1(x_2 + y_1) - \beta_2(x_1 + y_1), \\ J_2 &= (x_1 + x_2 + y_1 + y_2)^2. \end{aligned}$$

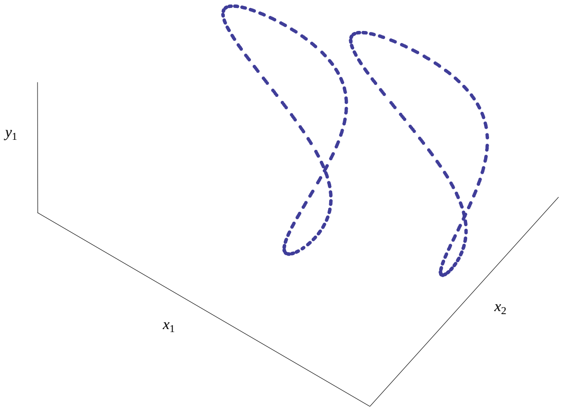


Figure : Projection of T_2 on $\mathbb{R}^3$

- 3-periodic initial value problem

The transfer map:

$$T_3(x_1, x_2, x_3, y_1, y_2, y_3) =$$

$$(y_1 + \frac{\alpha_1 - \beta_1}{x_1 + y_1}, y_2 + \frac{\alpha_2 - \beta_2}{x_2 + y_2}, y_3 + \frac{\alpha_3 - \beta_3}{x_3 + y_3}, x_2 - \frac{\alpha_2 - \beta_2}{x_2 + y_2}, x_3 - \frac{\alpha_3 - \beta_3}{x_3 + y_3}, x_1 - \frac{\alpha_1 - \beta_1}{x_1 + y_1}).$$

The Monodromy matrix :

$$M_3(x_1, x_2, x_3, y_1, y_2, y_3) =$$

$$L(y_3; \beta_3)L(x_3; \alpha_3)L(y_2; \beta_2)L(x_2; \alpha_2)L(y_1; \beta_1)L(x_2; \alpha_1),$$

From the trace of $M_3(x_1, x_2, x_3, y_1, y_2, y_3)$ we derive three functional independent first integrals.

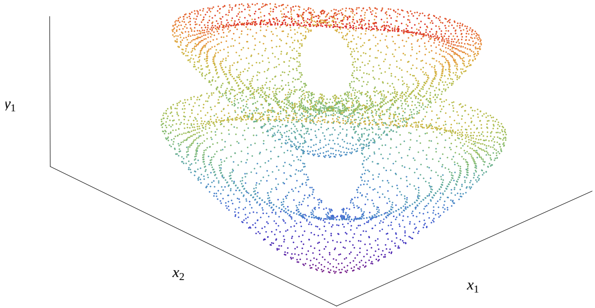


Figure : Projection of T_3 on $\mathbb{R}^3$

Poisson structure on polynomial Lax matrices

We denote by $\mathcal{L}_m^n$ the set of $m \times m$, $n-$degree polynomial matrices

$$L(\zeta) = X_0 + \zeta X_1 + \ldots + \zeta^n X_n, \quad X_i \in Mat_{m\times m}(\mathbb{K}), \ \zeta \in \mathbb{K}$$

The Sklyanin bracket:

$$\{L(\zeta) \overset{\otimes}{,} L(\eta)\} = [\frac{r}{\zeta - \eta}, L(\zeta) \otimes L(\eta)], \quad r(x \otimes y) = y \otimes x.$$

For $L(\zeta) = [a_{ij}(\zeta)] \in \mathcal{L}_m^n$

$$\{L(\zeta) \overset{\otimes}{,} L(\eta)\} = \begin{pmatrix} A_{11} & \ldots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \ldots & A_{mm} \end{pmatrix}, A_{ij} = \begin{pmatrix} \{a_{ij}(\zeta), a_{11}(\eta)\} & \ldots & \{a_{ij}(\zeta), a_{1m}(\eta)\} \\ \vdots & & \vdots \\ \{a_{ij}(\zeta), a_{2m}(\eta)\} & \ldots & \{a_{ij}(\zeta), a_{mm}(\eta)\} \end{pmatrix}$$

$$L(\zeta) \otimes L(\eta) = \begin{pmatrix} a_{11}(\zeta)L(\eta) & \ldots & a_{1m}(\zeta)L(\eta) \\ \vdots & & \vdots \\ a_{m1}(\zeta)L(\eta) & \ldots & a_{mm}(\zeta)L(\eta) \end{pmatrix}, \qquad dim\mathcal{L}_m^n = m^2(n+1)$$

For $L(\zeta) = X - \zeta A \in \mathcal{L}_2^1$, with

$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ the Poisson structure matrix is

$$J_A(X) = \begin{pmatrix} 0 & -x_2 a_1 + x_1 a_2 & x_3 a_1 - x_1 a_3 & x_3 a_2 - x_2 a_3 \\ * & 0 & x_4 a_1 - x_1 a_4 & x_4 a_2 - x_2 a_4 \\ * & * & 0 & -x_4 a_3 + x_3 a_4 \\ * & * & * & 0 \end{pmatrix}$$

with $J_A(X)_{ij} = \{x_i - \zeta a_i, x_j - \zeta a_j\}$ for $i, j = 1, ..., 4$.

Six Casimir functions on $\mathcal{L}^2$: $\quad a_i \quad i = 1, ..., 4$ and

$$f_0(X; A) = \det X, \;\; f_1(X; A) = a_4 x_1 - a_3 x_2 - a_2 x_3 + a_1 x_4,$$

i.e. the coefficients of

$\det L(\zeta) = f_2(X; A)\zeta^2 - f_1(X; A)\zeta + f_0(X; A)$ with $f_2(X; A) = \det A$.

$$L(\zeta) = X - \zeta A \in \mathcal{L}_2^1,\ X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix},\ A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix},\ detA \neq 0$$
$$f_0(X;A) = x_1x_4 - x_2x_3,\ \ f_1(X;A) = a_4x_1 - a_3x_2 - a_2x_3 + a_1x_4.$$

Let $L(x,\alpha) \in \mathcal{L}_2^1,\ x = (x_1, x_2),\ \alpha = (\alpha_1, \alpha_2)$ such that

$$f_0(L(x,\alpha);A) = \alpha_1,\ \ f_1(L(x,\alpha);A) = \alpha_2$$

Proposition

- *The equation* $L(u,\alpha)L(v,\beta) = L(y,\beta)L(x,\alpha)$*, admits a unique solution with respect to* $u = (u_1, u_2),\ v = (v_1, v_2)$.
- *The map* $R_{\alpha,\beta} : (x,y) \mapsto (u,v)$ *is a quadrirational YB map.*
- *The map* $R_{\alpha,\beta}$ *is Poisson with respect to the Sklyanin bracket.*

We consider $L(\zeta) = X - \zeta A$, with $A = \begin{pmatrix} \varepsilon & 1 \\ 0 & \varepsilon \end{pmatrix}$.

Casimir functions: $f_0(X) = \det X$, $f_1(X) = \varepsilon(x_{11} + x_{22}) - x_{21}$

We set $f_0(X) = \alpha$, $f_1(X) = 1$ and solve with respect to x_{12} and x_{21} to derive the strong Lax matrix

$$L(x_1, x_2, \alpha) = \begin{pmatrix} x_1 + \varepsilon\zeta & \frac{x_1 x_2 - \alpha}{\varepsilon(x_1 + x_2) - 1} + \zeta \\ \varepsilon(x_1 + x_2) - 1 & x_2 + \varepsilon\zeta \end{pmatrix},$$

The map $R_{\alpha,\beta} : (x_1, x_2, y_1, y_2) \mapsto (u_1, u_2, v_1, v_2)$ with u_i, v_i the unique solution of

$$L(u_1, u_2, \alpha) L(v_1, v_2, \beta) = L(y_1, y_2, \beta) L(x_1, x_2; \alpha)$$

is a symplectic YB map with respect to

$$\{x_1, x_2\} = -1 + \varepsilon(x_1 + x_2), \ \{y_1, y_2\} = -1 + \varepsilon(y_1 + y_2), \ \{x_i, y_j\} = 0.$$

Degenerate YB maps arising as limits of non-degenerate

In the previous example, for $\varepsilon \to 0$,

$$R_{\alpha,\beta}(x_1, x_2, y_1, y_2) = (y_1 + \frac{\alpha - \beta}{x_1 + y_2}, y_2, x_1, x_2 + \frac{\alpha - \beta}{x_1 + y_2})$$

symplectic YB map with respect to

$$\{x_1, x_2\} = -1,\ \{y_1, y_2\} = -1,\ \{x_i, y_j\} = 0$$

and (weak) Lax matrix

$$L(x_1, x_2; \alpha) = \begin{pmatrix} x_1 & \alpha - x_1 x_2 - \zeta \\ -1 & x_2 \end{pmatrix}.$$

Refactorization of $m \times m$ binomial matrices

Let A, B be invertible mxm and matrices such that $AB = BA$ and $L_1(\bar{x}, \bar{\alpha}), L_2(\bar{y}, \bar{\beta})$ generic elements of $\mathcal{C}^1_m(A, \bar{\alpha})$ and $\mathcal{C}^1_m(B, \bar{\beta})$. Then there is a unique map $R_{\bar{\alpha},\bar{\beta}} : (\bar{x}, \bar{y}) \mapsto (\bar{u}, \bar{v})$, such that

$$L_1(\bar{u}, \bar{\alpha})L_2(\bar{v}, \bar{\beta}) = L_2(\bar{y}, \bar{\beta})L_1(\bar{x}, \bar{\alpha}).$$

If $L_1 = L_2$, then $R_{\bar{\alpha},\bar{\beta}}$ is a parametric quadrirational YB map.

- Integrable mappings by considering initial value problems on lattices $(\{TrM_n(\bar{x}, \bar{y}, \zeta), TrM_n(\bar{x}, \bar{y}, \eta)\} = 0)$.
- For $m > 2$, $\mathcal{S}_{\bar{\alpha},\bar{\beta}}$ can be reduced to a lower dimensional symplectic map.
- Entwining Yang-Baxter maps.

Symplectic YB maps on $\mathcal{L}_3^1$

By restriction to four dimensional symplectic leaves of $\mathcal{L}_3^1$ with $A = B = I$, we derive

$$L(\bar{x}, \bar{\alpha}) = L(x_1, x_2, X_1, X_2; \alpha_1, \alpha_2) =$$

$$\begin{pmatrix} \alpha_1 + \alpha_2 - x_1 X_1 - \zeta & -X_1 x_2 & X_1 \\ -x_1 X_2 & \alpha_1 + \alpha_2 - x_2 X_2 - \zeta & X_2 \\ -x_1(x_1 X_1 + x_2 X_2 - 3\alpha_2) & -x_2(x_1 X_1 + x_2 X_2 - 3\alpha_2) & \alpha_1 - 2\alpha_2 + x_1 X_1 + x_2 X_2 - \zeta \end{pmatrix}$$

The equation

$$L(\bar{u}; \bar{\alpha}) L(\bar{v}; \bar{\beta}) = L(\bar{y}; \bar{\beta}) L(\bar{x}; \bar{\alpha})$$

admits a unique solution with respect to

$$\bar{u} = (u_1, u_1, U_1, U_2), \;\; \bar{v} = (v_1, v_2, V_1, V_2)$$

$$u_1 = y_1 - \frac{(\alpha_1 - 2\alpha_2 - \beta_1 + 2\beta_2)(x_1 - y_1)}{d}, \quad u_2 = y_2 - \frac{(\alpha_1 - 2\alpha_2 - \beta_1 + 2\beta_2)(x_2 - y_2)}{d}$$

$$v_1 = x_1 + \frac{(\alpha_1 + \alpha_2 - \beta_1 - \beta_2)(x_1 - y_1)}{d}, \quad v_2 = x_2 + \frac{(\alpha_1 + \alpha_2 - \beta_1 - \beta_2)(x_2 - y_2)}{d},$$

$$U_1 = \frac{x_2 X_1 + y_2 Y_1 - v_2(X_1 + Y_1)}{u_2 - v_2}, \quad U_2 = \frac{x_1 X_2 + y_1 Y_2 - v_1(X_2 + Y_2)}{u_1 - v_1},$$

$$V_1 = \frac{x_1 X_1 + y_1 Y_1 - u_1(X_1 + Y_1)}{v_1 - u_1}, \quad V_2 = \frac{x_2 X_2 + y_2 Y_2 - u_2(X_2 + Y_2)}{v_2 - u_2},$$

$$d = 2\alpha_2 - \alpha_1 + \beta_1 + \beta_1 + y_1 X_1 + y_2 X_2 - x_1 X_1 - x_2 X_2.$$

$$R_{\bar{\alpha},\bar{\beta}} : ((x_1, x_2, X_1, X_2), (y_1, y_2, Y_1, Y_2)) \mapsto ((u_1, u_2, U_1, U_2)(v_1, v_2, V_1, V_2)),$$

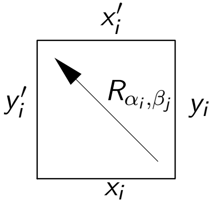
is a symplectic Yang-Baxter map with respect to

$$\omega = dx_1 \wedge dX_1 + dx_2 \wedge dX_2 + dy_1 \wedge dY_1 + dy_2 \wedge dY_2.$$

For any YB map $R_{\alpha,\beta}$, we consider $\tilde{S}_{i,j} : \mathcal{X}^n \times \mathcal{X}^n \to \mathcal{X}^n \times \mathcal{X}^n$,

$$\tilde{S}_{i,j}(x_1, ..., x_n, y_1, ..., y_n) = (x_1, ..., x_i', ...x_n, y_1, ..., y_j', ..., y_n),$$

with $(x_i', y_j') := R_{\alpha_i,\beta_j}(x_i, y_j)$



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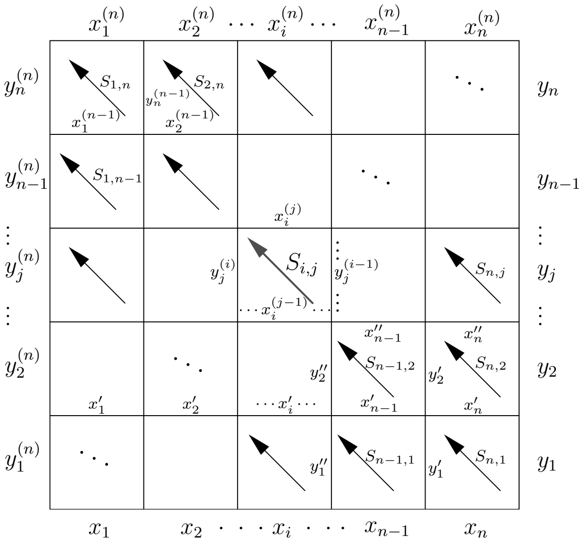
We also consider the map $\mathbf{R}^n_{\bar{\alpha},\bar{\beta}} : \mathcal{X}^n \times \mathcal{X}^n \to \mathcal{X}^n \times \mathcal{X}^n$, defined by

$$\mathbf{R}^n_{\bar{\alpha},\bar{\beta}} = \mathcal{S}_{2n-1} \circ \mathcal{S}_{2n-2} \circ ... \circ \mathcal{S}_2 \circ \mathcal{S}_1,$$

$$\mathcal{S}_k = \begin{cases} \circ_{i=1}^{k} \ \tilde{S}_{n-k+i,i} \ , & k = 1, 2, ..., n, \\ \circ_{i=1}^{2n-k} \ \tilde{S}_{i,i+k-n} \ , & k = n+1, ..., 2n-1 \end{cases}$$

The map $\mathbf{R}^n_{\bar{\alpha},\bar{\beta}}$



- The map $\mathbf{R}^n_{\bar{\alpha},\bar{\beta}}$ is a parametric YB map on $\mathcal{X}^n \times \mathcal{X}^n$ with parameters $\bar{\alpha} = (\alpha_1, ..., \alpha_n)$, $\bar{\beta} = (\beta_1, ..., \beta_n)$

- If $L(x, \alpha)$ is a Lax matrix of $R_{\alpha,\beta}$, then

$$\mathcal{L}(x_1, ..., x_n, \alpha_1, ..., \alpha_n) = L(x_n, \alpha_n)L(x_{n-1}, \alpha_{n-1})...L(x_1, \alpha_1)$$

  is a Lax matrix of $\mathbf{R}^n_{\bar{\alpha},\bar{\beta}}$

- If $R_{\alpha,\beta} : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a Poisson map then $\mathbf{R}^n_{\bar{\alpha},\bar{\beta}} : \mathcal{X}^n \times \mathcal{X}^n \to \mathcal{X}^n \times \mathcal{X}^n$ is a Poisson map

$$\{\mathcal{L}(x_1, ..., x_n, \zeta) \overset{\otimes}{,} \mathcal{L}(x_1, ..., x_n, \eta)\} = [\frac{r}{\zeta - \eta}, \mathcal{L}(x_1, ..., x_n, \zeta) \otimes \mathcal{L}(x_1, ..., x_n, \eta)]$$

Integrals are derived from the trace of the monodromy matrix

$$M(x_1, ..., x_n, y_1, ..., y_n) = \mathcal{L}(y_1, ..., y_n, \beta_1, ..., \beta_n)\mathcal{L}(x_1, ..., x_n, \alpha_1, ..., \alpha_n)$$

which are in involution with respect to the Sklyanin bracket.

We consider the YB map $R_{\alpha,\beta} : \mathbb{K}^4 \times \mathbb{K}^4 \to \mathbb{K}^4 \times \mathbb{K}^4$
$R_{\bar{\alpha},\bar{\beta}} : ((x_1, x_2, X_1, X_2), (y_1, y_2, Y_1, Y_2)) \mapsto ((u_1, u_2, U_1, U_2)(v_1, v_2, V_1, V_2))$,
with Lax matrix $L(\mathbf{x}, \alpha) = L(x_1, x_2, X_1, X_2, \alpha_1, \alpha_2)$

$$= \begin{pmatrix} \alpha_1 + \alpha_2 - x_1 X_1 - \zeta & -X_1 x_2 & X_1 \\ -x_1 X_2 & \alpha_1 + \alpha_2 - x_2 X_2 - \zeta & X_2 \\ -x_1(x_1 X_1 + x_2 X_2 - 3\alpha_2) & -x_2(x_1 X_1 + x_2 X_2 - 3\alpha_2) & \alpha_1 - 2\alpha_2 + x_1 X_1 + x_2 X_2 - \zeta \end{pmatrix}$$

For any $n \in \mathbb{N}$, the map $\mathbf{R}^n_{\bar{\alpha},\bar{\beta}} : \mathbb{K}^{4n} \times \mathbb{K}^{4n} \to \mathbb{K}^{4n} \times \mathbb{K}^{4n}$ is a Poisson YB map with respect to

$$\pi = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial X_1} + \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial Y_1} + ... + \frac{\partial}{\partial x_{2n}} \wedge \frac{\partial}{\partial X_{2n}} + \frac{\partial}{\partial y_{2n}} \wedge \frac{\partial}{\partial Y_{2n}}.$$

$\mathbf{R}^n_{\bar{\alpha},\bar{\beta}}$ is integrable for $n = 1, 2$.

Towards a general description-Classification

Classification of Lax matrices in five cases.

- Classification of the binomial 2×2 Lax matrices in five cases.

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Sklyanin conjecture

Any symplectic leaf of $\mathcal{L}_2^n$ can be decomposed (in a non-unique way) into a product of a constant matrix K and linear matrix polynomials.

Sklyanin (1999) Bäcklund transformations and Baxter's Q-operator

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Any symplectic leaf of $\mathcal{L}_2^n$ can be decomposed (in a non-unique way) into a product of a constant matrix K and linear matrix polynomials.

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- Thank You

$(n+m)m$ Casimir functions : The m^2 elements of the highest degree term X_n and mn functions f_0, $f_1, \ldots, f_{mn-1}$ defined as the coefficients of the polynomial

$$\det L(\zeta) = f_0(\bar{x}) + f_1(\bar{x})\zeta + \ldots + f_{mn-1}(\bar{x})\zeta^{mn-1} + f_{mn}(\bar{x})\zeta^{mn}$$

with $\bar{x} = (X_0, \ldots, X_n)$ and $f_{mn}(\bar{x}) = \det X_n$.

level sets:

$$\mathcal{C}_m^n(A, \bar{\alpha}) = \{L(\bar{x}, \bar{\alpha}) \in \mathcal{L}_m^n \ / \ f_0(\bar{x}) = \alpha_0, \ldots, f_{mn-1}(\bar{x}) = \alpha_{mn-1}, X_n = A\},$$

$\bar{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{mn-1})$, $A \in Mat_{m\times m}$. $\dim(\mathcal{C}_m^n(\bar{\alpha})) = mn(m-1)$.

Sklyanin bracket on $\mathcal{L}_m^n \times \mathcal{L}_m^n$, for any $(L_1(\zeta), L_2(\zeta)) \in \mathcal{L}_m^n \times \mathcal{L}_m^n$.

$$\{L_1(\zeta) \overset{\otimes}{,} L_1(\eta)\} = [\tfrac{r}{\zeta-\eta}, L_1(\zeta) \otimes L_1(\eta)],$$
$$\{L_2(\zeta) \overset{\otimes}{,} L_2(\eta)\} = [\tfrac{r}{\zeta-\eta}, L_2(\zeta) \otimes L_2(\eta)], \quad \{L_1(\zeta) \overset{\otimes}{,} L_2(\eta)\} = 0,$$

- We consider $\mathcal{L}_3^1$, $dim(\mathcal{L}_3^1) = 9$, three Casimir functions $f_0, \ f_1, \ f_2$. For $A = B = I$ we derive a Poisson YB map

$$R_{\bar{\alpha},\bar{\beta}} : \mathbb{K}^6 \times \mathbb{K}^6 \to \mathbb{K}^6 \times \mathbb{K}^6, \ \ \bar{\alpha} = (\alpha_0, \alpha_1, \alpha_2), \ \bar{\beta} = (\beta_0, \beta_1, \beta_2).$$

- $\mathcal{M}_3^1 = \{L(\zeta) \in \mathcal{L}_3^1 \ / \ rank(L(\zeta)) = 4\} \ \ dim(\mathcal{M}_l^3) = 6.$ Two functionally independent Casimir functions on $\mathcal{M}_3^1$

$$4f_0{f_2}^3 - {f_1}^2{f_2}^2 + 4{f_1}^3 - 18f_0f_1f_2 + 27{f_0}^2 = 0.$$

We consider

$$L(\bar{x}, \bar{\alpha}) \in \mathcal{S} = \{L(\zeta) \in \mathcal{M}_3^1 \ / \ f_i(L(\zeta)) = \alpha_i, \ \ i = 0, 1, 2\}$$

with $4\alpha_0{\alpha_2}^3 - {\alpha_1}^2{\alpha_2}^2 + 4{\alpha_1}^3 - 18\alpha_0\alpha_1\alpha_2 + 27{\alpha_0}^2 = 0$