On a refactorisation of the QRT map

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Contents

Reversible Yang-Bater maps

Let $\mathbb{X}$ any set and R a map from $\mathbb{X} \times \mathbb{X}$ to itself

$$R: \quad \mathbb{X} \times \mathbb{X} \ni (x, y) \to (X, Y) \in \mathbb{X} \times \mathbb{X}.$$

Let $R_{ij}: \ \mathbb{X}^n \to \mathbb{X}^n$ where $\mathbb{X}^n = \mathbb{X} \times \mathbb{X} \times \ldots \times \mathbb{X}$ the map that acts as R on the i, and j factors of $\mathbb{X}^n$ and as identity to the others.

Example (n=3 $\mathbb{X} = \mathbb{CP}^1$)

For $n = 3$ let $R: \ (X, Y) = (f(x, y), g(x, y))$, we have

$$\begin{aligned}
R_{12}: &\quad (X, Y, Z) = (f(x, y), g(x, y), z) \\
R_{13}: &\quad (X, Y, Z) = (f(x, z), y, g(x, z)) \\
R_{23}: &\quad (X, Y, Z) = (x, f(y, z), g(y, z)), \quad \text{also} \\
R_{21}: &\quad (X, Y, Z) = (g(y, x), f(y, x), z).
\end{aligned}$$

Note $R_{21} = P \circ R_{12} \circ P$ where $P: \ (X, Y) = (y, x)$.
The map R is called Yang-Baxter if

$$R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12}.$$

If in addition $R_{12} \circ R_{21} = Id$, R is a reversible Yang-Baxter map.

The F-list[2] of Quadrirational Yang-Baxter maps

$$X = \alpha y P, \quad Y = \beta x P, \quad P = \frac{(1-\beta)x + \beta - \alpha + (\alpha - 1)y}{\beta(1-\alpha)x + (\alpha - \beta)xy + \alpha(\beta - 1)y}, \qquad (F_I)$$

$$X = \frac{y}{\alpha} P, \quad Y = \frac{x}{\beta} P, \quad P = \frac{\alpha x - \beta y + \beta - \alpha}{x - y}, \qquad (F_{II})$$

$$X = \frac{y}{\alpha} P, \quad Y = \frac{x}{\beta} P, \quad P = \frac{\alpha x - \beta y}{x - y}, \qquad (F_{III})$$

$$X = yP, \quad Y = xP, \quad P = 1 + \frac{\beta - \alpha}{x - y}, \qquad (F_{IV})$$

$$X = y + P, \quad Y = x + P, \quad P = \frac{\alpha - \beta}{x - y}, \qquad (F_V)$$

[2]Adler, Bobenko, Suris 2004

The QRT Mapping

The QRT mapping is defined by the composition of two non-commuting involutions i_1, i_2, which both preserve the same biquadratic integral

$$I(x,y) = \frac{\mathbf{X}^T A_0 \mathbf{Y}}{\mathbf{X}^T A_1 \mathbf{Y}},$$

where $\mathbf{X}$, $\mathbf{Y}$ are vectors $\mathbf{X^T} = (x^2, x, 1)$, $\mathbf{Y} = (y^2, y, 1)^T$ and A_0, A_1 are two 3×3 matrices,

$$A_i = \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ \delta_i & \epsilon_i & \zeta_i \\ \kappa_i & \lambda_i & \mu_i \end{pmatrix}.$$

From $I(\tilde{x}, y) - I(x,y) = 0$ and $I(x, \overline{y}) - I(x,y) = 0$ we have

$$i_1 : \begin{cases} \tilde{x} = \dfrac{f_1(y) - f_2(y)x}{f_2(y) - f_3(y)x} \\ \tilde{y} = y \end{cases}, \quad i_2 : \begin{cases} \overline{x} = x \\ \overline{y} = \dfrac{g_1(x) - g_2(x)y}{g_2(x) - g_3(x)y} \end{cases},$$

where

$$(f_1(y), f_2(y), f_3(y))^T = (A_0 \mathbf{Y}) \times (A_1 \mathbf{Y}),$$

$$(g_1(x), g_2(x), g_3(x))^T = (A_0^T \mathbf{X}) \times (A_1^T \mathbf{X}).$$

The QRT[3] map preserves a linear pencil of biquadratics curves $h(x, y; t) = \mathbf{X}^T A_0 \mathbf{Y} - t\mathbf{X}^T A_1 \mathbf{Y} = 0$.

- **Base points of the linear pencil of curves** $B(x, y; t) = 0$
  The set of base points is the set of points (x, y) that are contained on all curves of the linear pencil of biquadratic curves $(B(x, y; t))$. Is given by

$$x = \frac{f_1(y)}{f_2(y)} = \frac{f_2(y)}{f_3(y)}, \quad \text{or} \quad y = \frac{g_1(x)}{g_2(x)} = \frac{g_2(x)}{g_3(x)}. \tag{1}$$

- **Singular points of the QRT map**
  Singular points of the QRT map are the points (x, y) where the QRT map is not defined (both numerators and denominators of the QRT map are zero of infinity). The singular points of the QRT map are exactly the base points of the linear pencil of bi-quadratic curves that the map bi-rationally preserves.

[3]Quispel, Roberts, Thompson 1988

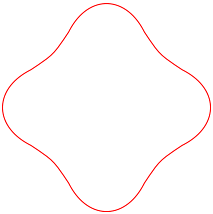
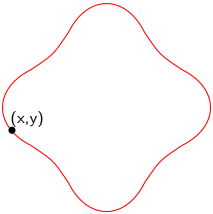


Figure: The QRT map[4]



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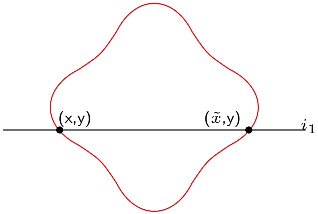

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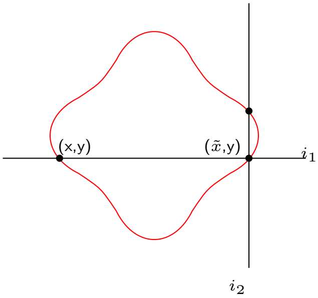

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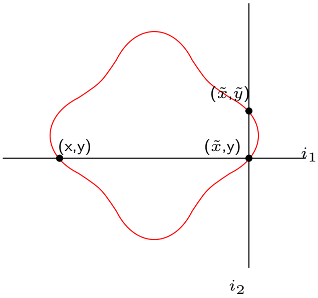

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[4]Tsuda 2004

Refactorisation of the QRT actions

For the integral $I(x,y) = \dfrac{\mathbf{X}^T A_0 \mathbf{Y}}{\mathbf{X}^T A_1 \mathbf{Y}}$, or equivalently for the pencil of bi-quadratic curves $B(x,y;t) := \mathbf{X}^T A_0 \mathbf{Y} - t\, \mathbf{X}^T A_1 \mathbf{Y}$ where $\mathbf{X}$, $\mathbf{Y}$ are vectors $\mathbf{X} = (x^2, x, 1)^T$, $\mathbf{Y} = (y^2, y, 1)^T$ and A_0, A_1 are two 3×3 matrices,

$$A_0 = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \zeta \\ \kappa & \lambda & \mu \end{pmatrix}, \quad A_1 = \begin{pmatrix} |A_0|_{33} & 2|A_0|_{32} & |A_0|_{31} \\ 2|A_0|_{23} & 4|A_0|_{22} & 2|A_0|_{21} \\ |A_0|_{13} & 2|A_0|_{12} & |A_0|_{11} \end{pmatrix} \quad (2)$$

where $|A_0|_{ij}$ the $(ij)-$minor determinant of the matrix A_0, we have:

- the QRT involutions i_1, i_2 which preserve $I(x,y)$,

- the primitive-QRT involutions $j_1,\ j_2,\ k_1,\ k_2$, which anti-preserve $I(x,y)$

$$j_1:\begin{cases} X=-\dfrac{2(\kappa y^2+\lambda y+\mu)+(\delta y^2+\epsilon y+\zeta)x}{\delta y^2+\epsilon y+\zeta+2(\alpha y^2+\beta y+\gamma)x}\\ Y=y\end{cases}$$
$$j_2:\begin{cases} X=-\dfrac{2(|A_0|_{13}y^2+|A_0|_{12}y+|A_0|_{11})+(|A_0|_{23}y^2+|A_0|_{22}y+|A_0|_{21})x}{|A_0|_{23}y^2+|A_0|_{22}y+|A_0|_{21}+2(|A_0|_{33}y^2+|A_0|_{32}y+|A_0|_{31})x}\\ Y=y\end{cases},$$

$$k_1:\begin{cases} X=x\\ Y=-\dfrac{2(\gamma x^2+\zeta x+\mu)+(\beta x^2+\epsilon x+\lambda)y}{\beta x^2+\epsilon x+\lambda+2(\alpha x^2+\delta y+\kappa)x}\end{cases}$$
$$k_2:\begin{cases} X=x\\ Y=-\dfrac{2(|A_0|_{31}x^2+|A_0|_{21}x+|A_0|_{11})+(|A_0|_{32}x^2+|A_0|_{22}x+|A_0|_{12})y}{|A_0|_{32}x^2+|A_0|_{22}x+|A_0|_{12}+2(|A_0|_{33}x^2+|A_0|_{23}x+|A_0|_{13})y}\end{cases},$$

Proof.

1. The solution of $I(X, y) - I(x, y) = 0$ and $I(x, Y) - I(x, y) = 0$, apart the trivial solutions $X = x$ and $Y = y$, gives respectively the QRT involutions i_1, i_2.
2. There is

$$I(x, y) = \frac{\mathbf{X}^T A_0 \mathbf{Y}}{\mathbf{X}^T A_1 \mathbf{Y}} = \frac{a(y)x^2 + b(y)x + c(y)}{a_1(y)x^2 + b_1(y)x + c_1(y)} = \frac{A(x)y^2 + B(x)y + C(x)}{A_1(x)y^2 + B_1(x)y + C_1(x)}.$$

The quadratic equation $I(X, y) + I(x, y) = 0$ has two rational solutions for X when its discriminant is a perfect square. This is true when

$$bb_1 = 2(a_1 c + a c_1).$$

Similarly for the equation $I(x, Y) + I(x, y) = 0$

$$BB_1 = 2(A_1 C + A C_1).$$

Quartic polynomials in y and x respectively. Equating their coefficients to zero essentially we arrive to 8 equations. A solution of these equations after an appropriate gauge is exactly the form of the matrix A_1.

From primitive involutions to QRT maps

For the primitive-QRT involutions $j_i,\ k_i$ $i = 1, 2$ the following holds:

- $j_1 \circ j_2 = i_1,\ \ k_1 \circ k_2 = i_2$, where i_1, i_2 the QRT involutions associated to the parameter matrices $A_0,\ A_1$ of (2). Hence the QRT mapping $\phi = i_2 \circ i_1 = k_1 \circ k_2 \circ i_1 \circ i_2$ refactorises as the product of the primitive-QRT involutions.

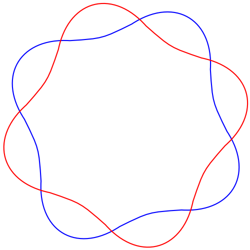

Figure: The QRT map as a composition of primitive involutions

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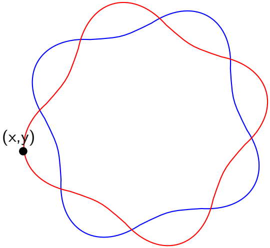

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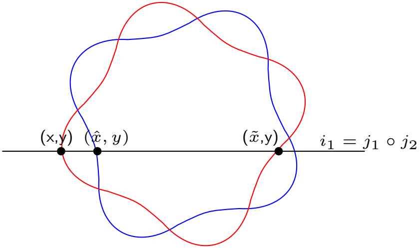

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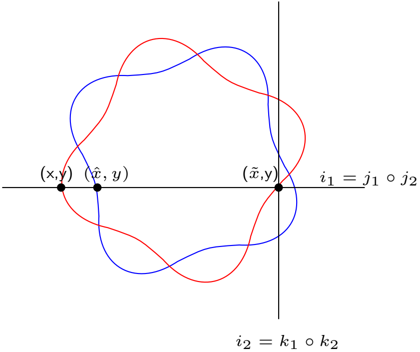

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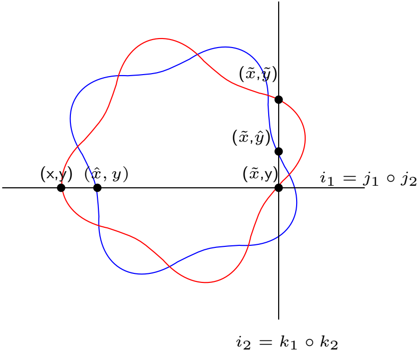

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- They form a group $G = \{j_1, j_2, k_1, k_2 | j_i^2 = k_i^2 = (j_1 \circ j_2)^2 = (k_1 \circ k_2)^2 = (j_i \circ k_{i+1})^2 = id,\ i = 1, 2\}$. There is $G \subset G'$, with $G' = \{J, R, S | J^2 = R^2 = S^2 = (R \circ S)^2 = (J \circ S)^4 = (R \circ S \circ R)^4 = id\}$, $G' \subset T(2, 4, \infty)$ where

$$J : (x, y; A_0) \mapsto \left(-\frac{2(\kappa y^2 + \lambda y + \mu) + (\delta y^2 + \epsilon y + \zeta)x}{\delta y^2 + \epsilon y + \zeta + 2(\alpha y^2 + \beta y + \gamma)x}, y; A_0 \right),$$

$$R : (x, y; A_0) \mapsto \left(y, x; A_0^T \right),$$

$$S : (x, y; A_0) \mapsto (y, x; A_2),$$

where

$$A_2 = \frac{1}{det(A_0)} \begin{pmatrix} 4|A_0|_{33} & 2|A_0|_{23} & 4|A_0|_{13} \\ 2|A_0|_{32} & |A_0|_{22} & 2|A_0|_{12} \\ 4|A_0|_{31} & 2|A_0|_{21} & 4|A_0|_{11} \end{pmatrix}.$$

- There is $j_1 = J$, $j_2 = R \circ S \circ J \circ R \circ S$, $k_1 = R \circ J \circ R$, $k_2 = S \circ J \circ S$.
- G' acts on the invariant I as

$$I \circ J = -I, \quad I \circ R = I \quad I \circ S = 1/I$$

Using the generators of the group G we can construct the following maps

$$\Upsilon_i = j_i \circ j_{i+1} \circ k_i, \;\; \Phi_i = k_i \circ k_{i+1} \circ j_i, \;\; \Psi_i = j_i \circ k_i \;\; \Omega_i = j_i \circ k_{i+1} \;\; i = 1, 2.$$

On the group G' the mappings above read

$$\begin{aligned}
&\Upsilon_1 = (J \circ R \circ S)^2 R \circ J \circ R, && \Phi_1 = (R \circ J \circ S)^2 \circ J,\\
&\Upsilon_2 = (J \circ R \circ S)^2 S \circ J \circ S, && \Phi_1 = (R \circ J \circ S)^2 \circ R \circ S \circ J \circ R \circ S,\\
&\Psi_1 = (J \circ R)^2, && \Omega_1 = (J \circ S)^2,\\
&\Psi_2 = (R \circ S \circ J \circ S)^2, && \Omega_2 = (R \circ S \circ J \circ R)^2.
\end{aligned}$$

- The maps $\Upsilon_i, \Phi_i, \Psi_i$, are of infinite order
- Mappings Ω_i are involutions and furthermore quadrirational maps.

Example (The mapping Ψ_1)

There is $\Psi_1 = (J \circ R)^2$, where $J \circ R$ reads:

$$J \circ R : (x, y; A_0) \mapsto \left(-\frac{2(\gamma x^2 + \zeta x + \mu) + (\beta x^2 + \epsilon x + \lambda)y}{\beta x^2 + \epsilon x + \lambda + 2(\alpha x^2 + \delta x + \kappa)y}, x; A_0^T \right).$$

For this map the parameters vary as $A_0 \mapsto A_0^T$, hence we can integrate to obtain that α, ϵ, μ are constants and

$$\begin{array}{ll} \beta = c_1(1 + (-1)^n) + c_2(1 - (-1)^n), & \delta = c_1(1 - (-1)^n) + c_2(1 + (-1)^n) \\ \gamma = c_3(1 + (-1)^n) + c_4(1 - (-1)^n), & \kappa = c_3(1 - (-1)^n) + c_4(1 + (-1)^n) \\ \lambda = c_5(1 + (-1)^n) + c_6(1 - (-1)^n), & \zeta = c_5(1 - (-1)^n) + c_6(1 + (-1)^n) \end{array},$$

where $c_i, i = 1 \ldots, 6$ constants as well. We have the non-autonomous 3-point map

$$y_{n+1} = -\frac{2(\gamma y_n^2 + \zeta y_n + \mu) + (\beta y_n^2 + \epsilon y_n + \lambda) y_{n-1}}{\beta y_n^2 + \epsilon y_n + \lambda + 2(\alpha y_n^2 + \delta y_n + \kappa) y_{n-1}}.$$

Setting $A_0 = A_0^T$, this map becomes autonomous and exactly the $(1,1)$ reduction of Viallet's Q_V integrable partial difference equation. Further specialisation of the parameters leads to the $(1,1)$ reduction of the Adler's $Q4$ partial difference equation.

Quadrirational maps

- A rational map $\mathbb{CP}^1 \times \mathbb{CP}^1 \ni (x,y) \to (X,Y) \in \mathbb{CP}^1 \times \mathbb{CP}^1$ is called quadrirational if the maps $(X,Y) \to (x,y)$, $(X,y) \to (x,Y)$, and $(x,Y) \to (X,y)$ are rational maps as well.

Mappings Ω_i are quadrirational and they explicitly read:

$$\Omega_1: \; j_1 \circ k_2: \begin{cases} X = -\dfrac{2(\kappa y^2+\lambda y+\mu)+(\delta y^2+\epsilon y+\zeta)x}{\delta y^2+\epsilon y+\zeta+2(\alpha y^2+\beta y+\gamma)x} \\ Y = -\dfrac{2(|A_0|_{31}x^2+|A_0|_{21}x+|A_0|_{11})+(|A_0|_{32}x^2+|A_0|_{22}x+|A_0|_{12})y}{|A_0|_{32}x^2+|A_0|_{22}x+|A_0|_{12}+2(|A_0|_{33}x^2+|A_0|_{23}x+|A_0|_{13})y} \end{cases};$$

$$\Omega_2: \; j_2 \circ k_1: \begin{cases} X = -\dfrac{2(|A_0|_{13}y^2+|A_0|_{12}y+|A_0|_{11})+(|A_0|_{23}y^2+|A_0|_{22}y+|A_0|_{21})x}{|A_0|_{23}y^2+|A_0|_{22}y+|A_0|_{21}+2(|A_0|_{33}y^2+|A_0|_{32}y+|A_0|_{31})x} \\ Y = -\dfrac{2(\gamma x^2+\zeta x+\mu)+(\beta x^2+\epsilon x+\lambda)y}{\beta x^2+\epsilon x+\lambda+2(\alpha x^2+\delta y+\kappa)y} \end{cases}$$

Note that

- $\Omega_1^2 = \Omega_2^2 = id$
- $\Omega_1 = j_1 \circ k_2 = J \circ S \circ J \circ S = (J \circ S)^2 = \omega_1^2$ and $\Omega_2 = j_2 \circ k_1 = R \circ S \circ J \circ R \circ S \circ R \circ J \circ R = (R \circ S \circ J \circ R)^2 = \omega_2^2$.
- There is $(\omega_1)^4 = (\omega_2)^4 = id$, so $\omega_1 = J \circ S$ and $\omega_2 = R \circ S \circ J \circ R$ are period 4 maps.

Example (The F_I Yang-Baxter map and its dual)

Choosing the parameter matrix A_0 as

$$A_0 = \begin{pmatrix} 0 & p-q & q(1-p) \\ 0 & 2p(q-1) & 0 \\ p(1-p) & p(p-q) & 0 \end{pmatrix},$$

The integral reads

$$I(x,y) = \frac{-qx^2 + pqx^2 - p^2y + pqy + 2pxy - 2pqxy - px^2y + qx^2y - py^2 + p^2y^2}{pqx - q^2x - qx^2 + q^2x^2 + 2qxy - 2pqxy - py^2 + pqy^2 + pxy^2 - qxy^2}$$

The map Ω_1 becomes exactly the (F_I) Yang-Baxter map, while its dual Ω_2 is a new map that we will denote it as $(\hat{F}_I)$

$$R_{xy}: \quad X = pyP, \quad Y = qxP, \quad P = \frac{(1-q)x + q - p + (p-1)y}{q(1-p)x + (p-q)xy + p(q-1)y}, \quad (F_I)$$

and the dual map $(\hat{F}_I)$

$$L_{xy}: \quad \begin{array}{l} X = y + W(x,y,p,q), \\ Y = x + W(y,x,q,p), \\ W(x,y,p,q) = \dfrac{(q-p)\left(q(x+y-2xy) + y^2(x+y-2)\right)}{q(p-q)2qx(q-1) + y\left(2q - 2pq + (p-q)y\right)} \end{array}$$

- (F_I) is a Yang-Baxter map but $(\hat{F}_I)$ is not. So (F_I) satisfies:

$$R_{ij}^2 = id, \quad (R_{x_y} \circ R_{xz} \circ R_{yz})^2 = id$$

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- Is there a common property that both maps share?

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- Is there a common property that both maps share?
- Yes both ω_i, *remember* $\Omega_i = (\omega_i)^2$, satisfy a modified Yang-Baxter relation

$$M_{ij}^4 = id, \quad (M_{x_y} \circ M_{xz} \circ M_{yz})^4 = id.$$

- For (F_I) there is

$$\omega_1 \equiv r_{xy}: \begin{array}{ll} X = qxW(y,x,q,p), & Y = x \\ P = q, & Q = p \\ W(x,y,p,q) = \frac{(1-q)x+q-p+(p-1)y}{q(1-p)x+(p-q)xy+p(q-1)y} & \end{array}$$

- For $(\hat{F}_I)$ there is

$$\omega_2 \equiv l_{xy}: \begin{array}{ll} X = x + W(y,x,q,p), & Y = x \\ P = q, & Q = p \\ W(x,y,p,q) = \frac{(q-p)\big(q(x+y-2xy)+y^2(x+y-2)\big)}{q(p-q)2qx(q-1)+y(2q-2pq+(p-q)y)} & \end{array}$$

	A_0	R_{xy}	r_{xy}
F_I	$\begin{pmatrix} 0 & p-q & q(1-p) \\ 0 & 2p(q-1) & 0 \\ p(1-p) & p(p-q) & 0 \end{pmatrix}$	$X = pyW(x,y,p,q),\ Y = qxW(x,y,p,q)$ $\quad W(x,y,p,q) = \frac{(1-q)x+q-p+(p-1)y}{q(1-p)x+(p-q)xy+p(q-1)y}$	$X = qxW(y,x,q,p),\quad Y = x$ $P = q,\quad Q = p$
F_{II}	$\begin{pmatrix} 0 & 0 & p \\ 0 & -2p & 0 \\ q & (p-q) & 0 \end{pmatrix}$	$X = y/pW(x,y,p,q),\ Y = x/qW(x,y,p,q)$ $\quad W(x,y,p,q) = \frac{px-qy+q-p}{x-y}$	$X = x/qW(y,x,q,p),\quad Y = x$ $P = q,\quad Q = p$
F_{III}	$\begin{pmatrix} 0 & 0 & p \\ 0 & -2p & 0 \\ q & 0 & 0 \end{pmatrix}$	$X = y/pW(x,y,p,q),\ Y = x/qW(x,y,p,q)$ $\quad W(x,y,p,q) = \frac{px-qy}{x-y}$	$X = x/qW(y,x,q,p),\quad Y = x$ $P = q,\quad Q = p$
F_{IV}	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ -1 & (p-q) & 0 \end{pmatrix}$	$X = yW(x,y,p,q),\ Y = xW(x,y,p,q)$ $\quad W(x,y,p,q) = 1 - \frac{p-q}{x-y}$	$X = xW(y,x,q,p),\quad Y = x$ $P = q,\quad Q = p$
F_V	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & q-p \end{pmatrix}$	$X = y + W(x,y,p,q),\ Y = x + W(x,y,p,q)$ $\quad W(x,y,p,q) = \frac{p-q}{x-y}$	$X = x + W(y,x,q,p),\quad Y = x$ $P = q,\quad Q = p$

- $R_{ij}^2 = id \qquad (R_{xy} \circ R_{xz} \circ R_{yz})^2 = id,$
- $r_{ij}^4 = id \qquad (r_{xy} \circ r_{xz} \circ r_{yz})^4 = id,$

	A_0	L_{xy}	l_{xy}
$\hat{F}_I$	$\begin{pmatrix} 0 & p-q & q(1-p) \\ 0 & 2p(q-1) & 0 \\ p(1-p) & p(p-q) & 0 \end{pmatrix}$	$X = y + W(x,y,p,q)$, $Y = x + W(y,x,q,p)$, $W(x,y,p,q) = \frac{(q-p)\left(q(x+y-2xy) + y^2(x+y-2)\right)}{q(p-q)2qx(q-1) + y\,(2q-2pq+(p-q)y)}$	$X = x + W(y,x,q,p)$, $Y = x$ $P = q$, $Q = p$
$\hat{F}_{II}$	$\begin{pmatrix} 0 & 0 & p \\ 0 & -2p & 0 \\ q & (p-q) & 0 \end{pmatrix}$	$X = y + W(x,y,p,q)$, $Y = x + W(x,y,p,q)$ $W(x,y,p,q) = \frac{(q-p)(x+y-2xy)}{p-q-2px+2qy}$	$X = x + W(y,x,q,p)$, $Y = x$ $P = q$, $Q = p$
$\hat{F}_{III}$	$\begin{pmatrix} 0 & 0 & p \\ 0 & -2p & 0 \\ q & 0 & 0 \end{pmatrix}$	$X = qyW(x,y,p,q)$, $Y = pxW(x,y,p,q)$ $W(x,y,p,q) = \frac{x-y}{px-qy}$	$X = pxW(y,x,q,p)$, $Y = x$ $P = q$, $Q = p$
$\hat{F}_{IV}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ -1 & (p-q) & 0 \end{pmatrix}$	$X = y + W(x,y,p,q)$, $Y = x + W(x,y,p,q)$ $W(x,y,p,q) = \frac{(q-p)(x+y)}{p-q-2x-2y}$	$X = x + W(y,x,q,p)$, $Y = x$ $P = q$, $Q = p$
$\hat{F}_V$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & q-p \end{pmatrix}$	$X = y + W(x,y,p,q)$, $Y = x + W(x,y,p,q)$ $W(x,y,p,q) = -\frac{p-q}{x-y}$	$X = x + W(y,x,q,p)$, $Y = x$ $P = q$, $Q = p$

- $L_{ij}^2 = id \quad (L_{xy} \circ L_{xz} \circ L_{yz})^2 \neq id,$
- $l_{ij}^4 = id \quad (l_{xy} \circ l_{xz} \circ l_{yz})^4 = id,$

- Modified entwining Yang-Baxter relations are satisfied as well eg.

$$r_{ij}^4 = l_{ij}^4 = id, \quad (l_{xy} \circ r_{xz} \circ l_{yz})^4 = id,$$

Conclusions

- Refactorisation of the QRT involutions
- Primitive involutions
  - QRT/non-QRT maps
  - Inner non-QRT maps. Recently Roberts produced outer non-QRT maps.
  - Quadrirational maps
  - After the refactorisation of each primitive involution to 2 consecutive ones, we arrive at a set of 8 involutions.