# Exactly Solvable models in Ultracold Physics

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# OUTLINE

**1-** INTRODUCTION

*review some experimental results*

**2-** INTEGRABLE BEC MODELS

*main emphasis: present the mathematical construction*

**3-** ULTRACOLD ATOMIC FERMI GASES

*main emphasis: discuss the physical properties*

**4-** WILSON RATIO

*obtain an analytical expression and discuss physical aspects*

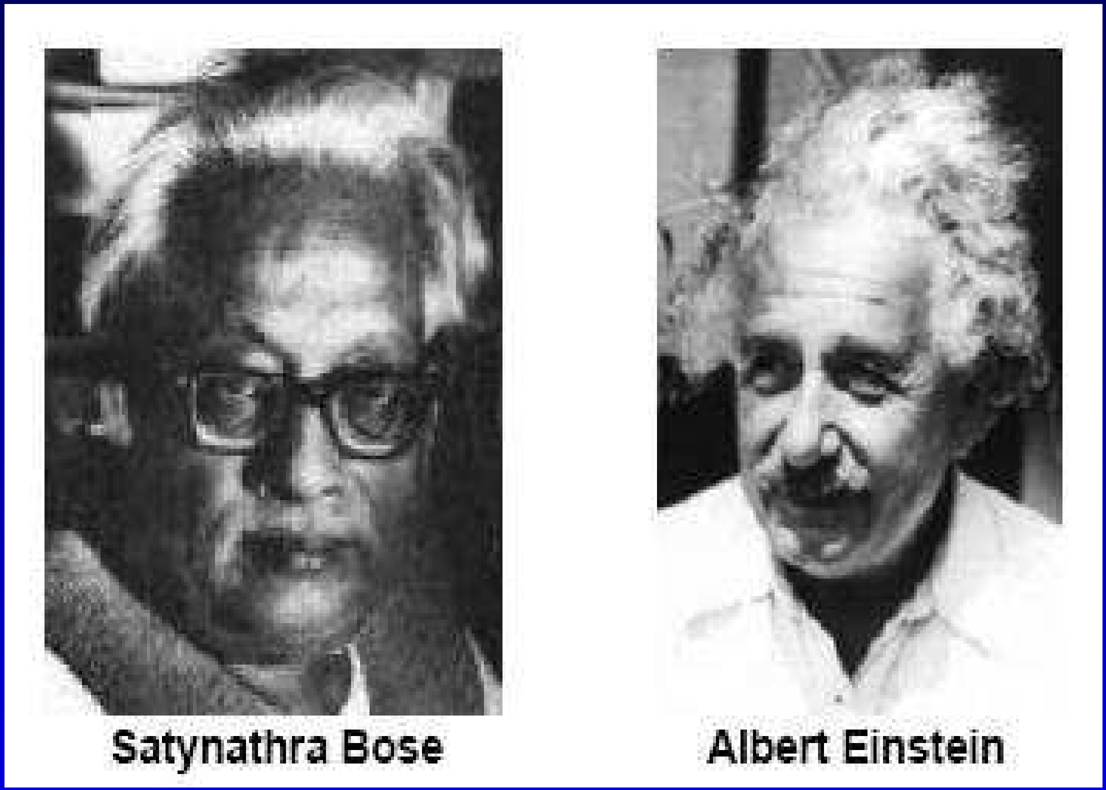
**5-** CONCLUSIONS

*outlook of the area*

# 1-INTRODUCTION

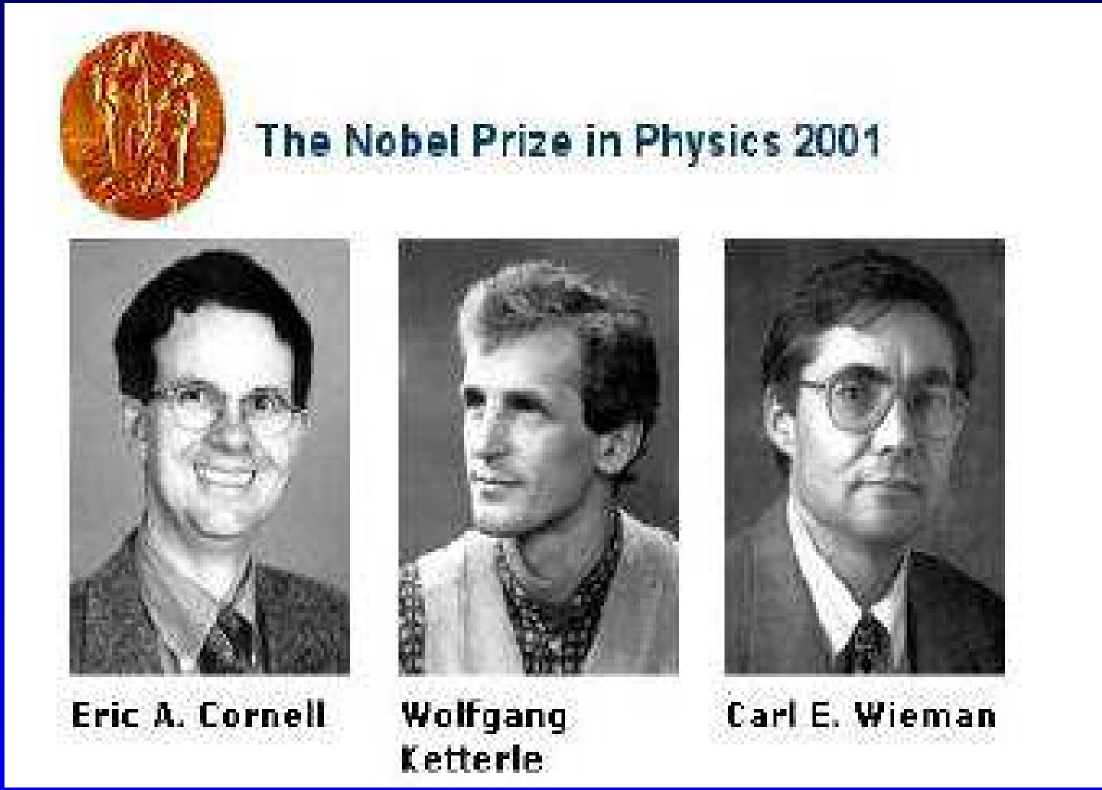
# Theoretical prediction:

- S. N. Bose (1924)
- A. Einstein (1924-1925)

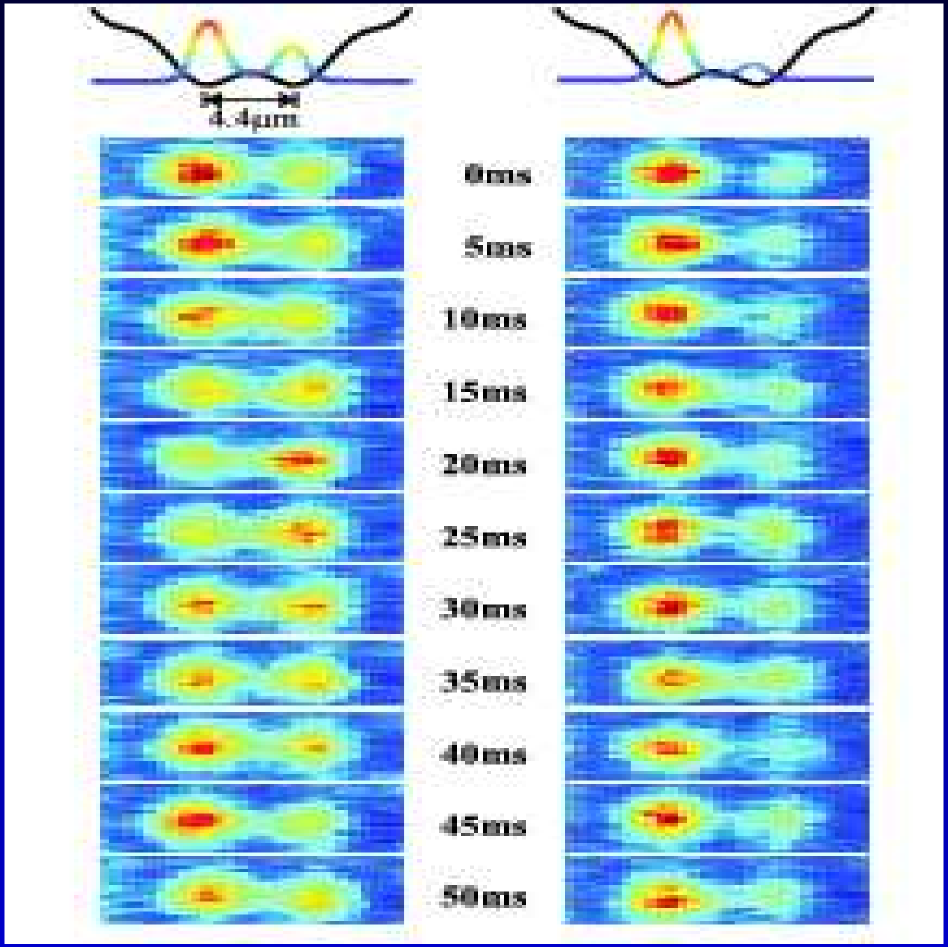


# Experimental realization (70 years later):

- E. A. Cornell *et al*. (1995 ) $\rightarrow {}^{87}Rb$
- W. Ketterle *et al*. (1995) $\rightarrow {}^{23}Na$
- C. C. Bradley *et al*. (1995) $\rightarrow {}^{7}Li$



# Direct observation of tunneling and self-trapping:

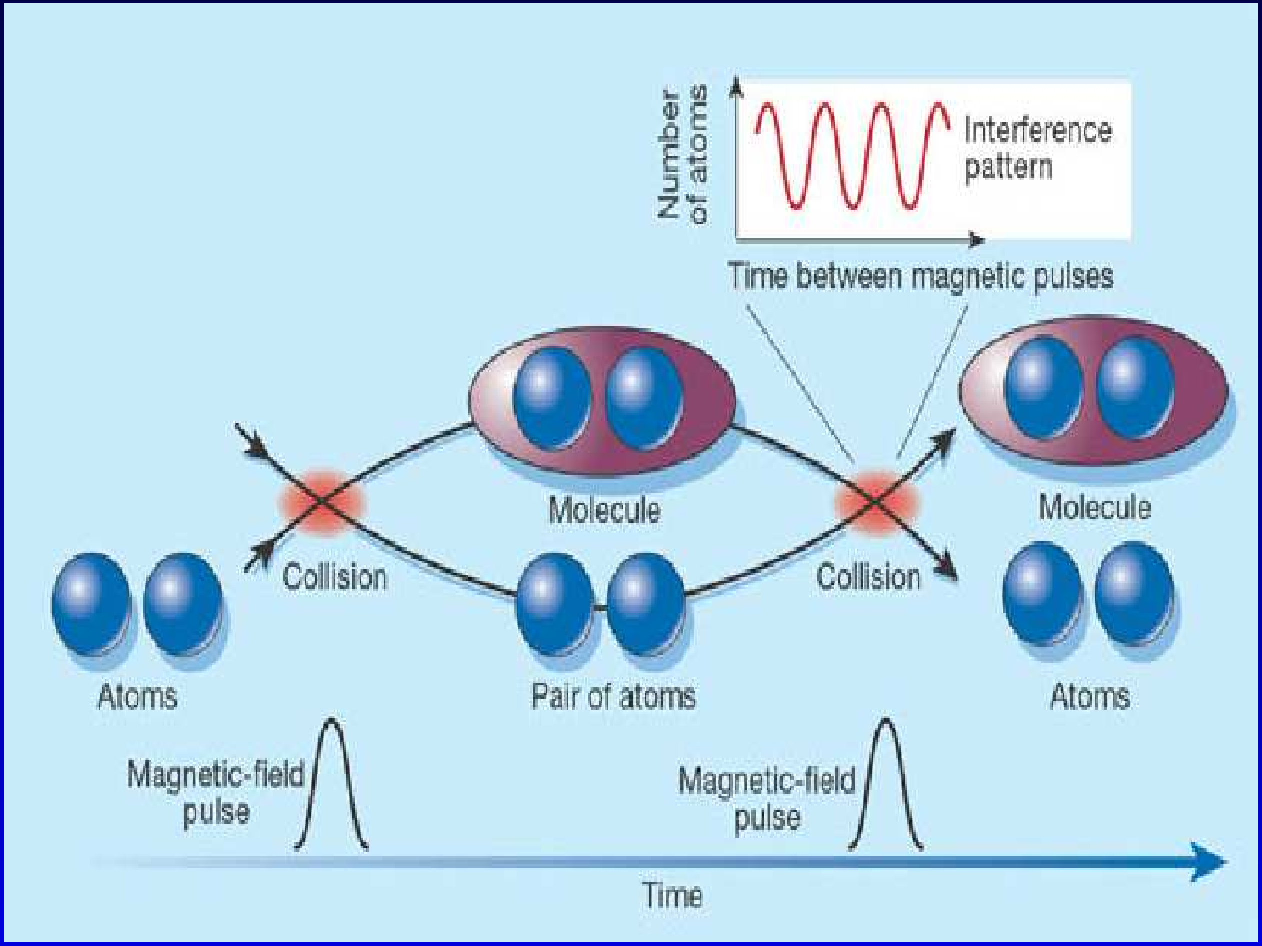



*Albiez, M. et al.*, Phys. Rev. Lett. **95** (2005) 010402

In Section 2: Integrable model that describes qualitatively tunneling X self-trapping

# Atom-molecule BEC

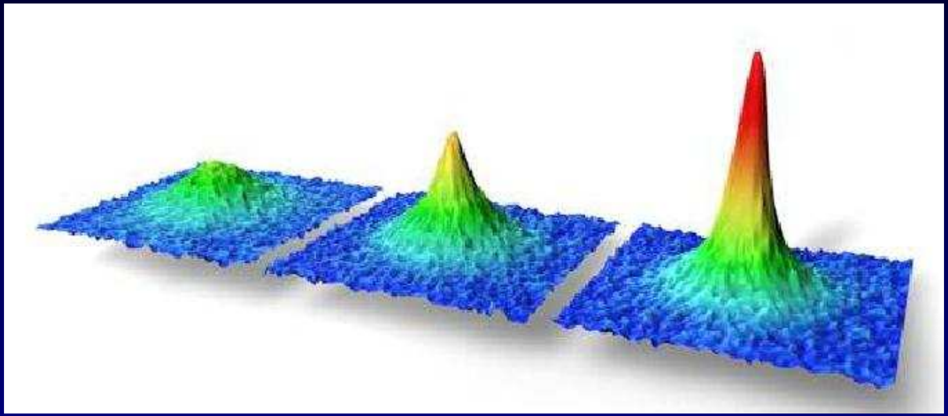
*After the experimental realization of a BEC using atoms, a significant effort was made to produce a stable BEC in a gas of molecules*




*Zoller, P.*, Nature **417** (2002) 493

*Donley, E. A.*, Nature **417** (2002) 529

# Fermionic pair condensation

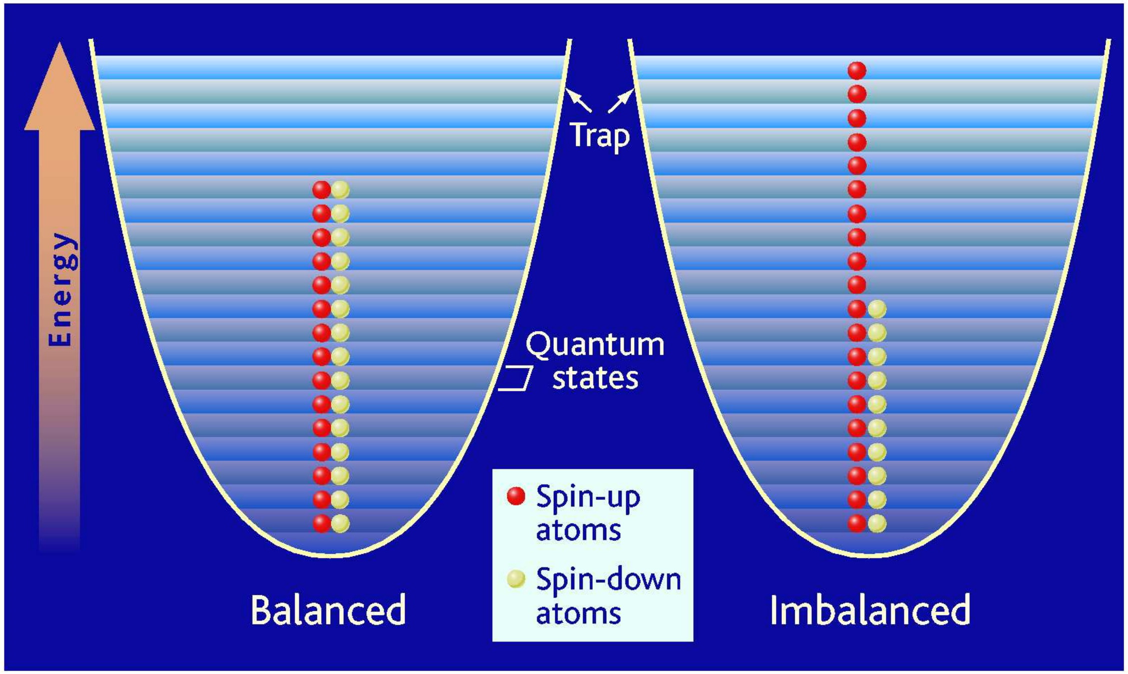


*D. Jin et al, Nature 424 (2003)*

The first ultracold Fermi gas of $^{40}K$ atoms was created in 1999 by the group of D. Jin at JILA. A breakthrough in the area was the creation of a molecular condensate in an ultracold degenerate Fermi gas by the groups of D. Jin, W. Ketterle and R. Grimm in 2003. After that the condensation of fermionic pairs was detected and proved to be a superfluid.

# Fermions with Polarization

Superfluidity can persist with polarization? $P = \frac{N_\uparrow - N_\downarrow}{N_\uparrow + N_\downarrow}$ $0 \leq P \leq 1$




**Find a partner.** When atoms spinning one way outnumber those spinning the other, they still can pair and flow freely—perhaps like matter in a neutron star.

*A. Cho, Science 319 (2008)*

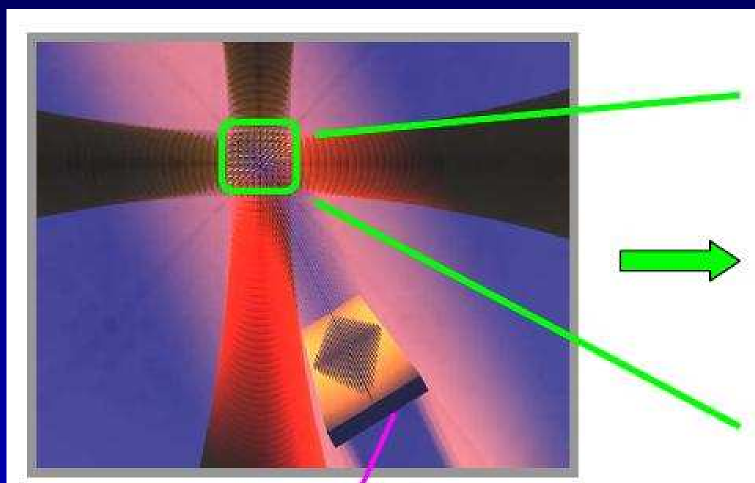
Superfluidity may still occur in a mismatched system and some theories with unusual pairing, exotic phases have been proposed: Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) states, etc.

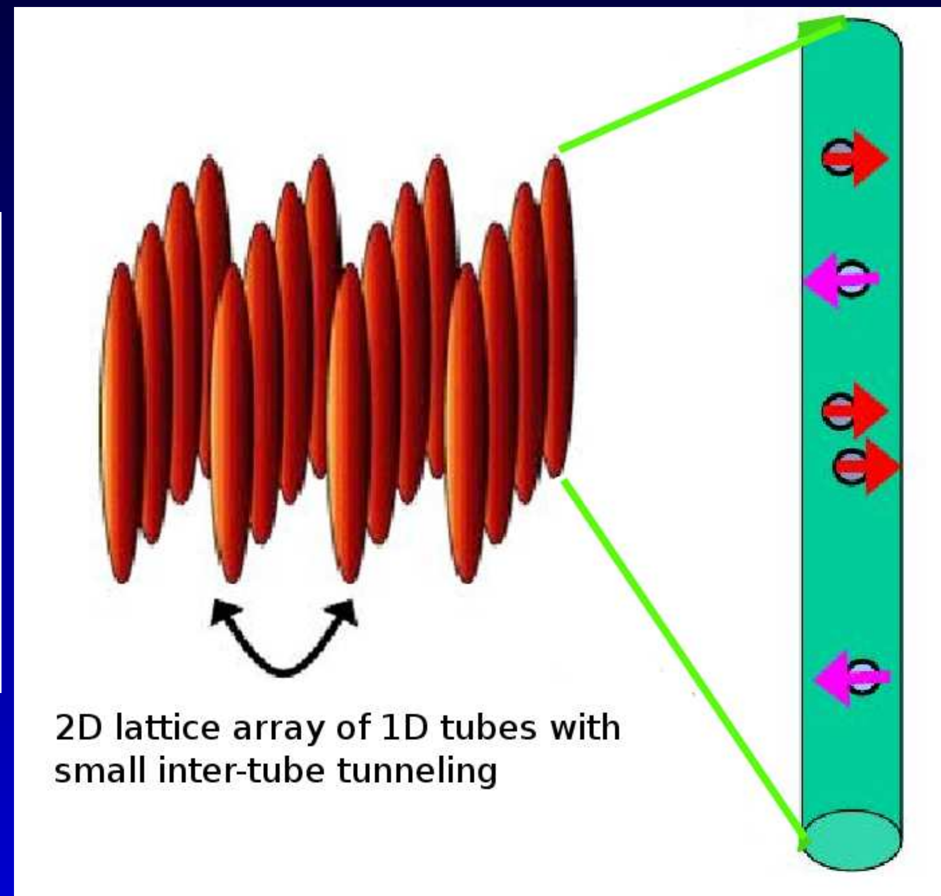
Experimental studies in ultracold Fermi gases with unequal spin population have been conducted, searching for these new phases (groups at MIT and RICE - 3D)

# Two-component Fermi gases: 1D experiments

Crossed beam optical trap

1D gas with spin imbalance






*Spin-imbalance in a one-dimensional Fermi gas, R. Hulet et al, Nature 467 (2010) 568*

Basic result: observation of a partially polarized core (FFLO-like phase) surrounded by either a completely paired BCS superfluid or a fully polarized Fermi gas, depending on the polarization

Section 3: Integrable model that predicts these 3 phases - qualitative and quantitative agreement

# 2- INTEGRABLE MODELS OF BEC

## Two-site Bose Hubbard Hamiltonian:

$$H = \frac{K}{8}(N_1 - N_2)^2 - \frac{\Delta\mu}{2}(N_1 - N_2) - \frac{\mathcal{E}_J}{2}(a_1^\dagger a_2 + a_2^\dagger a_1)$$

- $N_i = a_i^\dagger a_i$: number of atoms in the well $(i = 1, 2)$
- $K$: atom-atom interaction term
- $\Delta\mu$: external potential
- $\mathcal{E}_J$: tunneling strength

*G. Milburn et al, Phys. Rev. A* **55** *(1997) 4318; A. Leggett, Rev. Mod. Phys.* **73** *(2001) 307 A. Foerster, J. Links and H.Q. Zhou, Class. and Quant. Nonlinear Integ. Systems (2003)*

- The quantum dynamics of the model exhibits tunneling X self-trapping - experiment of Albiez et al - 2005
- The model describes recent experiments of Oberthaler et al on bifurcations and entanglement - 2011/12

# Integrability and exact solution:

- R-matrix:

$$R(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$b(u) = \frac{u}{u+\eta} \qquad c(u) = \frac{\eta}{u+\eta}$$

- Yang-Baxter algebra:

$$R_{12}(x-y)R_{13}(x)R_{23}(y) = R_{23}(y)R_{13}(x)R_{12}(x-y)$$

- Monodromy-matrix:

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

- Yang-Baxter algebra:

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v)$$

- Realization of the monodromy matrix:

$$L(u) = \pi(T(u)) = L_1^a(u+w)L_2^a(u-w)$$

$$L_i^a(u) = \begin{pmatrix} u + \eta N_i & a_i \\ a_i^\dagger & \eta^{-1} \end{pmatrix} \qquad i = 1, 2$$

- Transfer matrix:

$$\tau(u) = \pi(Tr(T(u))) = \pi(A(u) + D(u))$$

- Integrability:

$$[\tau(u), \tau(v)] = 0 \longrightarrow [H, \tau(v)] = 0$$

- Hamiltonian and transfer matrix:

$$H = \kappa \left( \tau(u) - \frac{1}{4}(\tau'(0))^2 - u\tau'(0) - \eta^{-2} + w^2 - u^2 \right)$$

*with the identification:*

$$\frac{K}{4} = \frac{\kappa\eta^2}{2}, \qquad \frac{\Delta\mu}{2} = -\kappa\eta w, \qquad \frac{\mathcal{E}_{\mathcal{J}}}{2} = \kappa$$

$$H = \frac{K}{8}(N_1 - N_2)^2 - \frac{\Delta\mu}{2}(N_1 - N_2) - \frac{\mathcal{E}_J}{2}(a_1^\dagger a_2 + a_2^\dagger a_1)$$

A generalization to a 2-site Bose Hubbard model with non-linear tunneling is being investigated

*Foerster, Links, Marquette and Mattei and Roditi, 2013- in preparation*

Applying the algebraic Bethe ansatz method:

- Energy:

$$E = -\kappa(\eta^{-2}\prod_{i=1}^{N}\left(1+\frac{\eta}{v_i-w}\right)-\frac{\eta^2N^2}{4}-w\eta N-\eta^{-2})$$

- Bethe Ansatz Equations:

$$\eta^2(v_i^2-w^2)=\prod_{j\neq i}^{N}\frac{v_i-v_j-\eta}{v_i-v_j+\eta}$$

The solutions of the BAE can be used to study the Quantum Phase Transitions of the model

*D. Rubeni, A. Foerster, E. Mattei and I. Roditi, NPB 2012*

## INTEGRABLE GENERALISED MODELS:

Basic idea:

**We can construct integrable generalised models in the BEC context exploring different representations of some algebra, such as the $gl(N)$ algebra and $gl(M/N)$ superalgebra.**

- Three-coupled BEC model
- Models for homo and hetero atom-molecular BEC
- ......

**see talk by: JON LINKS**

*A. Foerster, J. Links and H.Q. Zhou*, Class. and Quant. Nonlinear Integ. Systems (2003);

*A. Foerster and E. Ragoucy*, Nuclear Phys. **B777** (2007) 373;

*A. Tonel, G. Santos, A. Foerster, I. Roditi, Z. Santos*, Physical Review **A79** (2009) 013624;

# 3 - ULTRACOLD ATOMIC FERMI GASES

## 1D 2-component attractive Fermi gas with polarization:

- Hamiltonian

$$\mathcal{H} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g_{1D}\sum_{1\le i<j\le N}\delta(x_i - x_j) - \frac{H}{2}(N_\uparrow - N_\downarrow)$$

- $N$ spin $1/2$ fermions of mass $m$
- constrained by PBC to a line of length $L$
- $H$: external field
- $g_{1D} = \frac{\hbar^2 c}{m}$: 1D interaction strength:
  attractive for $g_{1D} < 0$ and repulsive for $g_{1D} > 0$
  *HERE: ATTRACTIVE REGIME*
- $\gamma \equiv \frac{c}{n}\,(n = \frac{N}{L})$ : dimensionless interaction;

# Bethe ansatz method

*C.N. Yang, PRL* **19***(1967)1312; M. Gaudin, Phys. Lett.* **24** *(1967) 55*

- Energy:

$$E = \frac{\hbar^2}{2m}\sum_{j=1}^{N} k_j^2,$$

- BAE:

$$\exp(\mathrm{i}k_j L) = \prod_{\ell=1}^{M} \frac{k_j - \Lambda_\ell + \mathrm{i}\, c/2}{k_j - \Lambda_\ell - \mathrm{i}\, c/2}$$

$$\prod_{\ell=1}^{N} \frac{\Lambda_\alpha - k_\ell + \mathrm{i}\, c/2}{\Lambda_\alpha - k_\ell - \mathrm{i}\, c/2} = -\prod_{\beta=1}^{M} \frac{\Lambda_\alpha - \Lambda_\beta + \mathrm{i}\, c}{\Lambda_\alpha - \Lambda_\beta - \mathrm{i}\, c}$$

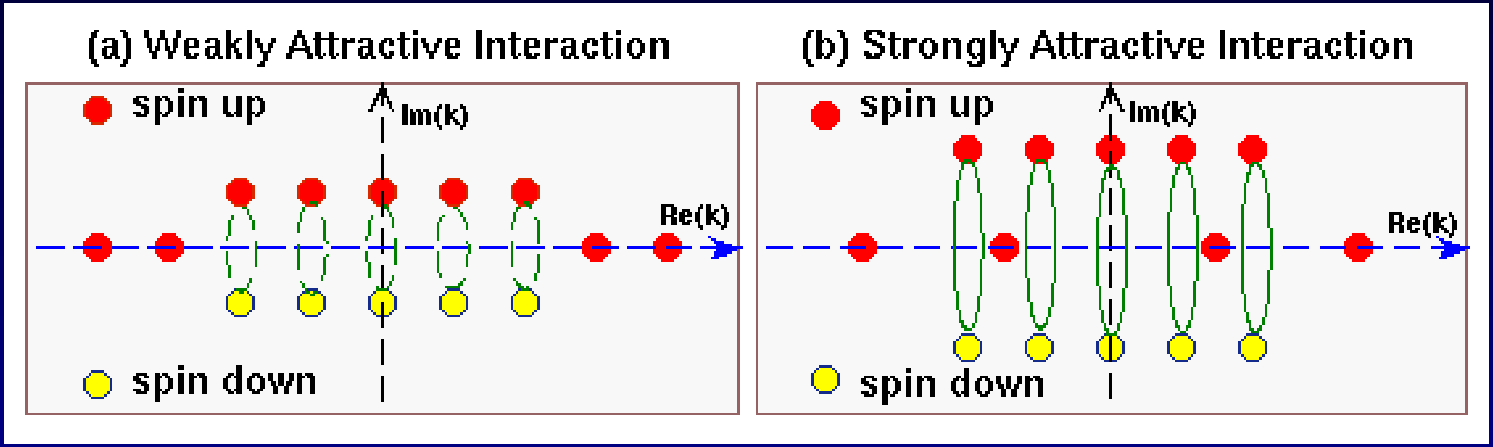
$\{k_j,\ j = 1, \ldots, N\}$ are the quasimomenta for the fermions;

$\{\Lambda_\alpha,\ \alpha = 1, \ldots, M\}$ are the rapidities for the internal spin degrees of freedom

The solutions to the BAE give the GS properties and provide a clear pairing signature

# BA-root configuration for the GS

BA configuration of quasimomenta k in the complex plane for the GS: for a given polarization, the system is described by bound states ("Cooper pairs") and unpaired fermions




Weak Regime

$$\frac{E}{L} \approx \frac{\hbar^2 n^3}{2m}\left(-\frac{|\gamma|}{2}(1-P^2)+\frac{\pi^2}{12}+\frac{\pi^2}{4}P^2\right)$$

Strong Regime

$$\frac{E}{L} \approx \frac{\hbar^2 n^3}{2m}\left\{-\frac{\gamma^2(1-P)}{4}+\frac{P^3\pi^2}{3}\left(1+\frac{4(1-P)}{|\gamma|}\right)+\frac{\pi^2(1-P)^3}{48}\left(1+\frac{1-P}{|\gamma|}+\frac{4P}{|\gamma|}\right)\right\}$$

## Thermodynamical Bethe Ansatz - TBA

- elegant method to study thermodynamical properties
- convenient formalism to analyse QPT at $T = 0$
- 1969 C. N. Yang and C. P. Yang, "Yang-Yang approach"
- 1972 M. Takahashi, string hypothesis
- thermodynamic limit: $L \to \infty$, $N \to \infty$ with $N/L$ finite:
  - consider a distribution function for the BA-roots;
  - the equilibrium state is determined by the condition of minimizing the Gibbs free energy: $G = E - HM^z - \mu N - TS$

# TBA - equations:

*set of coupled nonlinear integral equation:*

$$\begin{aligned}
\epsilon^{\rm b}(k) &= 2(k^2-\mu-\frac{1}{4}c^2)+Ta_2*\ln(1+{\rm e}^{-\epsilon^{\rm b}(k)/T})\\
&\quad +Ta_1*\ln(1+{\rm e}^{-\epsilon^{\rm u}(k)/T})\\
\epsilon^{\rm u}(k) &= k^2-\mu-\frac{1}{2}H+Ta_1*\ln(1+{\rm e}^{-\epsilon^{\rm b}(k)/T})\\
&\quad -T\sum_{n=1}^{\infty}a_n*\ln(1+\eta_n^{-1}(k))\\
\ln\eta_n(\lambda) &= \frac{nH}{T}+a_n*\ln(1+{\rm e}^{-\epsilon^{\rm u}(\lambda)/T})\\
&\quad +\sum_{n=1}^{\infty}T_{nm}*\ln(1+\eta_m^{-1}(\lambda))
\end{aligned}$$

*The dressed energies:* $\epsilon^{\rm b}(k) := T\ln(\sigma^h(k)/\sigma(k))$ *and* $\epsilon^{\rm u}(k) := T\ln(\rho^h(k)/\rho(k))$ *for paired and unpaired fermions; the function* $\eta_n(\lambda) := \xi^h(\lambda)/\xi(\lambda)$ *is the ratio of string densities.*
*The Gibbs free energy per unit length (Takahashi's book):*

$$G=-\frac{T}{\pi}\int_{-\infty}^{\infty}dk\ln(1+{\rm e}^{-\epsilon^{\rm b}(k)/T})-\frac{T}{2\pi}\int_{-\infty}^{\infty}dk\ln(1+{\rm e}^{-\epsilon^{\rm u}(k)/T})$$

# Limit $T \to 0$: dressed energy equations

$$\epsilon^{\mathrm{b}}(\Lambda) = 2(\Lambda^2 - \mu - \frac{c^2}{4}) - \int_{-B}^{B} a_2(\Lambda - \Lambda')\epsilon^{\mathrm{b}}(\Lambda')d\Lambda' - \int_{-Q}^{Q} a_1(\Lambda - k)\epsilon^{\mathrm{u}}(k)dk$$

$$\epsilon^{\mathrm{u}}(k) = (k^2 - \mu - \frac{H}{2}) - \int_{-B}^{B} a_1(k - \Lambda)\epsilon^{\mathrm{b}}(\Lambda)d\Lambda$$

$$a_m(x) = \frac{1}{2\pi}\frac{m|c|}{(m\,c/2)^2 + x^2},\ \epsilon^{\mathrm{b}}(\pm B) = \epsilon^{\mathrm{u}}(\pm Q) = 0$$

The Gibbs free energy per unit length at zero temperature is given by

$$G(\mu, H) = \frac{1}{\pi}\int_{-B}^{B} \epsilon^{\mathrm{b}}(\Lambda)d\Lambda + \frac{\mathbf{1}}{\mathbf{2}\pi}\int_{-\mathbf{Q}}^{\mathbf{Q}} \epsilon^{\mathrm{u}}(\mathbf{k})\mathbf{dk}$$

From the Gibbs free energy per unit length we have the relations

$$-\partial G(\mu, H)/\partial\mu = n, \quad -\partial G(\mu, H)/\partial H = m_z = nP/2$$

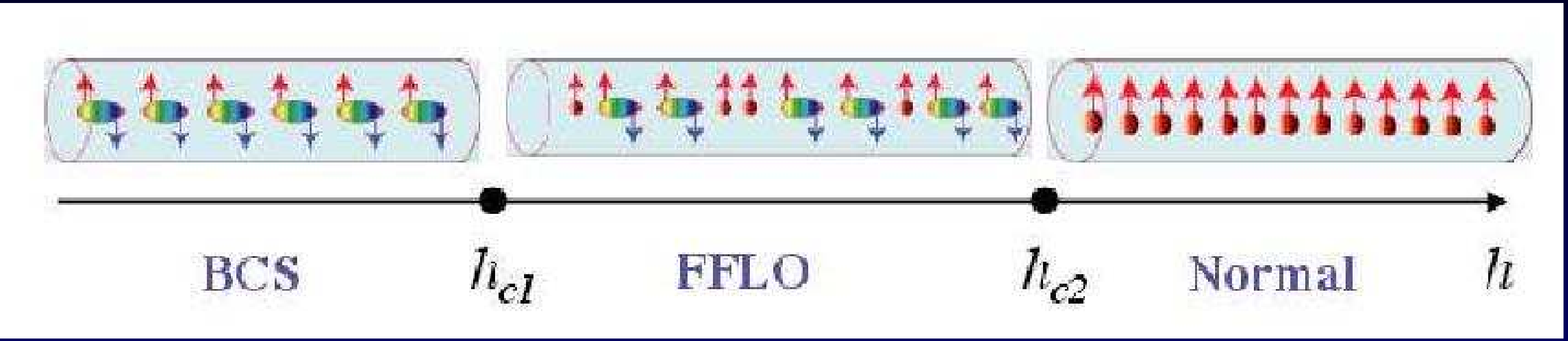
# Strong attraction

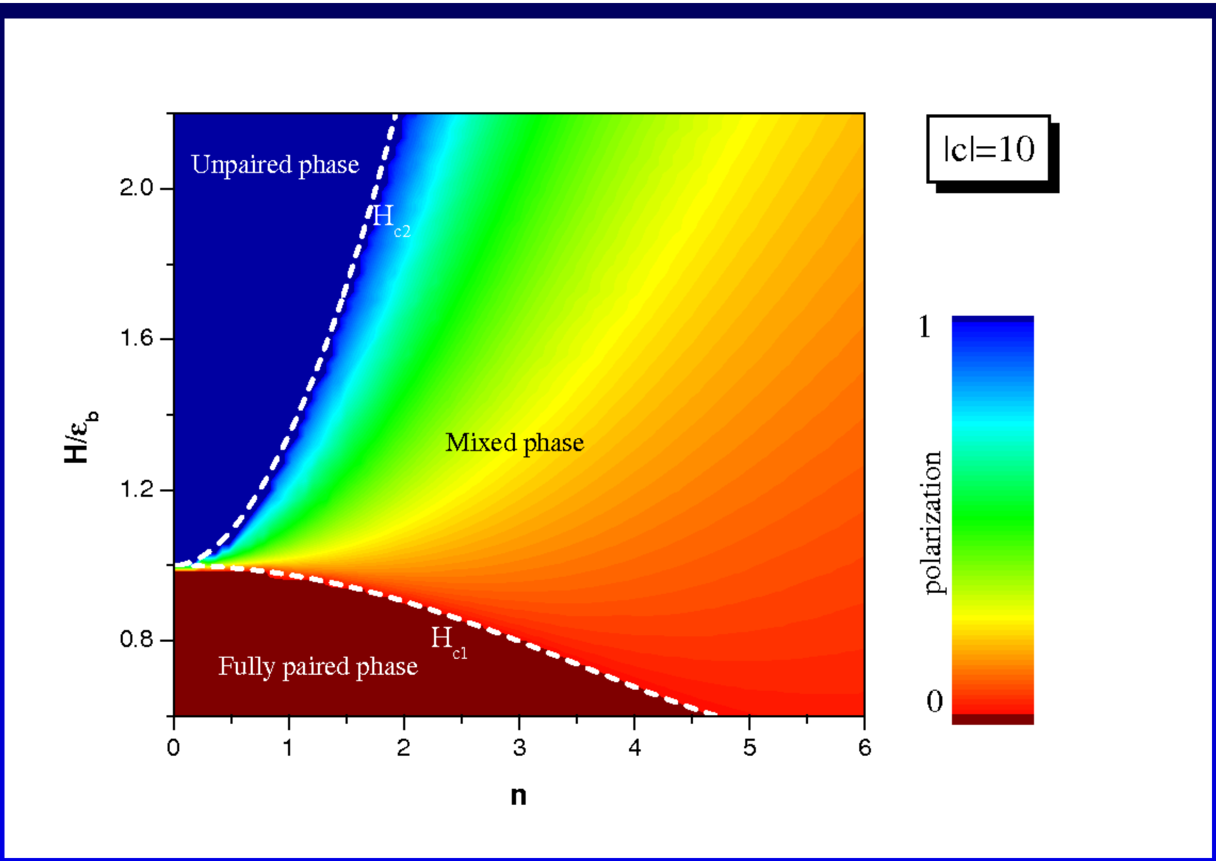
$$
\begin{aligned}
H &= \frac{\hbar^2 n^2}{2m}\left\{\frac{\gamma^2}{2}+2P^2\pi^2\left(1+\frac{4(1-P)}{|\gamma|}-\frac{4P}{3|\gamma|}\right)\right. \\
&\qquad \left. -\frac{\pi^2(1-P)^2}{8}\left(1+\frac{4P}{|\gamma|}\right)\right\}.
\end{aligned}
$$

CRITICAL FIELDS:

$$
\begin{aligned}
H_{c1} &= \frac{\hbar^2 n^2}{2m}\left(\frac{\gamma^2}{2}-\frac{\pi^2}{8}\right) \\
H_{c2} &= \frac{\hbar^2 n^2}{2m}\left[\frac{\gamma^2}{2}+2\pi^2\left(1-\frac{4}{3|\gamma|}\right)\right]
\end{aligned}
$$

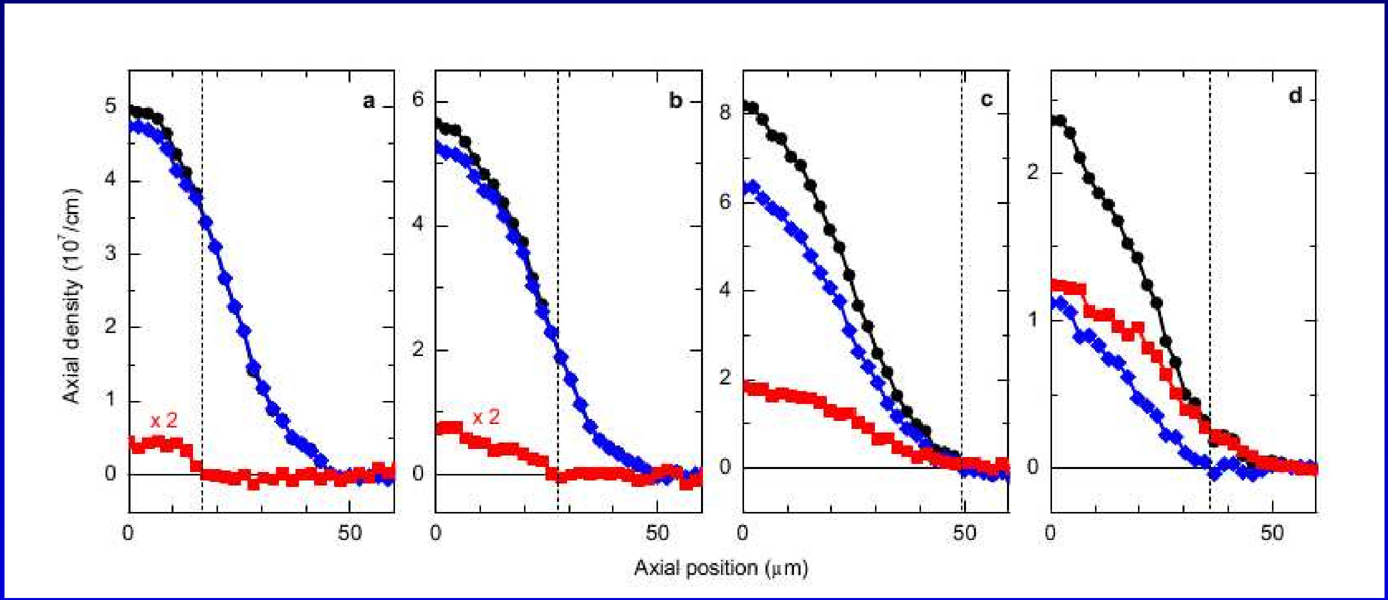
## Phase diagram and schematic representation:

- 


*He, Foerster, Guan, Batchelor, NJP2009*

- For T=0: The TBA predicts 3 phases in the strong coupling regime: a superfluid phase,a fully polarized phase and a partially polarized phase, in agreement with experiments
- For low T: Theoretical predictions from low-T TBA + LDA (solid lines) in the strong coupling regime are in quantitative agreement with experimental measurements (circles) of density profiles of a 2-spin mixture of ultracold $^6$Li atoms in 1D tubes




*Liao et al. Nature 2010; the black (blue) circles are the density of fermions in the state $|1>$ ($|2>$) and the red squares the difference between these two states; the solid lines are the predictions from TBA+LDA; the polarizations are 0.015, 0.055, 0.10 and 0.33.*

# 4-Wilson Ratio

# Universal Ratios

In condensed matter, dimensionless ratios of quantities that take universal values can provide deep physical insights. Some examples include:

- Wiedemann-Franz ratio
- Sommerfeld-Wilson ratio
- Kadowaki-Woods ratio
- Korringa ratio

Why universal ratios are important?

- show that the same particles are responsible for the two different quantities that form the ratio;
- provide significant constraints on theories;
- demonstrate universality.

# Wilson Ratio

The Wilson ratio is defined as the ratio of the magnetic susceptibility $\chi$ to specific heat $c_v$ divided by temperature $T$

$$R_W = \frac{4}{3}\left(\frac{\pi k_B}{\mu_B g}\right)^2 \frac{\chi}{c_v/T}$$

- It quantifies the interaction effect and spin fluctuations;
- Examples: $R_W = 1$ for noninteracting or weakly correlated electrons in metals; $R_W = 2$ in Kondo regime for impurity problem; $R_W = 2$ for the spin-1/2 Heisenberg chain
- The Wilson ratio has recently been measured in a spin 1/2-ladder compound $(C_7H_{10}N)_2C_uB_{r2}$: $R_W = 4K$, where $K$ is the TLL parameter *Ninios et al, PRL2012*
- where $k_B$ is the Boltzmann constant, $\mu_B$ is the Bohr magneton, $g$ is the Lande factor;

## Wilson Ratio for the 2-component attractive Fermi gas

Key result:

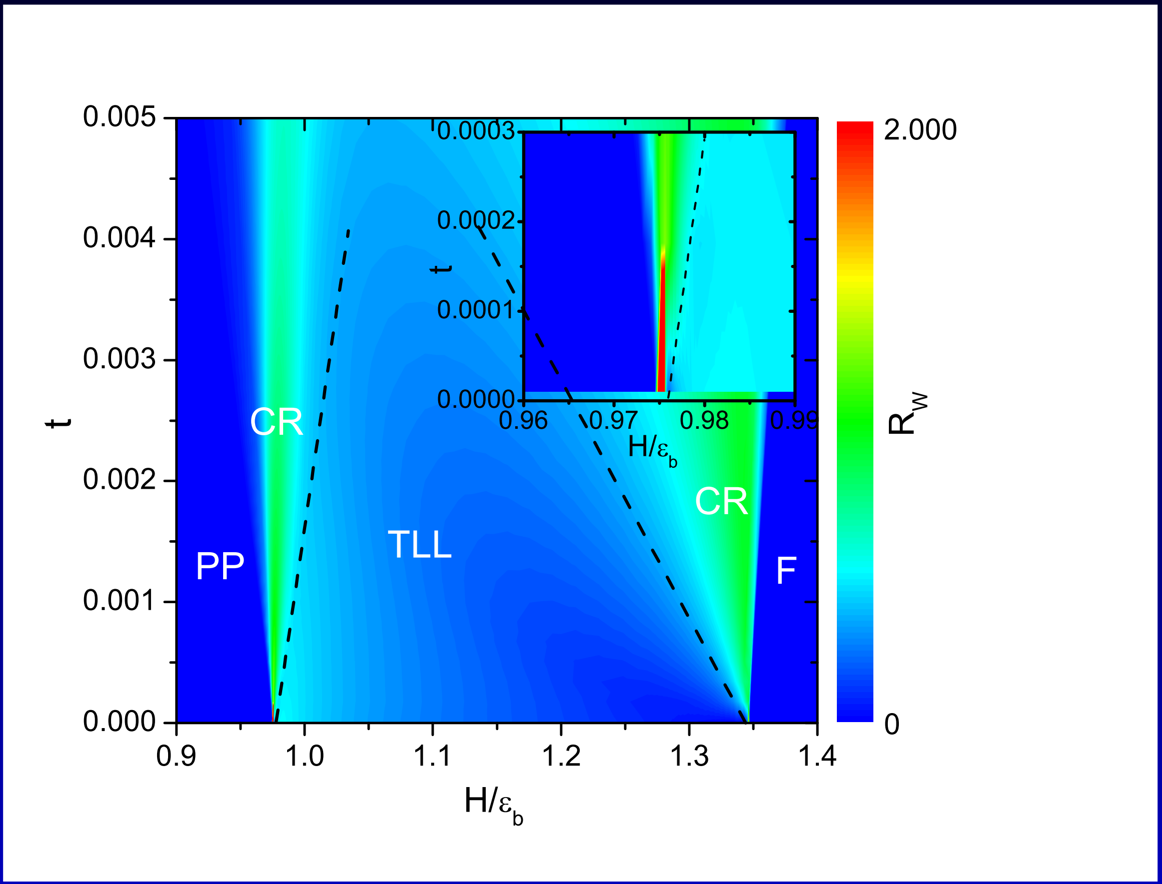
$$R_W = \frac{4}{\left(v_N^{\mathrm{b}} + 4v_N^{\mathrm{u}}\right)\left(\frac{1}{v_s^{\mathrm{b}}} + \frac{1}{v_s^{\mathrm{u}}}\right)}$$

in terms of the density stiffness $v_N^{\mathrm{b,u}}$ and sound velocity $v_s^{\mathrm{b,u}}$ for pairs b and unpaired fermions u. They can be calculated from the ground state energy:

$$\begin{aligned}
v_N^{\mathrm{b}} &= \frac{\hbar\pi n_2}{2m}\left[1 + \frac{4}{|c|}(n - 3n_2) + \frac{3}{c^2}(4n^2 - 24nn_2 + 30n_2^2)\right] \\
v_N^{\mathrm{u}} &= \frac{\hbar\pi n_1}{m}\left[1 + \frac{4}{|c|}(n - 2n_1) + \frac{4}{c^2}(3n^2 + 10n_1^2 - 12nn_1)\right] \\
v_s^{\mathrm{b}} &= \frac{\hbar\pi n_2}{2m}\left[1 + \frac{2(2n_1 + n_2)}{|c|} + \frac{3(2n_1 + n_2)^2}{c^2}\right] \\
v_s^{\mathrm{u}} &= \frac{\hbar\pi n_1}{m}\left[1 + \frac{8n_2}{|c|} + \frac{48n_2^2}{c^2}\right]
\end{aligned}$$

*Guan, Yin, Foerster, Batchelor, Lee, Lin, PRL 2013*

# Contour Plot of the Wilson Ratio:

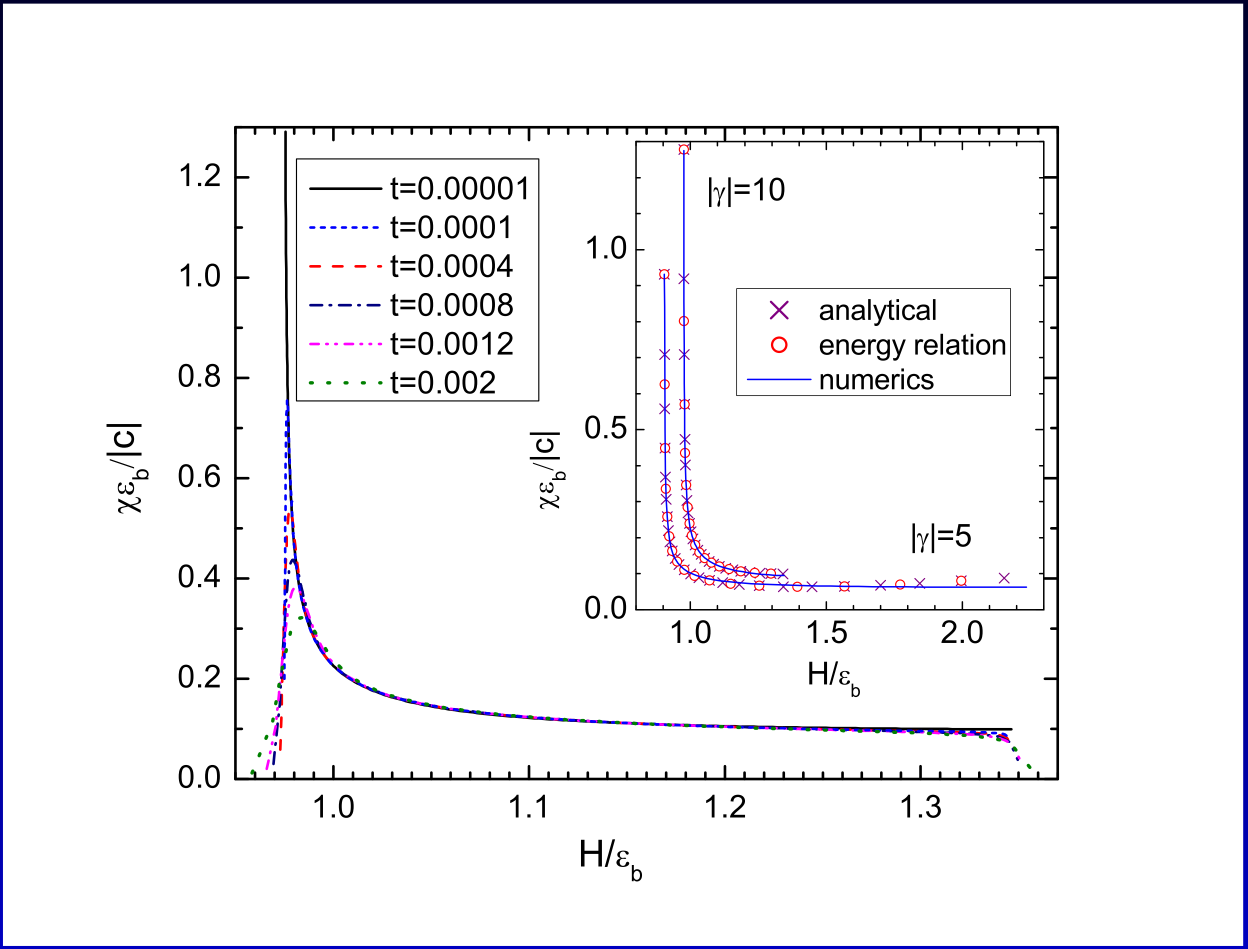



Contour plot of $R_W$ for $|\gamma| = 10$ as a function of the temperature and magnetic field using the TBA. In the region below the dashed lines, $R_W$ is temperature independent. $R_W = 0$ for the paired (PP) and ferromagnetic (F) phases. Near the critical points, the ratio reveals anomalous enhancement. The inset shows the enhancement at the lower critical point.

# Basic idea:

- The main ingredients: susceptibility and specific heat are obtained from the TBA-equations ( coupled equations with temperature)
- Procedure: From the TBA equations we derive an equation of state in the strong coupling and low temperature regime and from it we derive the susceptibility and specific heat by standard thermodynamics

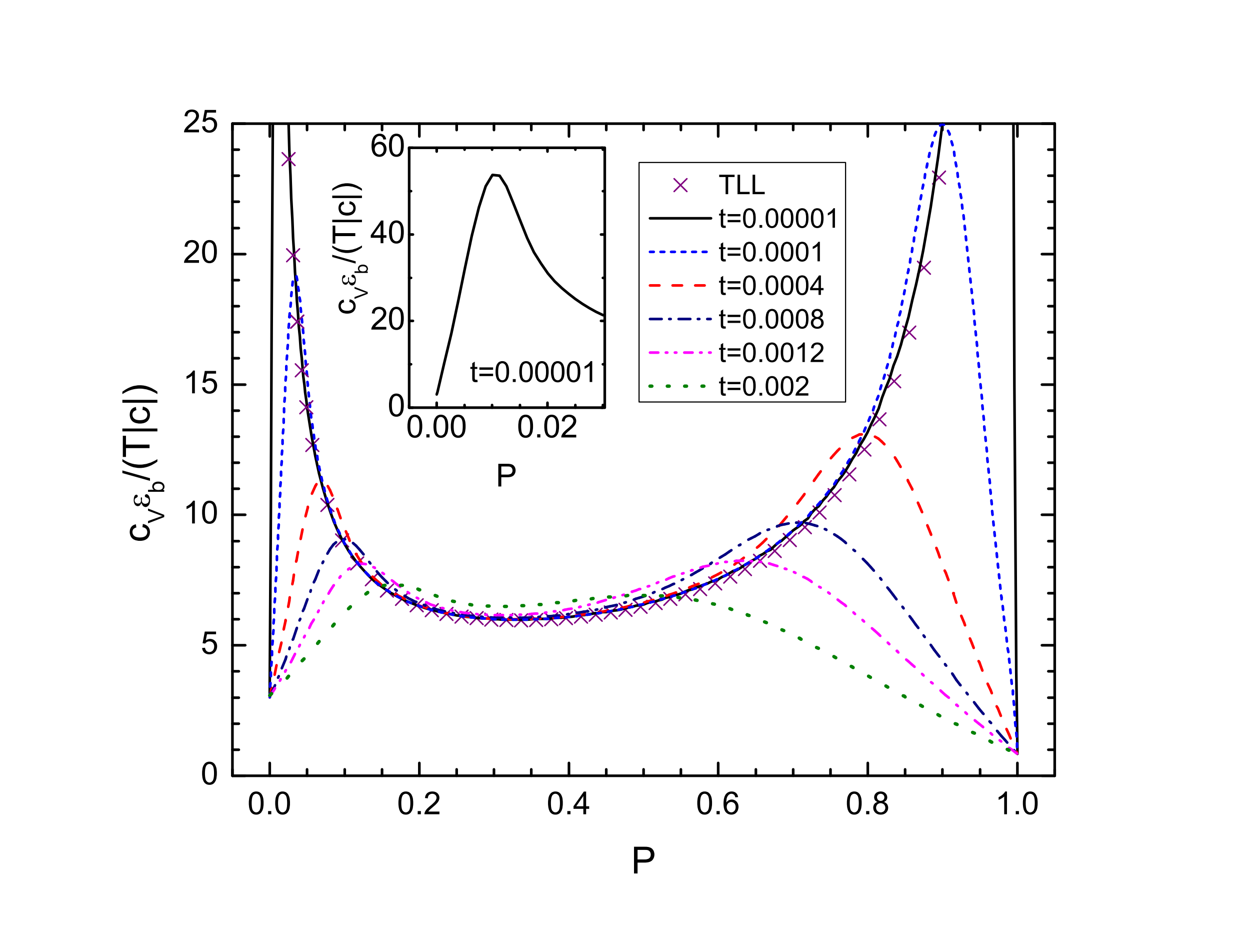
# Susceptibility




$$\frac{1}{\chi} = \frac{1}{\chi_{\rm u}} + \frac{1}{\chi_{\rm b}}, \quad \chi_{\rm b} = 1/(\hbar\pi v_N^b) \quad \chi_{\rm u} = 1/(4\hbar\pi v_N^{\rm u})$$

Dimensionless susceptibility vs magnetic field for $|\gamma| = 10$. The analytic result (red crosses) agrees with the numerical result obtained from the equation of state (TBA). The susceptibility is basically independent of temperature.

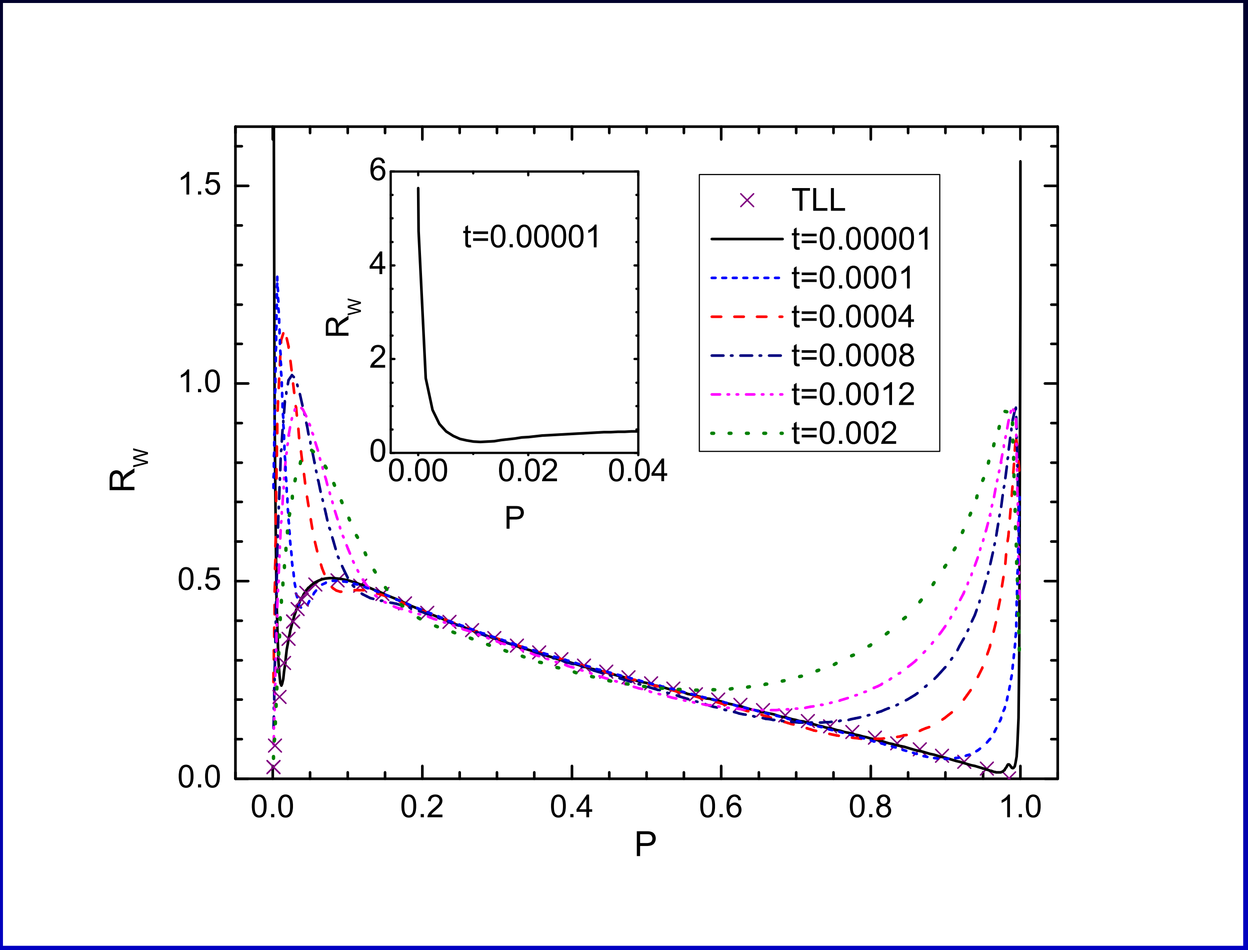
# Specific Heat




$$c_v = \frac{\pi k_B^2 T}{3\hbar}\left(\frac{1}{v_s^{\mathrm{b}}} + \frac{1}{v_s^{\mathrm{u}}}\right).$$

Dimensionless specific heat vs polarization for $|\gamma| = 10$. The analytic result (red crosses) agrees with the numerical result obtained from the equation of state (TBA). By increasing $t$ a deviation from analytic result can be seen, indicating the breakdown of the 2-component TLL.

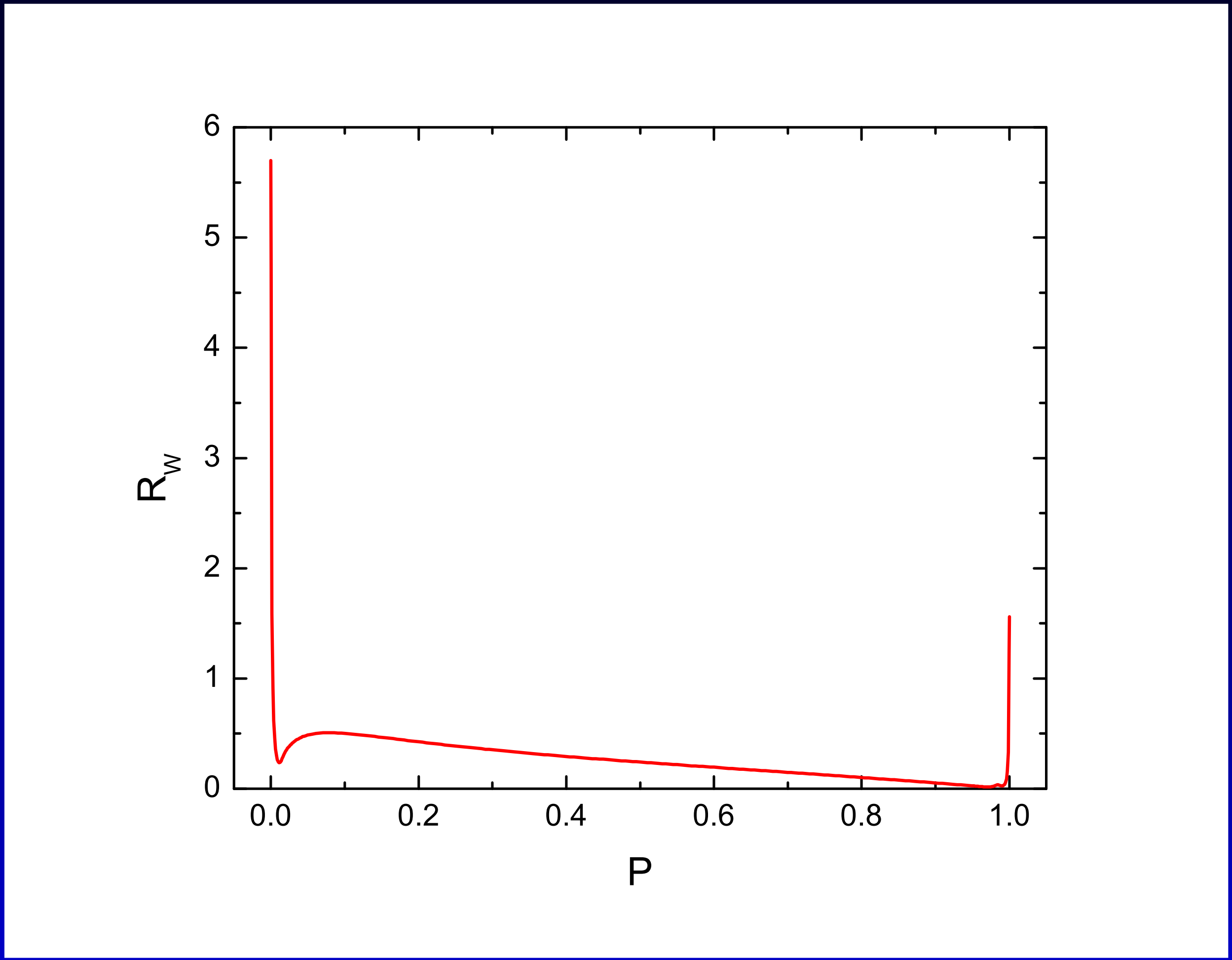
# Wilson Ratio




$$R_W = \frac{4}{\left(v_N^{\mathrm{b}} + 4v_N^{\mathrm{u}}\right)\left(\frac{1}{v_s^{\mathrm{b}}} + \frac{1}{v_s^{\mathrm{u}}}\right)}$$

The analytic result (red crosses) agrees with the numerical result obtained from the equation of state (TBA). Anomalous behaviour is found near $P = 0$ and $P = 1$.

# Wilson Ratio




$R_W$ vs polarization for $|\gamma| = 10$ at $T = 0.00001\epsilon_b$. The ratio exhibits anomalous enhancement near the two critical points due to the sudden change of the density of states, where the values $R_W = 5.53$ and $R_W = 1.52$ agree with the values obtained from the analytic expression.

**The WR of 1D Fermi gases can be measured in the lab**

# 5-CONCLUSIONS

## Exactly solvable models finding their way into the lab:

*Advanced experimental techniques in trapping and cooling atoms in 1D have provided the realization of exactly solved models in the lab. Some examples:*

- the Lieb-Liniger Bose gas

  *T. Kinoshita et al Science 2004, PRL 2005, Nature 2006; A. van Amerongen et al PRL2008; T. Kitagawa et al PRL 2010; J. Armijo et al PRL 2010*

- the super Tonks-Girardeau gas

  *E. Haller et al Science 2009*

- the degenerate spin-1/2 Fermi gas

  *Y. Liao et al Nature 2010; S. Jochim et al, Science 2011,PRL 2012: Deterministic preparation of few-fermion system; 2 fermions in a 1D harmonic trap*

- the two-component spinor Bose gas

  *J. van Druten et al arXiv:1010.4545*

# Concluding remarks:

- The Wilson ratio provides a measurable parameter to quantify different phases in 1D interacting fermions; it exhibits anomalous enhancement at the two critical points due to the sudden change in the density of states.
- Exactly solvable models are no longer *toy models*!
  They provide a precise description of quantum phases, thermodynamics, density profiles which are applicable to experiments with ultracold atoms confined to 1D tubes;
- It is clear that the Bethe Ansatz will continue to prosper as a valuable tool in the description of ultracold atoms.

## Collaborators

1. ***Prof. Murray T. Batchelor, ANU-Australia
2. ***Prof. Xiwen Guan, ANU-Australia and Wuhan-China
3. ***Prof. Jon Links, UQ-QLD-Australia
4. Prof. Eric Ragoucy, LAPTH-France
5. Prof. Itzhak Roditi, CBPF-Brazil
6. Prof. Arlei Tonel, Unipampa-Brazil
7. Dr. Ioannis Brouzos, Uni-Ulm-Germany
8. Dr. Gilberto S. Filho, CBPF-Brazil
9. ***Dr. Carlos Kuhn, ANU-Australia
10. ***Dr. Ian Marquette, UQ-QLD-Australia
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# THANK YOU !!!

## Dimensionless ratios:

- The Wiedemann-Franz ratio is the ratio of the electronic contribution of the thermal conductivity ($\kappa$) to the electrical conductivity ($\sigma$)
- The Kadowaki-Woods ratio is the ratio of A, the quadratic term of the resistivity and $\gamma$, the linear term of the specific heat.
- The Korringa ratio $K$ is given by $K = 1/(T_1 T K^2)$, where $T_1$ is the nuclear spin-lattice relaxation time, $K$ the Knight shift and $T$ the temperature.

# Equation of state:

From the TBA equations we can derive the equation of state for strong coupling:

$$\tilde{p} := p/(|c|\varepsilon_b) = \tilde{p}^b + \tilde{p}^u$$

where the pressures of the bound pairs and unpaired fermions are:

$$\tilde{p}^b = -\frac{t^{\frac{3}{2}}}{2\sqrt{\pi}} F^b_{3/2}\left[1 + \frac{\tilde{p}^b}{8} + 2\tilde{p}^u\right] + O(c^4)$$

$$\tilde{p}^u = -\frac{t^{\frac{3}{2}}}{2\sqrt{2\pi}} F^u_{3/2}\left[1 + 2\tilde{p}^b\right] + O(c^4)$$

with the functions $F^b_n$, $F^u_n$, $f^b_n$, $f^u_n$ defined by $F^{b,u}_n := \mathrm{Li}_n\left(-\mathrm{e}^{X_{b,u}/t}\right)$ and $f^{b,u}_n := \mathrm{Li}_n\left(-\mathrm{e}^{\nu_{b,u}/t}\right)$, with $\nu_b = 2\tilde{\mu} + 1$, $\nu_u = \tilde{\mu} + h/2$. Here $\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} z^k/k^s$ is the polylog function, with $I_0(x) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{x}{2}\right)^{2k}$

$$\frac{X_b}{t} = \frac{\nu_b}{t} - \frac{\tilde{p}^b}{t} - \frac{4\tilde{p}^u}{t} - \frac{t^{\frac{3}{2}}}{\sqrt{\pi}}\left(\frac{1}{16} f^b_{5/2} + \sqrt{2} f^u_{5/2}\right)$$

$$\frac{X_u}{t} = \frac{\nu_u}{t} - \frac{2\tilde{p}^b}{t} - \frac{t^{\frac{3}{2}}}{2\sqrt{\pi}} f^b_{5/2} + \mathrm{e}^{-h/t}\mathrm{e}^{-K} I_0(K).$$

# Thermodynamics

The thermodynamics of the model can be calculated from the standard thermodynamic relations.

$$\begin{aligned} M^z &= \left(\frac{\partial p(\mu, H, T)}{\partial H}\right)_{T,\mu}, \qquad s = \left(\frac{\partial p(\mu, H, T)}{\partial T}\right)_{\mu,H}, \\ \chi &= \left(\frac{\partial M^z}{\partial H}\right)_{n,T}, \qquad c_v = T\left(\frac{\partial s}{\partial T}\right)_{n,p}. \end{aligned} \tag{1}$$

where the pressures of the unpaired fermions and bound pairs are:

$$p^r = \frac{rT}{2\pi}\int_{-\infty}^{\infty} dk \ln(1 + \mathrm{e}^{-\epsilon^{\mathrm{r}}(k)/T})$$

with $\epsilon^{\mathrm{r}}$ the dressed energies which can be written in terms of polylogarithmic functions and $r = 1$ for unpaired fermions and $r = 2$ for paired fermions

# Susceptibility and specific heat at finite T:

From the equation of state, obtained from the TBA equations:

- Susceptibility:

$$\tilde{\chi} = -\frac{1}{8\sqrt{2\pi}\Delta^3}\left\{\frac{1}{\sqrt{t}}F^u_{-\frac{1}{2}}\left[1+\frac{3\sqrt{t}}{2\sqrt{\pi}}F^{A_b}_{1/2}+\frac{2\sqrt{2}t}{\pi}F^b_{\frac{1}{2}}F^u_{\frac{1}{2}}\right]+\frac{2\sqrt{2}\sqrt{t}}{\pi}F^b_{-\frac{1}{2}}\left(F^u_{\frac{1}{2}}\right)^2\right\}$$

where $\Delta = 1-\frac{\sqrt{t}}{2\sqrt{\pi}}F^b_{\frac{1}{2}}-\frac{t\sqrt{2}}{\pi}F^b_{\frac{1}{2}}F^u_{\frac{1}{2}}+\frac{t^{\frac{3}{2}}}{16\sqrt{\pi}}F^b_{3/2}.$

- Specific heat:

$c_v = c_v^b + c_v^u$

$$\begin{aligned}\frac{c_v^b}{|c|} &= \frac{1}{\sqrt{\pi}}\left\{-\frac{3}{8}\sqrt{t}F^b_{\frac{3}{2}}+\sqrt{t}F^b_{\frac{1}{2}}\left(\frac{\tilde{\nu}_b}{2t}+\frac{5}{8t}(4\tilde{p}^u+\tilde{p}^b)+\frac{\sqrt{2}\tilde{\nu}_u}{\sqrt{\pi t}}F^u_{\frac{1}{2}}+\frac{\tilde{\nu}_b}{2\sqrt{\pi t}}F^b_{\frac{1}{2}}\right)\right.\\ &\quad\left.-\frac{1}{2\sqrt{t}}F^b_{-\frac{1}{2}}\left(\frac{\tilde{\nu}_b}{t}(4\tilde{p}^u+\tilde{p}^b)+\frac{\tilde{\nu}_b^2}{t}+\frac{2\sqrt{2}\tilde{\nu}_b\tilde{\nu}_u}{\sqrt{\pi t}}F^u_{\frac{1}{2}}+\frac{3\tilde{\nu}_b^2}{2\sqrt{\pi t}}F^b_{\frac{1}{2}}+\frac{\tilde{\nu}_b^2}{\sqrt{2\pi t}}F^u_{\frac{1}{2}}\right)\right\}\\ \frac{c_v^u}{|c|} &= \frac{1}{\sqrt{\pi}}\left\{-\frac{3}{8\sqrt{2}}\sqrt{t}F^u_{\frac{3}{2}}+\frac{\sqrt{t}}{2\sqrt{2}}F^u_{\frac{1}{2}}\left(\frac{\tilde{\nu}_u}{t}+\frac{5}{2t}\tilde{p}^b+\frac{2\tilde{\nu}_b}{\sqrt{\pi t}}F^b_{\frac{1}{2}}\right)\right.\\ &\quad\left.-\frac{1}{2\sqrt{2}\sqrt{t}}F^u_{-\frac{1}{2}}\left(\frac{\tilde{\nu}_u^2}{t}+\frac{2\tilde{\nu}_u}{t}\tilde{p}^b+\frac{2\tilde{\nu}_b\tilde{\nu}_u}{\sqrt{\pi t}}F^b_{\frac{1}{2}}+\frac{2\tilde{\nu}_u^2}{\sqrt{2\pi t}}F^b_{\frac{1}{2}}\right)\right\}.\end{aligned}$$

## Integrability:

$\tau(u)$ is a generating function of conserved quantities

- The condition:

$$[\tau(u), \tau(v)] = 0$$

- represents a set of conservation laws:

$$[c_n, c_m] = 0$$

- where:

$$log\tau = \sum c_n v^n$$

- in many cases:

$$c_0 = iP, \quad c_1 = H, \quad \ldots\ldots$$

## Geometric Ansatz: the basic idea

- Change to Jacobi coordinates, which allows to remove the CM-coordinate;
- Change of coordinates to hyperspherical coordinates: radial component $\lambda$ and the angular part $\vec{\theta} = \{\theta_1, \theta_2, ...\theta_{N-2}\}$.
- The relative Hamiltonian now takes the form:

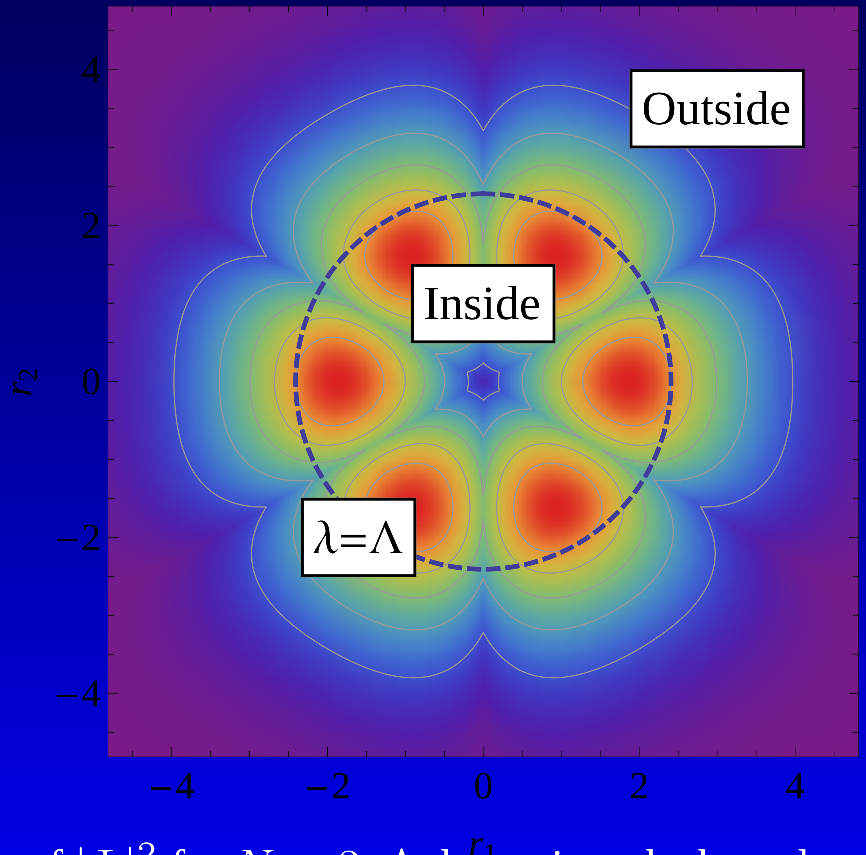
$$\hat{H}_{rel} = -\frac{\hbar^2}{2\mu}\nabla^2 + \frac{1}{2}\mu\omega^2\lambda^2 + c\sum_j \delta\left(d_j(\vec{\theta})\right).$$

For small $\lambda$, $\hat{H}_{rel}$ is approximately the one solved by the Bethe ansatz. For large $\lambda$ the behaviour is dominated by that of a harmonic oscillator.

Variational principle: $E_{GS} \leq \frac{\langle\psi|H_{rel}|\psi\rangle}{\langle\psi|\psi\rangle}$,

Geometric Ansatz for the trial wavefunction:

$$\Psi(\lambda,\vec{\theta}) = \begin{cases} \psi_B\left(\vec{\kappa},\lambda,\vec{\theta}\right) & \lambda < \Lambda \\ A(\vec{\theta})\exp\left(-\alpha(\vec{\theta})(\lambda^2-\Lambda^2)\right) & \lambda > \Lambda \end{cases}.$$




Schematic representation of $|\Psi|^2$ for $N = 3$. $\Lambda$ determines the boundary between 2 regions: inside (Bethe ansatz) and outside (asymptotic harmonic oscillator). The colors range from purple to red indicating respectively lower values and higher values of $|\Psi|^2$.

# Density Profiles

The equation of state can be reformulated within the local density approximation (LDA) by a replacement $\mu(x) = \mu(0) - \frac{1}{2}m\omega_x^2 x^2$ in which $x$ is the position and $\omega_x$ is the trapping frequency, the total particle number and the polarization are given by:

$$
\begin{aligned}
\frac{N}{a_x^2 c^2} &= \int_{-\infty}^{\infty} \tilde{n}(\tilde{x}) d\tilde{x}, \\
P &= \int_{-\infty}^{\infty} \tilde{n}_1(\tilde{x}) d\tilde{x} / (N/(a_x^2 c^2)).
\end{aligned}
$$

# Appendix:Bethe ansatz

A solvable or integrable quantum many-body system is one in which $N$-particle wave function may be explicitly constructed. In general, $N!$ plane waves are $N$-fold products of individual exponential phase factors $e^{\mathrm{i}k_i x_j}$, where the $N$ distinct wave numbers, $k_i$, are permuted among the $N$ distinct coordinates, $x_j$. Each of the $N!$ plane waves have an amplitude coefficient in each of regions. For example, in the domain $0 < x_{Q1} < x_{Q2} < \ldots < x_{QN} < L$, the wave function is written as

$$\psi = \sum_P A_{\sigma_1 \ldots \sigma_N}(P_1, \ldots, P_N | Q_1, \ldots, Q_N) \exp \mathrm{i}(k_{P1} x_{Q1} + \ldots + k_{PN} x_{QN})$$

- Continuity: $\psi_{x_{Qi} = x_{Qj}^-} = \psi_{x_{Qi} = x_{Qj}^+}$
- Schrödinger equation: $\mathcal{H}\psi = E\psi$
- two-body scattering relation: $A_{\sigma_1 \ldots \sigma_N}(P_i P_j | Q_i Q_j) = [Y_{ij}]_{\sigma_1 \ldots \sigma_N}^{\sigma'_1 \ldots \sigma'_N} A_{\sigma'_1 \ldots \sigma'_N}(P_j P_i | Q_i Q_j)$
- boundary conditions: $\psi(x_1, \ldots, x_i, \ldots, x_N) = \psi(x_1, \ldots, x_i + L, \ldots, x_N)$

## 1D tubes and experimental quantities:

Arrangement: Optical laser trap creates standing waves of slightly different frequencies in 2 orthogonal directions. The intersection of the standing waves creates 1D tubes.

Conditions for 1D:

a) highly elongated traps;

b) tunneling rate larger than the time of experiment.

- $N_1 \approx 120$, is the number of atoms per 1D tube in state $|1>$;
- $\frac{t}{k_B} \approx 17nK$ here $t$ is tunnelling rate;
- $\frac{\varepsilon_F}{k_B} \approx 1.2\mu K$, where $\varepsilon_F = N_1 \hbar\omega_z$ is 1D Fermi energy;
- $\omega_\perp/\omega_z = 1000$ where $\omega_z$ and $\omega_\perp$ are the axial and transverse confinement frequencies of an individual tube;
- $T \approx 175nK$.

It is necessary that $\varepsilon_F > t$ and $T > t$ to ensure that Fermi surface is 1D and avoid inter-tube tun.

# Three-coupled BEC model

$$\mathcal{H} = \Omega_2 \, (a_2^\dagger a_1 + a_1^\dagger a_2 + a_2^\dagger a_3 + a_3^\dagger a_2)$$
$$+\Omega \, (a_1^\dagger a_3 + a_3^\dagger a_1) + \mu \, n_1 + \mu \, n_3 + \mu_2 \, n_2,$$

- $(1)$: left well
- $(2)$: middle well
- $(3)$: right well
- $\Omega$: tunneling between the left and the right wells
- $\Omega_2$: left-middle and middle-right tunneling
- $\mu_2, \ \mu$: external potentials.

*A. Foerster and E. Ragoucy*, Nuclear Phys. **B777** (2007) 373;

## Appendix: Quantum dynamics:

- Temporal operator $U$:
  determines the time evolution of any physical quantity

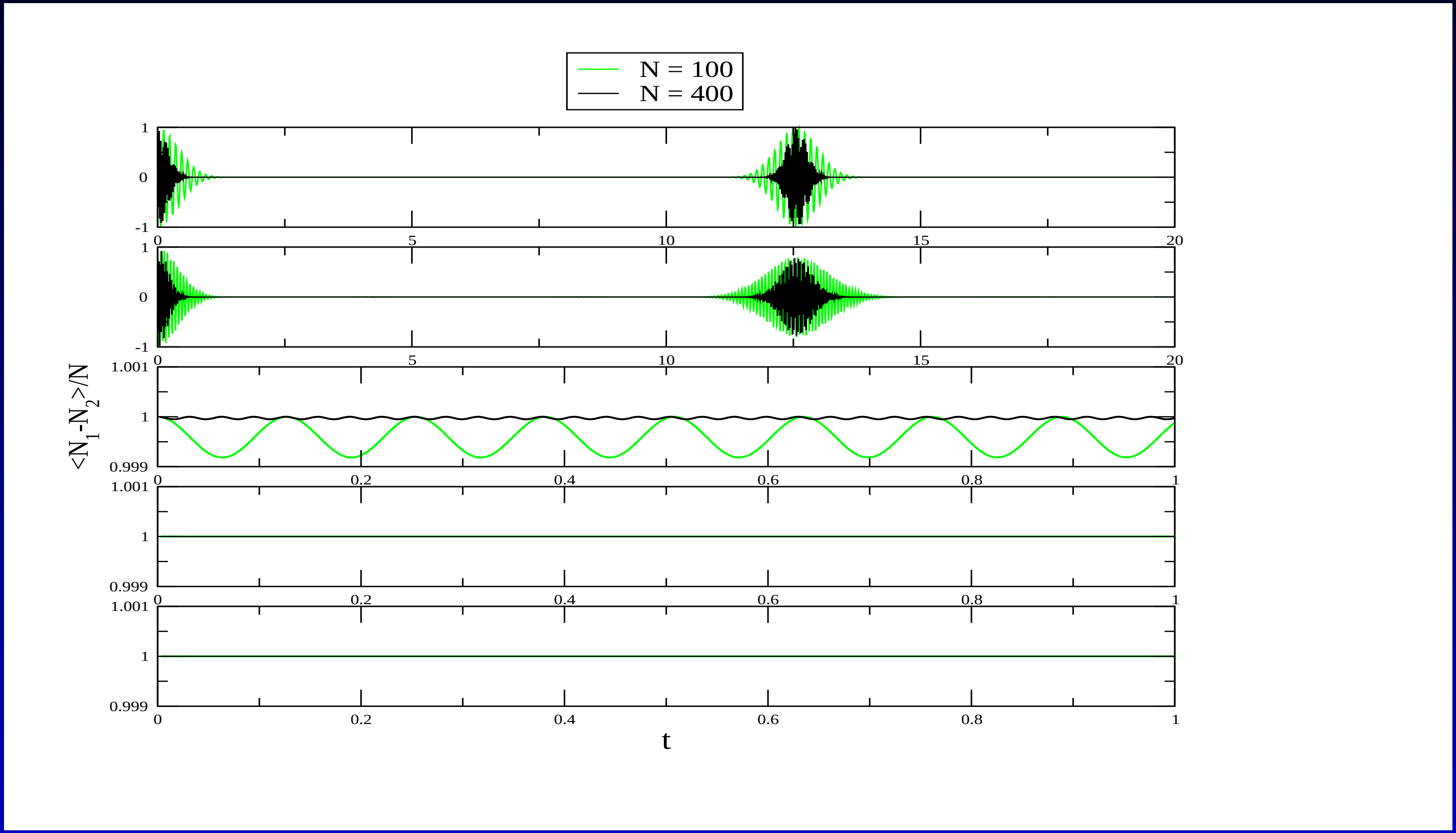
$$U = \sum_{n=0}^{N} e^{-i\lambda_n t} |\psi_n\rangle\langle\psi_n|$$

  $\{\lambda_n\}$ ; $\{|\psi_n\rangle\}$: eigenvalues and eigenvectors of $H$

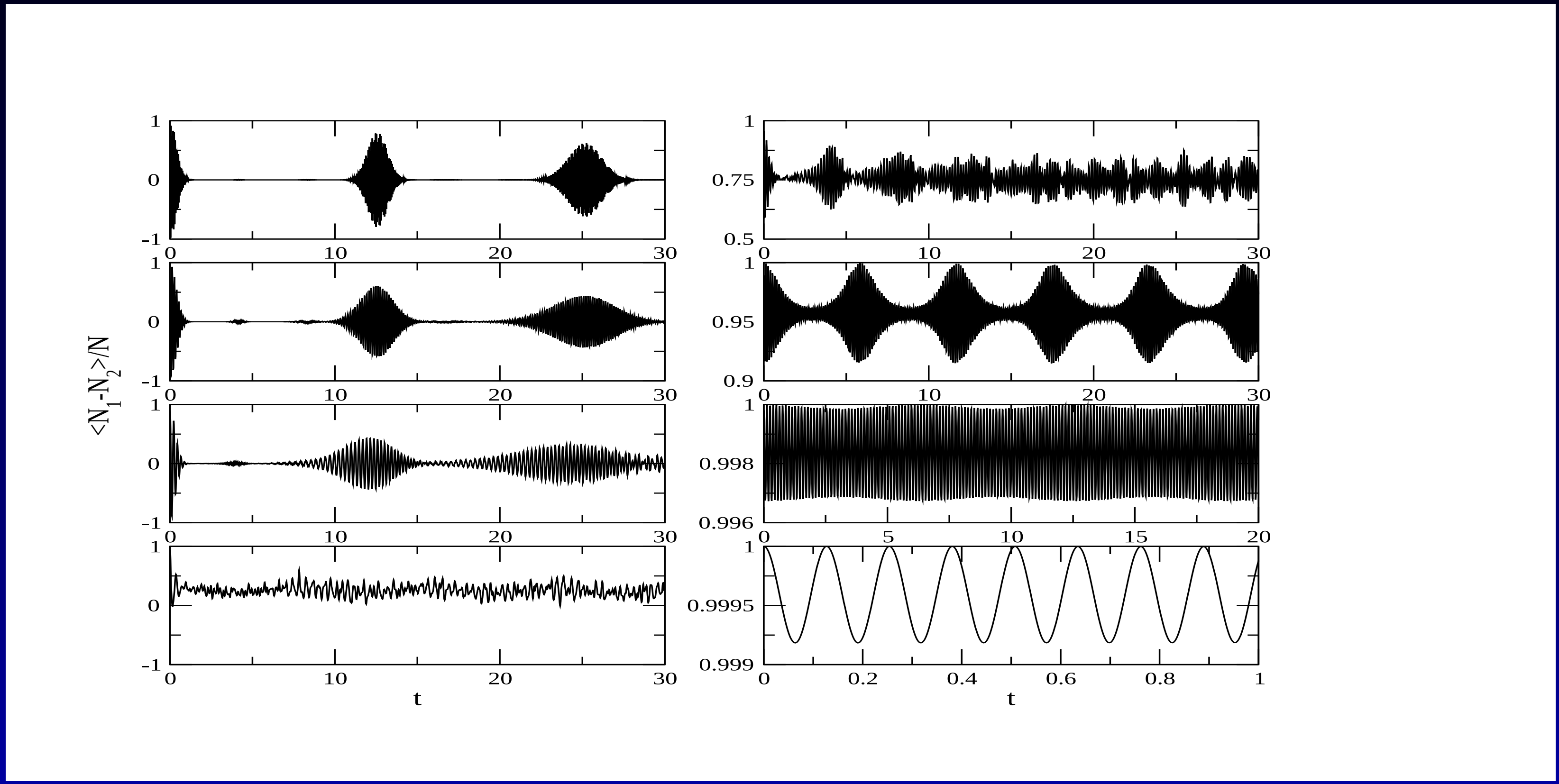
- Temporal evolution of any state: $|\psi(t)\rangle = U|\phi\rangle = \sum_{n=0}^{N} a_n e^{-i\lambda_n t}|\psi_n\rangle$,
  $a_n = \langle\psi_n|\phi\rangle$ and $|\phi\rangle$: initial state
- Expectation value of any operator $A$
  $\langle A\rangle = \langle\psi(t)|A|\psi(t)\rangle$
- Imbalance population
  $A = (N_1 - N_2)/N$

*Plot the time evolution of the expectation value of the imbalance population for different ratios of the coupling $K/\mathcal{E}_J$*

# Appendix: Dynamical regimes:




$$\frac{K}{\mathcal{E}_J} = \frac{\mathbf{1}}{\mathbf{N^2}}, \quad \frac{\mathbf{1}}{\mathbf{N}}, \quad \mathbf{1}, \quad \mathbf{N}, \quad \mathbf{N^2}$$




$$\frac{K}{\mathcal{E}_J} = \frac{1}{N},\ \frac{2}{N},\ \frac{3}{N},\ \frac{\mathbf{4}}{\mathbf{N}},\ \frac{5}{N},\ \frac{10}{N},\ \frac{50}{N},\ \mathbf{1}$$

$$\lambda_t = 2 \Rightarrow \frac{K}{\mathcal{E}_J} = \frac{4}{N}, ; \Delta\mu = 0$$

**Tunneling X Self-trapping**