Some inter-relations between random matrix ensembles

Peter Forrester,

M&S, University of Melbourne

Outline

- Superpositions and decimations
- Averages of characteristic polynomials
- Structure function
- Moments and resolvent

MATHEMATICS | PHYSICS

Log-Gases and Random Matrices

Peter J. Forrester

Random matrix theory, both as an application and as a theory, has evolved rapidly over the past fifteen years. *Log-Gases and Random Matrices* gives a comprehensive account of these developments, emphasizing log-gases as a physical picture and heuristic, as well as covering topics such as beta ensembles and Jack polynomials.

Peter Forrester presents an encyclopedic development of log-gases and random matrices viewed as examples of integrable or exactly solvable systems. Forrester develops not only the application and theory of Gaussian and circular ensembles of classical random matrix theory, but also of the Laguerre and Jacobi ensembles, and their beta extensions. Prominence is given to the computation of a multitude of Jacobians; determinantal point processes and orthogonal polynomials of one variable; the Selberg integral, Jack polynomials, and generalized hypergeometric functions; Painlevé transcendents; macroscopic electrostatistics and asymptotic formulas; nonintersecting paths and models in statistical mechanics; and applications of random matrix theory. This is the first textbook development of both nonsymmetric and symmetric Jack polynomial theory, as well as the connection between Selberg integral theory and beta ensembles. The author provides hundreds of guided exercises and linked topics, making *Log-Gases and Random Matrices* an indispensable reference work, as well as a learning resource for all students and researchers in the field.

Peter J. Forrester is professor of mathematics at the University of Melbourne.

"Encyclopedic in scope, this book achieves an excellent balance between the theoretical and physical approaches to the subject. It coherently leads the reader from first-principle definitions, through a combination of physical and mathematical arguments, to the full derivation of many fundamental results. The vast amount of material and impeccable choice of topics make it an invaluable reference."

—**Eduardo Dueñez,** University of Texas, San Antonio

"This self-contained treatment starts from the basics and leads to the 'high end' of the subject. Forrester often gives new derivations of old results that beginners will find helpful, and the coverage of comprehensive topics will be useful to practitioners in the field."

—**Boris Khoruzhenko,** Queen Mary, University of London

LONDON MATHEMATICAL SOCIETY MONOGRAPHS

Martin Bridson and Peter Sarnak, Series Editors

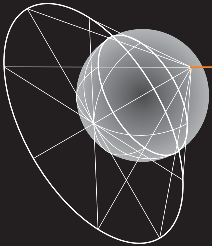
PRINCETON
press.princeton.edu

ISBN: 978-0-691-12829-0
90000
9 780691 128290

Forrester — Log-Gases and Random Matrices

34

PRINCETON



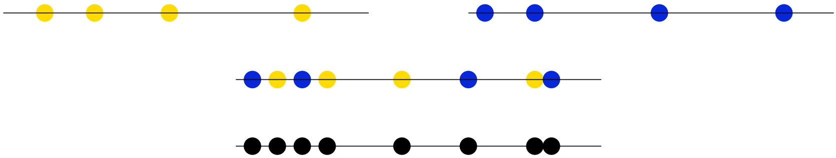
Log-Gases and Random Matrices

Peter J. Forrester

LONDON MATHEMATICAL SOCIETY MONOGRAPHS

Superimposed spectra

$$H = \begin{bmatrix} H_1 & \\ & H_2 \end{bmatrix}$$



$$p(x_1, \ldots, x_N) = \frac{1}{C} \prod_{l=1}^{N} e^{-x_l^2/2} \prod_{1 \le j < k \le N} |x_k - x_j|$$

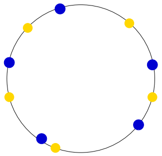
Superimposed spectra (cont)

Label the $2N$ points $x_1 < x_2 < \cdots < x_{2N}$. Must compute

$$\sum_{\substack{S\subset\{1,\ldots,2N\}\\|S|=N}} p(x_S)p(x_{\{1,\ldots,2N\}-S})$$

With $\Delta(\theta_S) = \prod_{1\leq j<k\leq N} \sin((\theta_{s_k} - \theta_{s_j})/2)$ it was proved by Gunson that

$$\sum_{\substack{S\subset\{1,\ldots,2N\}\\|S|=N}} \Delta(\theta_S)\Delta(\theta_{\{1,\ldots,2N\}-S}) = 2^N\Delta(\theta_{\{1,3,\ldots,2N-1\}})\Delta(\theta_{\{2,4,\ldots,2N\}})$$



Superimposed spectra (cont)

Suggests that the distribution of every second eigenvalue is special. Integrate $\{\theta_2, \theta_2, \ldots, \theta_{2N}\}$ over the region

$$R_N = \theta_1 < \theta_2 < \theta_3 < \theta_4 < \cdots < \theta_{2N-1} < \theta_{2N} < 2\pi + \theta_1$$

Using the Vandermonde identity

$$\prod_{1 \leq j < k \leq N} (x_j - x_k) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_N & x_{N-1} & x_{N-2} & \cdots & x_1 \\ x_N^2 & x_{N-1}^2 & x_{N-2}^2 & \cdots & x_1^2 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ x_N^{N-1} & x_{N-1}^{N-1} & x_{N-2}^{N-1} & \cdots & x_1^{N-1} \end{vmatrix}$$

can compute

$$\int_{R_N} d\theta_2 \cdots d\theta_{2N} \, \Delta(\theta_{\{2,4,\ldots,2N\}}) \propto \Delta(\theta_{\{1,3,\ldots,2N-1\}})$$

Dyson (1962) ex-conjecture

Let alt denote the operation of integration over every second eigenvalue.
Let $\cup$ denote the operation of random superposition.
We have

$$\mathrm{alt}\left(\mathrm{COE}_N \cup \mathrm{COE}_N\right) = \mathrm{CUE}_N$$

Consequence for gap probabilities

Let $E_N^{\mathrm{ME}}(0, J)$ denote the probability that there are no eigenvalues in the interval J of the matrix ensemble ME consisting of N eigenvalues. We have

$$E_N^{\mathrm{CUE}}(0; (-\theta, \theta)) =$$

$$E_N^{\mathrm{COE}}(0; (-\theta, \theta))\Big(E_N^{\mathrm{COE}}(0; (-\theta, \theta)) + E_N^{\mathrm{COE}}(1; (-\theta, \theta))\Big)$$

F & Rains (2001) (cont)

Question: For matrix ensembles with orthogonal symmetry, eigenvalue PDF of the form

$$\frac{1}{C_N}\prod_{l=1}^{N} f(x_l) \prod_{1\le j<k\le N} |x_k - x_j| =: \mathrm{OE}_N(f)$$

for what choices of f does

$$\mathrm{even}\left(\mathrm{OE}_N(f) \cup \mathrm{OE}_{N+1}(f)\right) = \mathrm{UE}_N(g)$$

for some g?

Must first obtain a Gunson type identity

$$\sum_{\substack{S\subset\{1,\dots,2N+1\}\\ |S|=N}} \Delta(x_S)\Delta(x_{\{1,\dots,2N+1\}-S}) = 2^N \Delta(x_{\{1,3,\dots,2N+1\}})\Delta(x_{\{2,4,\dots,2N\}})$$

where $\Delta(x_S) = \prod_{1\le j<k\le N}(x_{s_k} - x_{s_j})$.

F & Rains (2001) (cont)

Answer: (up to linear fractional transformation) the four classical weight functions:

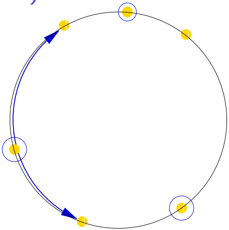
$$f(x) = \begin{cases} e^{-x^2/2}, & \text{Gaussian} \\ x^{(a-1)/2} e^{-x/2} \ (x>0), & \text{Laguerre} \\ (1-x)^{(a-1)/2}(1+x)^{(b-1)/2} \ (-1<x<1), & \text{Jacobi} \\ (1+ix)^{-(\alpha+1)/2}(1-ix)^{-(\bar\alpha+1)/2}, & \text{Cauchy} \end{cases}$$

$$g(x) = \begin{cases} e^{-x^2}, & \text{Gaussian} \\ x^{a} e^{-x} \ (x>0), & \text{Laguerre} \\ (1-x)^{a}(1+x)^{b} \ (-1<x<1), & \text{Jacobi} \\ (1+ix)^{-\alpha}(1-ix)^{-\bar\alpha}, & \text{Cauchy} \end{cases}$$

In particular

$$\text{even}\left(\mathrm{GOE}_{N+1} \cup \mathrm{GOE}_N\right) = \mathrm{GUE}_N$$

Mehta and Dyson (1963)



Using direct integration, showed

$$\mathrm{alt}\,(\mathrm{COE}_{2N}) = \mathrm{CSE}_N$$

Consequence for gap probabilities

We have

$$\begin{aligned} E_N^{\mathrm{CSE}}&(0;(-\theta,\theta)) \\ &= E_{2N}^{\mathrm{COE}}(0;(-\theta,\theta)) + \frac{1}{2}E_{2N}^{\mathrm{COE}}(1;(-\theta,\theta)) \\ &= \frac{1}{2}\Big(E_{2N}^{\mathrm{COE}}(0;(-\theta,\theta)) + \frac{E_{2N}^{\mathrm{CUE}}(0;(-\theta,\theta))}{E_{2N}^{\mathrm{COE}}(0;(-\theta,\theta))}\Big) \end{aligned}$$

Further new question:

For what choice of f does

$$\mathrm{even}\left(\mathrm{OE}_{2N+1}(f)\right) = \mathrm{SE}_N(g)$$

for some g?
Answer (FR 2001)

$$\mathrm{even}\left(\mathrm{OE}_{2N+1}(f)\right) = \mathrm{SE}_N((g/f)^2) \Leftrightarrow$$
$$\mathrm{even}\left(\mathrm{OE}_N(f) \cup \mathrm{OE}_{N+1}(f)\right) = \mathrm{UE}_N(g)$$

In particular, with $(f, g) = (e^{-x^2/2}, e^{-x^2})$

$$\mathrm{even}\left(\mathrm{GOE}_{2N+1}\right) = \mathrm{GSE}_N$$

A family of decimation relations (inspired by Bálint Virág)

Denote by $\mathrm{ME}_{\beta,N}(g(x))$ the PDF proportional to

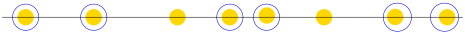
$$\prod_{l=1}^{N} g(x_l) \prod_{1\le j<k\le N} |x_k - x_j|^{\beta}$$

and let D_r denote the distribution of every r-th eigenvalue.
For the Gaussian case we have (F. 2009)

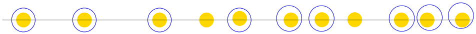
$$\mathrm{D}_{r+1}(\mathrm{ME}_{2/(r+1),(r+1)N+r}(e^{-x^2})) = \mathrm{ME}_{2(r+1),N}(e^{-(r+1)x^2})$$

e.g.

$$D_3(\mathrm{ME}_{2/3,3N+2}(e^{-x^2}) = \mathrm{ME}_{6,N}(e^{-3x^2})$$



$$D_4(\mathrm{ME}_{1/2,4N+3}(e^{-x^2}) = \mathrm{ME}_{8,N}(e^{-4x^2})$$



Consequences for asymptotic spacing distributions

Let $p_\beta^{\rm bulk,sp.}(n;s)$ denote the probability that in the bulk scaling limit there are n eigenvalues between 2 eigenvalues separated by distance s.
The decimation relations imply that for large s

$$E_{2/(r+1)}^{\rm bulk}((r+1)k+r;(r+1)s) \sim E_{2(r+1)}^{\rm bulk}(k;s).$$

A conjecture of Dyson, and of Fogler and Shklovskii (1995),

$$\log E_\beta^{\rm bulk}(n;(0,s)) \underset{s\to\infty}{\sim} -\beta\frac{(\pi s)^2}{16} + \Big(\beta n + \frac{\beta}{2} - 1\Big)\frac{\pi s}{2} + \Big\{\frac{n}{2}\Big(1 - \frac{\beta}{2} - \frac{\beta n}{2}\Big) + \frac{1}{4}\Big(\frac{\beta}{2} + \frac{2}{\beta} - 3\Big)\Big\}\log s$$

has this property.

Averages of characteristic polynomials

For the Gaussian β ensemble (Baker & F 1997)

$$\Big\langle \prod_{j=1}^{N} (c - \sqrt{\alpha} y_j)^n \Big\rangle_{\mathrm{ME}_{2/\alpha,N}(e^{-y^2})} = \Big\langle \prod_{j=1}^{n} (c - i y_j)^N \Big\rangle_{\mathrm{ME}_{2\alpha,n}(e^{-y^2})}.$$

Consequences

- The simplest case is $n = 1$. It tells us that the average of the characteristic polynomial for the Gaussian β ensemble is proportional to the Hermite polynomial $H_N(c)$.
- Suppose β is even. Then setting $n = \beta$ the LHS multiplied by $e^{-c^2/\alpha}$ is proportional to the eigenvalue density at $c/\sqrt{\alpha}$. Hence, for even β, this can be expressed as a β dimensional integral.
- Large N asymptotic analysis using the saddle point method gives oscillatory corrections to the Wigner semi-circle law, and the scaled density at the edge.

Explicit form of the scaled density at the edge

We have (Desrosiers & F (2006))

$$\lim_{N\to\infty} \frac{1}{\sqrt{2}N^{1/6}} \rho_{(1)}\Big(\sqrt{2N} + \frac{x}{\sqrt{2}N^{1/6}}\Big) = \frac{\Gamma(1+\beta/2)}{2\pi}\Big(\frac{4\pi}{\beta}\Big)^{\beta/2} \prod_{j=1}^{\beta} \frac{\Gamma(1+2/\beta)}{\Gamma(1+2j/\beta)} K_{\beta,\beta}(x),$$

where

$$K_{n,\beta}(x) := -\frac{1}{(2\pi i)^n} \int_{-i\infty}^{i\infty} dv_1 \cdots \int_{-i\infty}^{i\infty} dv_n \prod_{j=1}^{n} e^{v_j^3/3 - x v_j} \prod_{1 \le k < l \le n} |v_k - v_l|^{4/\beta}.$$

Asymptotics of the edge density

$$\rho_{(1)}^{\mathrm{soft},\beta}(x) \underset{x\to\infty}{\sim} \frac{1}{\pi}\frac{\Gamma(1+\beta/2)}{(4\beta)^{\beta/2}}\frac{e^{-2\beta x^{3/2}/3}}{x^{3\beta/4-1/2}} + \mathrm{O}\Big(\frac{1}{x^{3\beta/4+1}}\Big),$$

$$\rho_{(1)}^{\mathrm{soft},\beta}(x) \underset{x\to-\infty}{\sim} \frac{\sqrt{|x|}}{\pi} - \frac{\Gamma(1+\beta/2)}{2^{6/\beta-1}|x|^{3/\beta-1/2}}\cos\left(\frac{4}{3}|x|^{3/2} - \frac{\pi}{2}\Big(1-\frac{2}{\beta}\Big)\right)$$

This has consequence to the asymptotics of the right tail of the scaled distribution of the largest eigenvalue:

$$p_\beta^{\mathrm{soft}}(X) \underset{X\to\infty}{\sim} \rho_{(1)}^{\mathrm{soft},\beta}(X).$$

Averages of characteristic polynomials — circular ensemble

Let $\alpha = 2/\beta - 1$ and $\mu \in \mathbb{Z}^+$. We have

$$\Big\langle \prod_{l=1}^{N} |z - e^{i\theta_l}|^{2\mu} \Big\rangle_{\mathrm{CE}_{\beta,N}} \propto \Big\langle \prod_{l=1}^{\mu} \Big(1 - (1 - |z|^2) x_l \Big)^N \Big\rangle_{\mathrm{ME}_{4/\beta,\mu}(x^\alpha (1-x)^\alpha)}.$$

This can be generalized to allow a factor $|z - e^{i\theta_l}|^{2\mu_1}$ in the product on the LHS.
Hence for even β the two-point function can be written as a β-dimensional integral. It's proportional to (F. (1994))

$$(2 \sin \pi (r_1 - r_2)/L)^\beta e^{-\pi i \beta N (r_1 - r_2)} \int_{[0,1]^\beta} du_1 \cdots du_\beta$$
$$\times \prod_{j=1}^{\beta} (1 - (1 - e^{2\pi i (r_1 - r_2)}) u_j)^N u_j^{-1+2/\beta} (1 - u_j)^{-1+2/\beta} \prod_{j<k} |u_k - u_j|^{4/\beta}.$$

- The large N bulk scaled limit can be taken immediately.
- Can analyze the large N global expansion (no scaling of variables)

$$\left(\frac{2\pi}{N}\right)^2 \rho_{(2)}(0,\theta) = 1 - \frac{1}{\beta(2N\sin\theta/2)^2} + \frac{3(\beta-2)^2}{2\beta^3(2N\sin\theta/2)^4} - \cdots$$

Not suited to computing the structure function. In the bulk, for $\beta = p/q$ have

$$S(k;\beta) = \frac{|k|}{\pi\beta} f(|k|;\beta),$$

where for $|k| < 2\pi$

$$f(k;\beta) \propto \prod_{i=1}^{q} \int_0^\infty dx_i \prod_{j=1}^{p} \int_0^\infty dy_j \, Q_{p,q}^2 \hat{F}(q,p,\lambda|\{x_i,y_j\};k)\,\delta(1-Q_{p,q}),$$

with $\lambda = \beta/2$, $Q_{p,q} = 2\pi(\sum_{i=1}^{q} x_i + \sum_{j=1}^{p} y_j)$,

$$\hat{F}(q,p,\lambda|\{x_i,y_j\};k) = \frac{1}{\prod_{i=1}^{q}(x_i(1+kx_i/\lambda))^{1-\lambda}\prod_{j=1}^{p}(y_j(1-ky_j))^{1-1/\lambda}} \times \frac{\prod_{i<i'}|x_i - x_{i'}|^{2\lambda}\prod_{j<j'}|y_j - y_{j'}|^{2/\lambda}}{\prod_{i=1}^{q}\prod_{j=1}^{p}(x_i+\lambda y_j)^2}.$$

Functional equation for the structure function

From the exact form of $\rho_{(2)}^{\rm bulk}(0;x)$ have

$$S(k) = \begin{cases} \dfrac{|k|}{\pi} - \dfrac{|k|}{2\pi}\log\Big(1+\dfrac{|k|}{\pi}\Big), & |k| \le 2\pi,\ (\beta=1) \\ \dfrac{|k|}{2\pi}, & |k| \le 2\pi,\ (\beta=2) \\ \dfrac{|k|}{4\pi} - \dfrac{|k|}{8\pi}\log\Big(1-\dfrac{|k|}{2\pi}\Big), & |k| \le 4\pi,\ (\beta=4) \end{cases}$$

From the exact form for $S(k)$ for β rational can check that with

$$f(k;\beta) = \frac{\pi\beta}{|k|} S(k;\beta), \qquad 0 < k < \min(2\pi, \pi\beta)$$

and f defined by analytic continuation for $k < 0$,

$$f(k;\beta) = f\Big(-\frac{2k}{\beta};\frac{4}{\beta}\Big).$$

The simplest structure consistent with the functional equation is

$$\frac{\pi\beta}{|k|}S(k;\beta) = 1 + \sum_{j=1}^{\infty} p_j(\beta/2)\Big(\frac{|k|}{\pi\beta}\Big)^j, \qquad 0 < k < \min(2\pi, \pi\beta)$$

where $p_j(x)$ is a polynomial of degree j which satisfies the functional relation

$$p_j(1/x) = (-1)^j x^{-j} p_j(x).$$

Put $\kappa = \beta/2$, $y = |k|/\pi\beta$. We have (F., Jancovici, McAnally (2000))

$$\frac{\pi\beta}{|k|}S(k;\beta) = 1 + (\kappa-1)y + (\kappa-1)^2 y^2 + (\kappa-1)(\kappa^2 - \frac{11}{6}\kappa + 1)y^3$$
$$+ (\kappa-1)^2(\kappa^2 - \frac{3}{2}\kappa + 1)y^4 + (\kappa-1)(\kappa^4 - \frac{91}{30}\kappa^3 + \frac{62}{15}\kappa^2 - \frac{91}{30}\kappa + 1)y^5 + \cdot$$

Moments of the density and loop equations

For the Gaussian β ensemble, with the eigenvalues scaled so that the leading support is $(-1, 1)$, and with $\kappa = \beta/2$, let

$$m_{2l}(N, \kappa) = \int_{-\infty}^{\infty} x^{2l} \rho_{(1)}^{N}(x; \kappa)\, dx$$

It is known rigorously (Dumitriu and Edleman (2006)) that $m_{2l}(N, \kappa)$ is a polynomial of a degree $l + 1$ in N with constant term zero, satisfying

$$m_{2l}(N, \kappa) = (-1)^{l+1} \kappa^{-l-1} m_{2l}(-\kappa N, \kappa^{-1}).$$

$$\begin{aligned} m_0 &= N \\ m_2 &= N^2 + N(-1 + \kappa^{-1}) \\ m_4 &= 2N^3 + 5N^2(-1 + \kappa^{-1}) + N(3 - 5\kappa^{-1} + 3\kappa^{-2}) \\ &\vdots \end{aligned}$$

Consequences.
Let

$$W(x, N, \kappa) = \int_{-\infty}^{\infty} \frac{\rho_{(1)}^{N}(y;\kappa)}{x-y}\, dy$$

Then

$$W(x, N, \kappa) = -\kappa^{-1} W(x, -\kappa N, \kappa^{-1})$$

A linear differential equation of degree $2\kappa+1$ for $\kappa \in \mathbb{Z}^{+}$ can be derived for $Y := \rho_{(1)}^{N}(y;\kappa)$, e.g. for $\beta = 2$ (Haagerup and Thorbjornsen (2003))

$$\frac{1}{4N^2} Y''' + (1-y^2) Y' + yY = 0.$$

Can check that W satisfies an inhomogeneous form of the same equation. Hence must have that

$$\rho_{(1)}^{N}(x,\kappa) = -\kappa^{-1} \rho_{(1)}^{-\kappa N}(x,\kappa^{-1})$$

e.g. For $\beta = 1$ the density satisfies a 5th order homogeneous differential equation which is the same as that satisfied for $\beta = 4$ but with N replaced by $-N/2$.

On going research

- Linear differential equations for one-point functions/ averages of characteristic polynomials. e.g. What is the behaviour of

$$\left\langle \prod_{l=1}^{N} |z - e^{i\theta_l}|^{2\mu} \right\rangle_{\mathrm{CE}_{\beta,N}}$$

as $z \to 1$ for $\mu < 0$?

- Can the loop equation formalism be used to systematically generate the expansion

$$\left(\frac{2\pi}{N}\right)^2 \rho_{(2)}(0,\theta) = 1 - \frac{1}{\beta(2N\sin\theta/2)^2} + \frac{3(\beta-2)^2}{2\beta^3(2N\sin\theta/2)^4} - \cdots$$

- What is the q, t generalization of the family of Dixon-Anderson integrals used to derive the decimation identities?
- Duality formulas for random matrix ensembles with a source (Desrosiers).