Mixing time of the Swendsen-Wang process on the complete graph

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Collaborators

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Probability on Graphs

- Many problems in statistical mechanics are of the form:
  - Consider a sequence of finite graphs $G_n = (V_n, E_n)$ with:
    - $G_n \subset G_{n+1}$ and $|V_{n+1}| > |V_n|$
    - E.g. complete graphs K_n, or tori $\mathbb{Z}_n^d$
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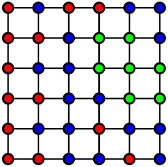

- $\Omega = [q]^V$ for fixed $q \in \{2, 3, 4 \ldots\}$
- $\pi(\sigma) = \dfrac{1}{Z} e^{-\beta H(\sigma)}$ for $\sigma \in \Omega$
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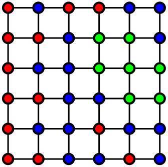

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    - $\beta = 1/$temperature
  - If $\beta \approx 0$ then $\pi(\cdot) \approx$ uniform on Ω ("Disorder")
  - If $\beta \gg 1$ preference for $u \sim v$ to have $\sigma_u = \sigma_v$ ("Order")
  - Phase transition between order and disorder at critical β_c

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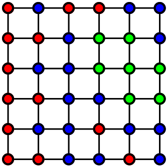

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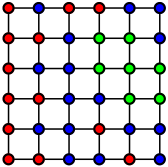

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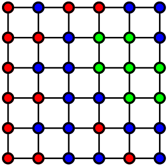

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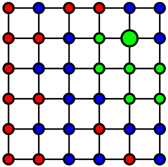

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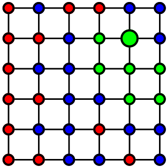

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$$\pi(\sigma'_v | \{\sigma_u\}_{u \in V \setminus v}) = \frac{e^{\beta \# \{u \sim v : \sigma'_v = \sigma_u\}}}{\sum_{\sigma_v \in [q]} e^{\beta \# \{u \sim v : \sigma_v = \sigma_u\}}}$$

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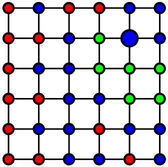

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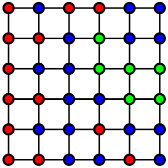

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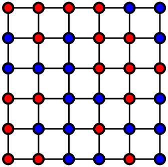
  - If $t_{\text{mix}} = O(\text{poly}(\log|\Omega|))$ we have **rapid mixing**
  - Otherwise, we have **torpid mixing**

Swendsen-Wang process

- Irreducible aperiodic Markov chain on $[q]^V$
- Stationary distribution is q-state Potts model



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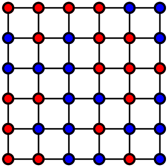

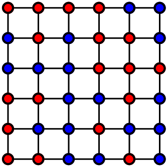

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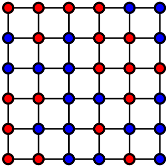

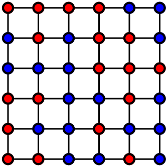

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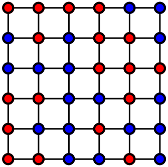

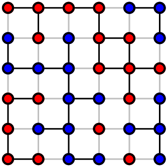

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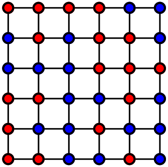

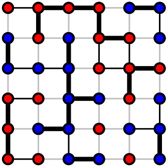

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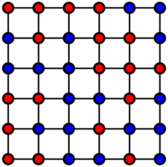

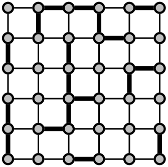

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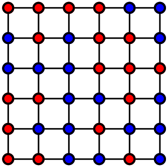

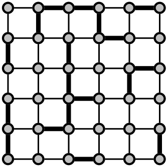

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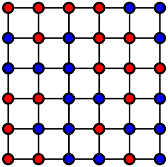

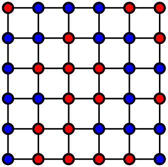

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  - Discontinuous transition for $q \geq 3$
- Potts energy depends only on magnetization $s(\sigma)$

$$-\beta H(\sigma) = \frac{\lambda}{2}\, n\, s(\sigma) \cdot s(\sigma) + \text{ constant}$$

- $s^i(\sigma) = |\sigma^{-1}(i)|/n =$ fraction of vertices coloured $i \in [q]$ by σ
- Let $\mathrm{SW}_n(\lambda, q) =$ SW process on K_n with parameters λ and q

  Given $\sigma_t \in [q]^n$, $\mathrm{SW}_n(\lambda, q)$ chooses σ_{t+1} as follows:
  - Independently for each $i \in [q]$ choose Erdös-Renyi graph $\mathcal{G}(\sigma_t^{-1}(i), \lambda/n)$. Let $A_{t+1} = \cup_{i \in [q]} \mathcal{G}(\sigma_t^{-1}(i), \lambda/n)$.
  - Independently and uniformly q-colour each component of (V, A_{t+1})

Note: edge probability in $\mathcal{G}(\sigma_t^{-1}(i), \lambda/n)$ is $\lambda/n = s^i(\sigma_t)\lambda/|\sigma_t^{-1}(i)|$

Rapid mixing for $q = 2$

Potts model on K_n has **continuous** phase transition when $q = 2$

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Theorem (Cooper, Dyer, Frieze & Rue 2000)

If $q = 2$ *then* $\mathrm{SW}_n(\lambda, q)$ *has mixing time*

$$t_{\mathrm{mix}} = O(\sqrt{n})$$

for all $\lambda \notin (\lambda_{\mathrm{c}} - \delta, \lambda_{\mathrm{c}} + \delta)$ *with* $\delta\sqrt{\log n} \to \infty$ *as* $n \to \infty$.

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Theorem (Long, Nachmias, Ning, & Peres 2012)

If $q = 2$ *then* $\mathrm{SW}_n(\lambda, q)$ *has mixing time*

$$t_{\mathrm{mix}} = \begin{cases} \Theta(1) & \lambda < \lambda_c \\ \Theta(n^{1/4}) & \lambda = \lambda_c \\ \Theta(\log n) & \lambda > \lambda_c \end{cases}$$

- **Ray, Tamayo, & Klein** (1989) conjectured $n^{1/4}$ at λ_c

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Theorem (Cuff, Ding, Louidor, Lubetzky, Peres, Sly 2012)

If $q \geq 3$ then the single-site Glauber process for the Potts model has

$$t_{\text{mix}} = \begin{cases} \Theta(n \log n) & \lambda < \lambda_o(q) \\ \Theta(n^{4/3}) & \lambda = \lambda_o(q) \\ \exp(\Omega(n)) & \lambda > \lambda_o(q) \end{cases}$$

*where $\lambda_o(q) < \lambda_c(q)$, so torpid mixing begins **before** transition*

Magnetization distribution

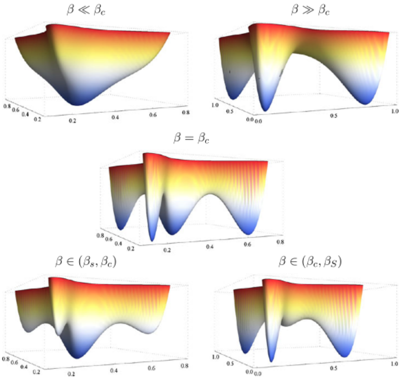

Figure: From Cuff *et. al* 2012

Large n distribution of $s(\sigma)$ known explicitly:

$$-\frac{1}{n}\log \mathbb{P}(s(\sigma)=a) \sim \phi_\lambda(a) - \inf_{a\in\Delta^{q-1}} \phi_\lambda(a)$$

$$\phi_\lambda(a) = \sum_{i=1}^{q}\left(a_i \log a_i - \frac{1}{2}\lambda a_i^2\right)$$

Minima of ϕ_λ correspond either to:

- **disordered** state: $s^i = 1/q$ for all $i \in [q]$
- **ordered** states: $s^i = \alpha > 1/q$ and $s^j = \frac{1-\alpha}{q-1}$ for $j \neq i$

$$\lambda_{\text{o}}(q) := \inf\{\lambda \geq 0 : \text{there exist ordered local minima of } \phi_\lambda\},$$
$$\lambda_{\text{d}}(q) := \sup\{\lambda \geq 0 : \text{the disordered state locally minimizes } \phi_\lambda\}.$$

Complete picture for $\mathrm{SW}_n(\lambda, q)$ with $q \geq 3$

Theorem (Lin & G. 2013)

If $q \geq 3$ *then* $\mathrm{SW}_n(\lambda, q)$ *has mixing time*

$$t_{\mathrm{mix}} = \begin{cases} \Theta(1) & \lambda < \lambda_{\mathrm{o}}(q) \\ \Theta(n^{1/3}) & \lambda = \lambda_{\mathrm{o}}(q) \\ \exp(\Omega(\sqrt{n})) & \lambda_{\mathrm{o}}(q) < \lambda < \lambda_{\mathrm{d}}(q) \\ \Theta(\log(n)) & \lambda \geq \lambda_{\mathrm{d}}(q) \end{cases}$$

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- Gore & Jerrum's torpid mixing result extends to a non-trivial interval $(\lambda_{\mathrm{o}}(q), \lambda_{\mathrm{d}}(q))$ containing $\lambda_{\mathrm{c}}(q)$
- Nothing special happens at $\lambda_{\mathrm{c}}(q)$
- Non-trivial scaling arises at $\lambda_{\mathrm{o}}(q)$
- Low and high temperature same as Ising case

Sketch of Proof

- If $Y_{t+1} := s^1_{t+1} - \mathbb{E}[s^1_{t+1}|\sigma_t]$ then

$$s^1_{t+1} \approx s^1_t + D(s^1_t) + Y_{t+1} \qquad (*)$$

  where

$$D_{\lambda,q}(x) := \theta(\lambda x)(1 - 1/q)x + 1/q - x$$

  - $\theta(\lambda)\, n = \mathbb{E}$(size of giant component) in Erdös-Renyi $\mathcal{G}(n, \lambda/n)$
  - $(Y_t)_{t\geq 0}$ is a sequence of martingale increments
  - $\mathrm{var}(Y_t|\sigma_t) = \Theta(n^{-1})$

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- Coupling arguments reduce mixing time to hitting time of s^1_t

Swendsen-Wang drift

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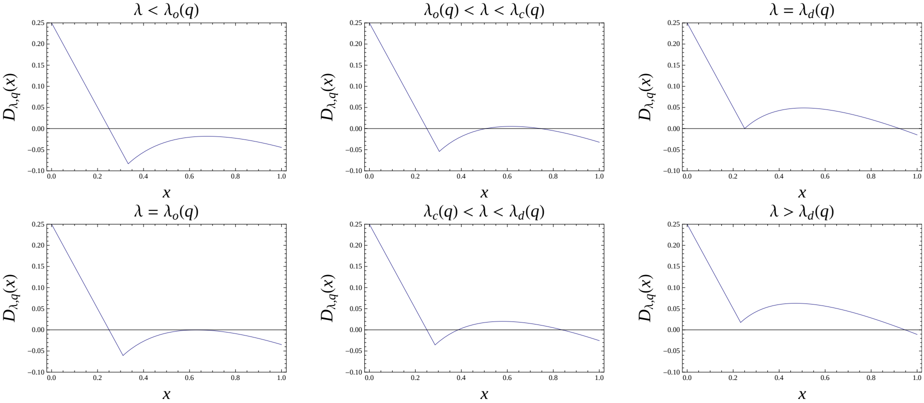

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Discussion

- Our hitting-time estimates for s_t^1 explain exponent values in mixing times for several other Potts/Ising processes
  - Mixing time exponents depend on:
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- Jon Machta (private communication) has conjectured mixing time asymptotics at λ_c for all real $q > 1$ for complete graph Chayes-Machta chain. Can this be proved?
- Can one say anything for the Glauber chain for the Fortuin-Kasteleyn model?