Generalized Verma and Wakimoto Modules

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- The level $\kappa_{\mathrm{crit}} = -\frac{1}{2}\kappa_{\mathrm{Kil}}$ is the *critical level*.

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  2. Smooth representations of affine Kac-Moody algebras play a central role in the Beilinson and Drinfeld's approach to the geometric Langlands program.

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- To answer these questions, we relate Verma modules to Wakimoto modules.

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- The critical shift is a “quantum correction" arising because of normal ordering of fields.

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