Superintegrability classical and quantum.

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Tremblay, Turbiner and Winternitz considered the classical Hamiltonian

$$H = p_r^2 + \frac{1}{r^2} p_\theta^2 + \alpha r^2 + \frac{\beta}{r^2 \cos^2 k\theta} + \frac{\gamma}{r^2 \sin^2 k\theta}$$

as well as the corresponding quantum problem with

$$\hat{H} = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \alpha r^2 + \frac{\beta}{r^2 \cos^2 k\theta} + \frac{\gamma}{r^2 \sin^2 k\theta}$$

(what did they show)

Classical Problem.

We can show that the classical Hamiltonian (for k rational) admits 3 functionally independent constants of the motion polynomial in the canonical momenta. There is already one constant

$$L_2 = p_\theta^2 + \frac{\beta}{\cos^2 k\theta} + \frac{\gamma}{\sin^2 k\theta}$$

as well as H.

To find an additional constant we construct functions $M(R, p_R)$ and $N(\theta, p_\theta)$ which satisfy $\{M, H\} = e^{-2R}$ and $\{N, H\} = e^{-2R}$ where $\{\,,\,\}$ is the Poisson bracket and $r = e^R$.
If we do this we obtain.

$$M = \frac{i}{4\sqrt{L_2}} B.$$

where

$$\sinh B = \frac{i(2L_2 e^{-2R} - H)}{\sqrt{H^2 - 4\alpha L_2}}$$

$$\cosh B = \frac{2\sqrt{L_2}\, e^{-2R} p_R}{\sqrt{H^2 - 4\alpha L_2}}$$

and.

$$N = \frac{-i}{4k\sqrt{L_2}} A$$

where

$$\sinh A = \frac{i(-\gamma + \beta - L_2 \cos(2k\theta))}{\sqrt{[(L_2 - \beta - \gamma)^2 - 4\beta\gamma]}}$$

$$\cosh A = \frac{\sqrt{L_2}\, \sin(2k\theta) p_\theta}{\sqrt{[(L_2 - \beta - \gamma)^2 - 4\beta\gamma]}}$$

From these relations we see that if $K = \frac{p}{q}$ (rational) then

$$\sinh\left(-4i p \sqrt{L_2}\,(N-M)\right)$$

and

$$\cosh\left(-4i p \sqrt{L_2}\,(N-M)\right)$$

are additional constants of the motion which are polynomial in the canonical momenta. (to within a possible factor $\sqrt{L_2}$)

Hence Superintegrability (classical).

Quantum Integrability for $\hat{H}$ has been recently proven using a "complicated" argument.

Let us look instead at the case of the caged anisotropic oscillator with classical Hamiltonian

$$H = p_x^2 + p_y^2 + \omega^2 (p^2 x^2 + q^2 y^2) + \frac{\alpha_1}{x^2} + \frac{\alpha_2}{y^2}$$

Similar methods to those given previously establish that H is classically superintegrable if p and q are integers. What about quantum superintegrability? We present an approach based on recurrence relations of the solutions.

To do this we write the quantum Hamiltonian as

$$H = \partial_x^2 + \partial_y^2 - \mu_1^2 x^2 - \mu_2^2 y^2 + \frac{\frac{1}{4} - a_1^2}{x^2} + \frac{\frac{1}{4} - a_2^2}{y^2}.$$

with suitable identifications

$\mu_1^2 = -p^2\omega^2$, $\mu_2^2 = -q^2\omega^2$, $\alpha_1 = \frac{1}{4} - a_1^2$ and $\alpha_2 = \frac{1}{4} - a_2^2$. We can find suitably general separable solutions of

$$H\Psi = E\Psi$$

by writing $\Psi = XY$

where
$$X = X_n = e^{-\frac{1}{2}\mu_1 x^2} x^{a_1+\frac{1}{2}} L_n^{a_1}(\mu_1 x^2)$$

and
$$Y = Y_m = e^{-\frac{1}{2}\mu_2 y^2} y^{a_2+\frac{1}{2}} L_m^{a_2}(\mu_2 y^2)$$

and $L_p^{\alpha}(z)$ is an associated Laguerre polynomial. We have in addition the eigenvalue equations.

$$\left(\partial_x^2 - \mu_1^2 x^2 + \frac{\frac{1}{4} - a_1^2}{x^2}\right) X_n = \lambda_1 X_n$$

$$\left(\partial_y^2 - \mu_2^2 y^2 + \frac{\frac{1}{4} - a_2^2}{y^2}\right) Y_m = \lambda_2 Y_m$$

where
$$\lambda_1 = -2\mu_1(2n + a_1 + 1)$$
$$\lambda_2 = -2\mu_2(2m + a_2 + 1)$$

and
$$E = -\lambda_1 - \lambda_2$$
$$= 2\mu(pn + qm + pa_1 + p + qa_2 + q)$$

with $\mu_1 = p\mu$, $\mu_2 = q\mu$

In order that E remain fixed $pn+qm$ must remain a constant. This is possible if $n \to n+q$, $m \to m-p$. when acting on $\Psi = X_n Y_m$. by using differential operators. this follows from the recurrence formulas

$$x \frac{d}{dx} L_p^{\alpha}(x) = p L_p^{\alpha}(x) - (p+\alpha) L_{p-1}^{\alpha}(x)$$

and

$$x \frac{d}{dx} L_p^{\alpha}(x) = (p+1) L_{p+1}^{\alpha}(x) - (p+1+\alpha-x) L_p^{\alpha}(x)$$

We can therefore raise and lower the indices m and n using differential operators. In particular we can perform the transformation $n \to n+q$, $m \to m-p$ that preserve E.

This can be seen from the formulas

$$D^+(\mu_1, x) X_n = (\partial_x^2 - 2x\mu_1\partial_x - \mu_1 + \mu_1^2 x^2 + \frac{\frac{1}{4} - a_1^2}{x^2}) X_n = -4\mu_1(n+1) X_{n+1}$$

$$D^-(\mu_2, y) Y_m = (\partial_y^2 + 2y\mu_2\partial_y + \mu_2 + \mu_2^2 y^2 + \frac{\frac{1}{4} - a_2^2}{y^2}) Y_m = -4\mu_2(m + a_2) Y_{m-1}$$

∴ If we consider $D^+(p\mu, x)^q \cdot D^-(q\mu, y)^p$ we have constructed a differential operator which performs the E preserving transformation

$$n \to n + q$$

$$m \to m - p.$$

Hence quantum superintegrability

What about Classical Models?

Consider $H = J_1^2 + J_2^2 + J_3^2 + \frac{\alpha}{z^2}$

defined on S_2. This is classically

super integrable.

$$A_1 = J_1^2 + \frac{\alpha}{2z^2}(1 + y^2 - x^2)$$

$$A_2 = J_1 J_2 - \frac{\alpha x y}{z^2}, \quad X = J_3.$$

There are "too many" constants.

We observe the relation

$$0 = A_1(H - A_1 - X^2) - A_2^2 - \frac{\alpha}{2}(X^2 + H) + \frac{\alpha^2}{4}$$

the Poisson algebra relations are

$$\{X, A_1\} = -2A_2$$

$$\{X, A_2\} = -H + X^2 + 2A_1$$

$$\{A_1, A_2\} = -X(2A_1 + \alpha)$$

Our question is "What does this imply for the Quantum Superintegrable case."

Indeed is this system quantum superintegrable.

It is not always this simple. As an example consider the quantum Hamiltonian on S_2.

$$H = \partial_\theta^2 + \cot\theta\,\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2 + \frac{\alpha}{\sin^2\theta\cos^2 K\phi}$$

with $K = \frac{p}{q}$ (rational). The corresponding classical Hamiltonian indicates classical superintegrability. What about the quantum case? If we put $\alpha = K^2\left(\frac{1}{4} - A^2\right)$ and $\psi = K\phi$ then we have solutions.

$$\Upsilon_0 = \Psi = P_n^{K(N+\frac{1}{2})}(\cos\theta)(\cos\psi)^{1/2} P_N^A(\sin\psi)$$

these solutions satisfy.

$$\left(\partial_\theta^2 + \cot\theta\,\partial_\theta - \frac{K^2(N+\frac{1}{2})^2}{\sin^2\theta}\right)\Theta$$

$$= -\underbrace{n(n+1)}_{=E}\,\Theta$$

$$L_2\Phi = \left(\partial_\phi^2 + \frac{K^2(\frac{1}{4}-A^2)}{\cos^2 K\phi}\right)\Phi = -K^2(N+\tfrac{1}{2})^2\Phi$$

To proceed further we note the recurrence relations.

$$D_\nu^+(x)P_\nu^\mu(x) = (1-x^2)\partial_x P_\nu^\mu(x) - (\nu+1)\,x\,P_\nu^\mu(x) = -(\nu-\mu+1)P_{\nu+1}^\mu(x)$$

$$D_\nu^-(x)P_\nu^\mu(x) = (1-x^2)\partial_x P_\nu^\mu(x) + \nu x P_\nu^\mu(x) = (\nu+\mu)P_{\nu-1}^\mu(x)$$

and.

$$C_\mu^+(x)P_\nu^\mu(x) = (1-x^2)^{1/2}\partial_x P_\nu^\mu(x) + \frac{\mu x}{(1-x^2)^{1/2}}\,P_\nu^\mu(x) = -P_\nu^{\mu+1}(x)$$

$$C_\mu^-(x)P_\nu^\mu(x) = (1-x^2)^{1/2}\partial_x P_\nu^\mu(x) - \frac{\mu x}{(1-x^2)^{1/2}}\,P_\nu^\mu(x) = (\mu+\nu)(\nu-\mu+1)P_\nu^{\mu-1}(x)$$

These relations enable ν and μ to be raised by ± 1 when acting on $P_\nu^\mu(x)$.

If we now let $N \to N+q$

then $$\Psi \to P_n^{K(N+\frac{1}{2})+p}(\cos\theta)(\cos\psi)^{1/2} \cdot P_{N+q}^{A}(\sin\psi)$$

If we take N to be arbitrary.

we can consider
$$\Upsilon_+ = P_n^{K(N+\frac{1}{2})+p}(x)\,P_{N+q}^{A}(y)$$

where $x=\cos\theta$, $y=\sin\psi$. This function can be obtained from Υ_0 via

$$\Delta_+\Upsilon_0 = C^+_{K(N+\frac{1}{2})+p-1}(x)$$
$$\ldots C^+_{K(N+\frac{1}{2})}(x)\,D^+_{N+q-1}(y)\ldots\ldots D^+_N(y)\Upsilon_0$$
$$\sim \Upsilon_+ = P_n^{K(N+\frac{1}{2})+p}(x)\,P_{N+q}^{A}(y)$$

Similarly we can consider $N \to N-q$

and obtain

$$\Delta_-\Upsilon_0 = C^-_{K(N+\frac{1}{2})-p+1}(x)\ldots\ldots C^-_{K(N+\frac{1}{2})}(x)$$
$$\cdot D^-_{N+1-q}(y)\ldots\ldots D^-_N(y)\Upsilon_0 \sim \Upsilon_-$$
$$= P_n^{K(N+\frac{1}{2})-p}(x)\,P_{N-q}^{A}(y)$$

we now look at

$$(\Delta_+ + \Delta_-)\, Y_0$$

This is seen to be a polynomial function of $N+\frac{1}{2}$ which is even.
This follows from the symmetries

$$C^+_\mu(x) = C^-_{-\mu}(x)$$

$$D^+_\nu(y) = D^-_{-\nu-1}(y)$$

We also know that

$$(N+\tfrac{1}{2})^2\,\Psi = \frac{1}{K^2}\left(\partial_\phi^2 + \frac{K^2(\frac{1}{4}-A^2)}{\sin^2 K\phi}\right)\Psi$$

$\therefore$ This system is quantum superintegrable.

A similar analysis works for the Hamiltonian introduced by Tremblay, Turbiner & Winternitz.

$$\hat{H} = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \alpha r^2 + \frac{\beta}{r^2\cos^2 K\theta} + \frac{\gamma}{r^2\sin^2 K\theta}$$

$$= \Delta + V$$

For the TTW system. typical solutions are (in polar coordinates)

$$\psi = e^{-\frac{\omega}{2}r^2}\, r^{K(2n+a+b+1)+\frac{1}{2}}$$

$$\times L_m^{K(2n+a+b+1)}(\omega r^2)\,(\sin K\phi)^{a+\frac{1}{2}}$$

$$\times (\cos K\phi)^{b+\frac{1}{2}}\, P_n^{ab}(\cos 2K\phi)$$

where. $\alpha = K^2(\frac{1}{4}-a^2)$, $\beta = K^2(\frac{1}{4}-b^2)$

$$= R_m^A(r)\, P_n^{ab}(x)$$

$x = \cos 2K\phi$. The energy is given by

$$E = -2\omega[2(m+nK+1) + (a+b+1)K]$$

and A is taken as

$$A = K(2n+a+b+1)$$

$L_m^\alpha(z)$ satisfies the equation of a Laguerre polynomial and $\left(\partial_\theta^2 + \frac{\alpha}{\sin^2 k\theta} + \frac{\beta}{\cos^2 k\theta}\right) \Theta$

$$= -k^2(2n+a+b+1)^2 \Theta$$

From the expression for E to maintain E we must fix m+nk. a transformation that does this is $n \to n+q, \quad m \to m-p.$

recall $k = \frac{p}{q}$. There is also the transformation

$$n \to n-q, \quad m \to m+p$$

functions $P_n^{ab}(x)$ could be Jacobi polynomials. Indeed.

$$J_n^+ P_n^{ab}(x) = (2n+a+b+2)\,\partial_x P_n^{ab}(x)$$
$$+ (n+a+b+1)\left(-(2n+a+b+2)x - (a-b)\right) P_n^{ab}(x)$$
$$= 2(n+1)(n+a+b+1) P_{n+1}^{ab}(x)$$

$$J_n^- P_n^{ab}(x)$$

$$= (2n+a+b)\partial_x P_n^{ab}(x) + n\left((2n+a+b)x - (a-b)\right)P_n^{ab}(x) = 2(n+a)(n+b)P_{n-1}^{ab}(x)$$

and the relations

$$K_{A,m}^+ R_m^A(r)$$

$$= \left(-2r(2n+a+b+2)\partial_r + 2(2n+a+b+1) + Er^2 - 2\lambda\right) R_m^A(r) \sim R_{m-1}^{A+2}(r)$$

$$K_{A,m}^- R_m^A(r)$$

$$= \left(2r(2n+a+b)\partial_r - 2(2n+a+b+1) + Er^2 - 2\lambda \right) R_m^A(r) \sim R_{m+1}^{A-2}(r)$$

we construct the two quantities

$$\Xi_+ = K^+_{A+2(p-1),\, m-(p-1)} \cdots \cdots K_{A,m}^+ J_{n+q-1}^+ \cdots J_n^+ R_m^A(r) P_n^{ab}(x)$$

$$\Xi_- = K^-_{A-2(p-1),\, m+p-1} \cdots\cdots$$

$$K^-_{A,m}\ J^-_{n-q+1} \cdots\cdots J^-_n\ R^A_m(r)\, P^{ab}_n(x)$$

We now form the operator

$$\Xi = \Xi_+ + \Xi_-$$

and recongise that this is a polynomial in n and symmetric under the interchange

$$n \rightarrow -n-a-b-1$$

It is therefore a polynomal in $(2n+a+b+1)^2$ and as a consequence a differential operator because

$$\lambda = -K^2(2n+a+b+1)^2$$

This argument establishes the quantum integrability of the TTW system.

One last example.

$$V = \alpha \frac{(x+iy)^6}{(x^2+y^2)^4}$$

In polar coordinates the Schrödinger equation has the form.

$$0 = \left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \frac{\alpha}{r^2} e^{6i\theta} - E\right)\Psi(r,\theta)$$

Typical solutions are

$$C_\Omega(\beta r)\, C_{\Omega/3}(\delta e^{3i\theta})$$

where $C_\Omega(z)$ is any solution of Bessel's equation, $E = -\beta^2$ and $\alpha = -9\delta^2$
Ω is determined from the θ separation equation.

$$\left(\partial_\theta^2 - 9\delta^2 e^{6i\theta} + \Omega^2\right)\Theta = 0$$

we constuct operators for which

$\Omega \to \Omega \pm 3$ and which are differential operators.

taking $w = \delta e^{3i\theta}$ we form.

$$\Phi_1 = \left(-\partial_r + \frac{\Omega+2}{r}\right)\left(-\partial_r + \frac{\Omega+1}{r}\right)$$

$$\cdot \left(-\partial_r + \frac{\Omega}{r}\right)\left(-\partial_w + \frac{\Omega}{3w}\right)\Psi_\Omega$$

$$\to \beta^3 \Psi_{\Omega+3}$$

and $\Phi_2 = \left(\partial_r + \frac{\Omega-2}{r}\right)\left(\partial_r + \frac{\Omega-1}{r}\right)$

$$\cdot \left(\partial_r + \frac{\Omega}{r}\right)\left(\partial_w + \frac{\Omega}{3w}\right)\Psi_\Omega$$

$$\to \beta^3 \Psi_{\Omega-3}$$

from this $\Phi = \Phi_1 + \Phi_2$ is an even function of Ω and hence a differential operator. (Properties)

I In each of our examples the differential operator commutes with H

II Quantum superintegrability is thus established.

This potential is classically superintegrable. Indeed,

$$K_1 = (p_x - i p_y)^3 + \cdots$$

$$K_2 = (x p_y - y p_x)(p_x - i p_y)^3 + \cdots$$

$$K_3 = (x p_y - y p_x)^2 + \cdots$$

The Poisson algebra relations are

$$\{K_1, K_2\}_{P.B} = 3i K_1^2$$

$$\{K_1, K_3\}_{P.B} = 6i K_2$$

$$\{K_2, K_3\}_{P.B} = 6i K_1 (K_3 + a)$$

together with the constraint

$$K_1^2 K_3 - K_2^2 + a(K_1^2 - H^2) = 0$$

If we choose a variety of possible one variable classical models.

(1) $K_3 = c, \quad K_1 = -\sqrt{\dfrac{aE^3}{a+c}} \cos(6\sqrt{c+a}\,\beta)$

$$K_2 = -i\sqrt{aE^3} \sin(6\sqrt{c+a}\,\beta)$$

(2) $K_1 = c, \quad K_2 = 3ic^2\beta$

$$K_3 = -8c^2\beta^2 + \frac{aE^3}{c^2} - a$$

(3) $K_2 = c, \quad K_1 = \dfrac{i}{3\beta}, \quad K_3 = -9(c^2 + aE^3)\beta^2 - a$

which suggests there are quantum analogoes which are difference operatorss (1) and differential operators (2) & (3).

The Quantum algebra is

$$[K_1, K_2] = 3iK_1^2, \quad [K_1, K_3] = 6iK_2 - 9K_1$$

$$[K_2, K_3] = 3i\{K_1, K_2\} + i(27 + 6a)K_1 + 9K_2$$

with the constraint.

$$\frac{1}{2}\{K_1, K_1, K_3\} - 3K_2^2 - \frac{9}{2}i\{K_1, K_2\} + \left(\frac{63}{2} + 3a\right)K_1^2 - 3aH^3 = 0$$

a one dimensional model of this algebra is

$$K_1 = \frac{-i}{3x}, \quad K_2 = \partial_x$$

$$K_3 = -9x^2\partial_x^2 - 27x\partial_x - (9 + a + 9aE^3x^2)$$

we can generalise these ideas to

$$V = a\frac{(x+iy)^{k-1}}{(x-iy)^{k-1}}$$

where k is <u>rational</u>.

The system we consider is in fact quantum superintegrable.

If we again write

$$\hat{H} = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2 + \frac{\alpha}{z^2}.$$

we obtain the quantum symmetries.

$$\hat{A}_1 = \hat{J}_1^2 + \frac{\alpha}{2z^2}(1+y^2-x^2)$$

$$\hat{A}_2 = \frac{1}{2}(\hat{J}_1\hat{J}_2 + \hat{J}_2\hat{J}_1) - \frac{\alpha xy}{z^2}$$

$$\hat{J}_3 = \hat{X}.$$

the quantum quadratic algebra is then.

$$[\hat{X},\hat{A}_1] = -2\hat{A}_2$$

$$[\hat{X},\hat{A}_2] = \hat{X}^2 + 2\hat{A}_1 - \hat{H}$$

$$[\hat{A}_1,\hat{A}_2] = -(\hat{A}_1\hat{X} + \hat{X}\hat{A}_1) - \left(\frac{1}{2}+\alpha\right)\hat{X}$$

these quantum symmetries are not all independent.

$$\frac{1}{3}(\hat{X}^2\hat{A}_1 + \hat{X}\hat{A}_1 X + \hat{A}_1\hat{X}^2) + A_1^2 + A_2^2$$

$$-\left(\frac{3}{2}\alpha + \frac{11}{12}\right)\hat{X}^2 + \hat{H}\left(-\hat{A}_1 + \frac{\alpha}{2} - \frac{1}{6}\right)$$

$$-\frac{2}{3}\hat{A}_1 - \frac{\alpha}{2}\left(\frac{\alpha}{2}+1\right) = 0$$

The basic problem of representation theory is to 'realise' this algebra in a variety of 'irreducible' ways.

One solution is

$$\tilde{A}_1 = t(t+1)^2\partial_t^2 + ((2-\alpha-m)t^2 + 2(1-m)t + \alpha - m)\partial_t + m(\alpha-1)t + \alpha(m+\tfrac{1}{2}) - (m+\tfrac{1}{2})$$

$$\tilde{A}_2 = it(1-t^2)\partial_t^2 + i[(\alpha+m-2)t^2 + (\alpha-m)]\partial_t - im(\alpha-1)t$$

$$X = i(t\partial_t - m),$$
$$H = \tfrac{1}{4} - (m+1-\alpha)^2$$

How is such a realisation arrived at ?

We can model the classical Poisson algebra by taking

$$X = c$$

$$A_1 = \frac{1}{2}(E - c^2) + \frac{1}{2}\{[c^2 - (E+\alpha)^2]\sin 2\beta + 2i\alpha\cos 2\beta]$$

$$A_2 = \frac{1}{2}[(c^2 - (E+\alpha)^2)\cos 2\beta - 2i\sqrt{\alpha}\sin 2\beta]$$

By considering $c \to \partial_c$, $\beta \to -c$ we can expect upon quantisation the realisation in terms of differential operators. If we instead consider

$$c \to c, \quad \beta \to \partial_c$$

we expect a realisation in terms of difference operators as

$$e^{a\partial_c} f(c) = f(c+a)$$

In fact we can obtain

$$A_1 = t^2 - \frac{\alpha}{2}, \quad X = h(t)T_i + m(t)T_{-i}$$

$$A_2 = -\frac{i}{2}(i + 2t)h(t)T_i + \frac{i}{2}(-i + 2t)T_i$$

$$T_\alpha f(t) = f(t+\alpha)$$

a convenient choice of functions $h(t)$, and $m(t)$ is

$$h(t) = i\,\frac{\left(\frac{1}{2} - a - it\right)\left(\mu + a - \frac{1}{2} - it\right)}{2t}$$

$$m(t) = -i\,\frac{\left(\frac{1}{2} - a + it\right)\left(\mu + a - \frac{1}{2} + it\right)}{2t}$$

where $\alpha = \frac{1}{4} - a^2$, $E = \frac{1}{4} - (\mu - 1 + a)^2$.

$$H = J^2 + \frac{\alpha}{Z^2}$$

$$L_E = A_1 + (e_1 - e_2) A_2 + X^2 = c.$$

$$L_E = c, \qquad A_2 = (e_1 - e_2) \sin(\sqrt{a_1} \beta, k)$$