Neveu Schwarz Indecomposables

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Outline

1. Background
   - Neveu Schwarz Algebra
   - Verma Modules
   - Singular Vectors
2. Fusion
   - Coproduct Formulae
   - An example
   - Towards a Classification of Indecomposables

Neveu Schwarz Algebra

$\mathbb{Z}_2$ graded Lie algebra

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Commutation Relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m,-n} \qquad m, n \in \mathbb{Z}$$

$$[L_m, G_n] = \left(\frac{m}{2} - n\right) G_{m+n} \qquad m \in \mathbb{Z}, n \in \mathbb{Z} + \frac{1}{2}$$

$$\{G_m, G_n\} = 2L_{m+n} + \frac{C}{3}(m^2 - \frac{1}{4})\delta_{m,-n} \qquad m, n \in \mathbb{Z} + \frac{1}{2}$$

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$$\mathrm{SVir} = \mathfrak{n}^- \oplus \underbrace{(\mathbb{C}L_0 \oplus \mathbb{C}C) \oplus \mathfrak{n}^+}_{\mathfrak{b}^+}$$

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This is a free module over $\mathbf{U}(\mathfrak{n}^-)$ with a basis of ordered monomials

$$\dots L_{-2}^{n_2} G_{-3/2}^{k_2} L_{-1}^{n_1} G_{-\frac{1}{2}}^{k_1} v_{h,c}$$

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We call $v_{h,c}$ a **highest weight vector.**

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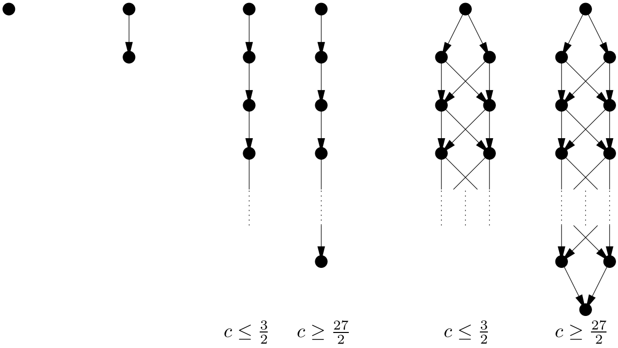
A singular vector is found at grade $\frac{rs}{2}$ and generates a Verma submodule, $V(h_{r,s}+\frac{rs}{2}, c)$.

Classification of SVir Submodule Structure



$Q_{r,s}$ module

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$Q_{r,s}$ is the quotient of $V(h_{r,s}, c)$ by the submodule generated by the singular vector at grade $\frac{rs}{2}$

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E.g. For the module $V(h_{3,1} = 0, c = 0)$ we have the singular vector

$$(L_{-1}G_{-1/2} - \frac{1}{2}G_{-3/2})v_{0,0}$$

at grade $3/2$. We will refer to the quotient as $Q_{3,1}$.

Fusion

Decompose $Q_{r_1,s_1} \otimes Q_{r_2,s_2} \sim$ OPE's in CFT

Vectors in module $\sim$ fields in CFT

We need an appropriate coproduct to make this connection

Coproduct Formulae

We have the mode expansion for each generator in SVir

$$G(w) = \sum_{n \in \mathbb{Z}+3/2} w^{n-3/2} G_{-n}, \qquad L(w) = \sum_{n \in \mathbb{Z}+2} w^{n-2} L_{-n}$$

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We consider the action of a mode on the product space as

$$\Delta(S_n)(v_1 \otimes v_2) = \oint_C dw w^n S(w) V(v_1, \zeta) V(v_2, z) \Omega \qquad S = L, G$$

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From this we derive two coproduct formulae for each generator S_n

$$\Delta(S_n) = \sum_{m=1-h}^{\infty} a_m (S_m \otimes \mathbf{1}) + \epsilon_1 \mathbf{1} \otimes S_n$$

$$\tilde{\Delta}(S_n) = S_n \otimes \mathbf{1} + \epsilon_1 \sum_{m=1-h}^{\infty} b_m (\mathbf{1} \otimes S_m)$$

Algebraic Formulation

We consider the fusion product as follows

$$(Q_{r_1,s_1} \times Q_{r_2,s_2})_f := (Q_{r_1,s_1} \otimes Q_{r_2,s_2})/(\Delta - \tilde{\Delta})$$

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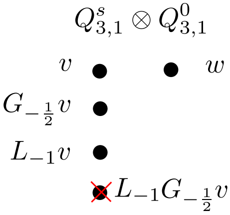
where we have the **special subspace** defined as follows

$$Q^s = Q/\left\langle L_{-m}, G_{-n} | m \geq 2, n \geq \tfrac{3}{2} \right\rangle Q$$

Fusion Example

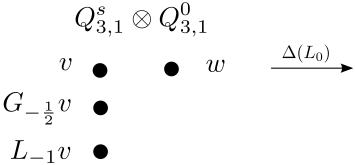


$$L_{-1}G_{-\frac{1}{2}}v = G_{-\frac{3}{2}}v$$

as singular vector set to zero.

Here we have a tensor basis consisting of $\{v \otimes w, G_{-\frac{1}{2}}v \otimes w, L_{-1}v \otimes w\}$. We calculate $\Delta(L_0)$ wrt this basis

Fusion Example



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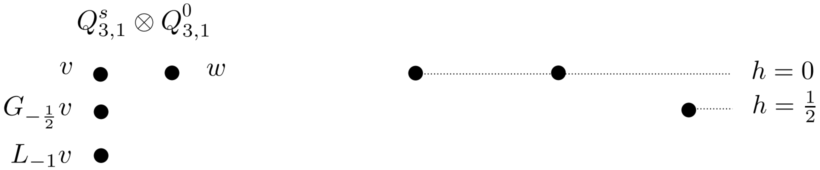

Fusion Example

$$Q^{s}_{3,1} \otimes Q^{1/2}_{3,1}$$

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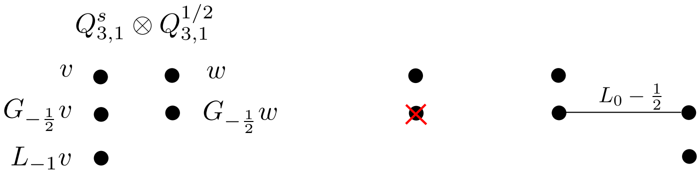

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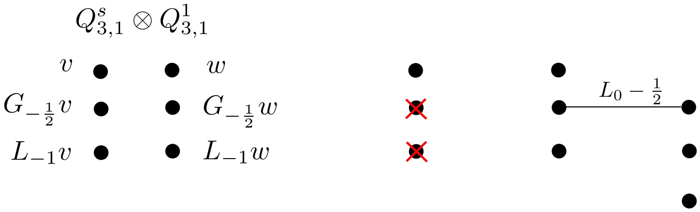

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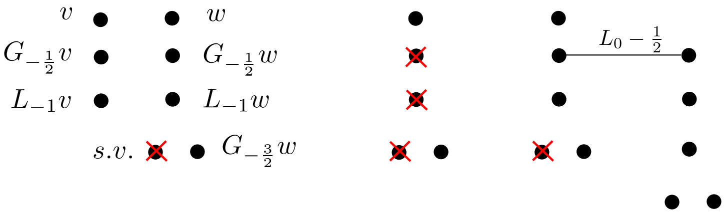

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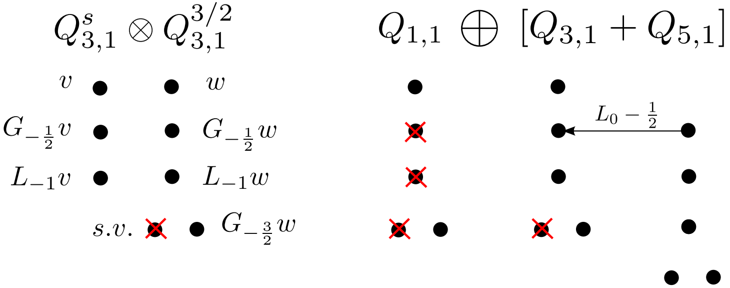


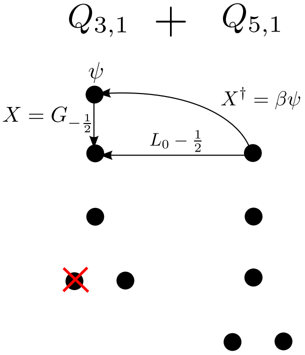
Category $\mathcal{O}$

1. M is finitely generated
2. For any $w \in M$, $U(\mathfrak{n}^+)w$ is finite dimensional.
3. C acts on M by multiplication by c
4. L_0 acts semisimply on M

We are interested in the fusion products that lead to representations outside Category $\mathcal{O}$. In particular, where L_0 can have Jordan cells of rank at most two.We would like to classify such indecomposables.

Identification of Indecomposables

In the previous example we saw the following structure



Identification of Indecomposables

Kac Table for c=0

		s				
		1	2	3	4	5
r	1	0	0.5625	1.5	28/9	5
	2	0/1	1/5	1/1	11/5	4/1
	3	0	1/9	0.5	14/9	3
	4	1/4	- 1/9	1/4	8/9	9/4
	5	0.5	1/9	0	5/9	1.5

Results

Table: c=0 module decomposition

$\times_f$	$Q_{1,1}$	$Q_{3,1}$	$Q_{5,1}$
$Q_{1,1}$	$Q_{1,1}$	$Q_{3,1}$	$Q_{5,1}$
$Q_{3,1}$		$Q_{1,1} \oplus [Q_{3,1} + Q_{5,1}]$	$[Q_{3,1} + Q_{5,1}] \oplus Q_{1,7}$
$Q_{5,1}$			$[Q_{1,1} + Q_{7,1}]_1 \oplus [Q_{3,1} + Q_{5,1}]_2 \oplus Q_{9,1}$

Table: c=0 beta numbers

$\times_f$	$Q_{1,1}$	$Q_{3,1}$	$Q_{5,1}$	$Q_{7,1}$
$Q_{1,1}$				
$Q_{3,1}$		$\beta = -1$	$\beta = -1$	$\beta = -15$
$Q_{5,1}$			$\beta_1 = -\frac{1}{4}, \beta_2 = -1$	

Table: c=3/2 module decomposition

$\times_f$	$Q_{3,1}$	$Q_{5,1}$
$Q_{3,3}$	$Q_{1,3}+Q_{3,3}+Q_{5,3}$	
$Q_{3,5}$	$Q_{1,5}+Q_{3,5}+Q_{5,5}$	
$Q_{3,7}$	$Q_{1,7}+Q_{3,7}+Q_{5,7}$	
$Q_{5,5}$		$Q_{1,5}+Q_{3,5}+Q_{5,5}+Q_{7,5}+Q_{9,5}$

Table: c=3/2 beta numbers

$\times_f$	$Q_{3,1}$	$Q_{5,1}$
$Q_{3,s}$	$\beta_i=0$	
$Q_{5,5}$		$\beta_i=0$

$$\left(Q_{r,1}\times Q_{r',1}\right)_f \cong \bigoplus_{r''=|r-r'|+1}^{r+r'-1} Q_{r'',1}$$

Results

Table: c= -5/2 module decomposition

$\times_f$	$Q_{3,1}$	$Q_{5,1}$
$Q_{3,1}$	$Q_{1,1} + Q_{5,1} \oplus Q_{3,1}$	$Q_{3,1} \oplus [Q_{5,1} + Q_{7,1}]$
$Q_{5,1}$		$[[Q_{1,1} + Q_{5,1}]_1 + Q_{7,1}]_2 \oplus Q_{3,1} \oplus Q_{9,1}$
$Q_{7,1}$	$[Q_{5,1} + Q_{7,1}] \oplus Q_{9,1}$	

Table: c=-5/2 beta numbers

$\times_f$	$Q_{3,1}$	$Q_{5,1}$
$Q_{3,1}$	$\beta = 0$	$\beta = -2$
$Q_{5,1}$		$\beta_1 = 0, \beta_2 = -2$
$Q_{7,1}$	$\beta = -2$	

$$(Q_{r,1} \times Q_{r',1})_f \cong \bigoplus^{r+r'-1} Q_{r'',1}$$

Things to do

1. Repeat these computations for the Ramond Sector
2. Classify indecomposables in the language of module extensions