ANZAMP Conference 2013

Alternative Tableau and the Asymmetric Simple Exclusion Process

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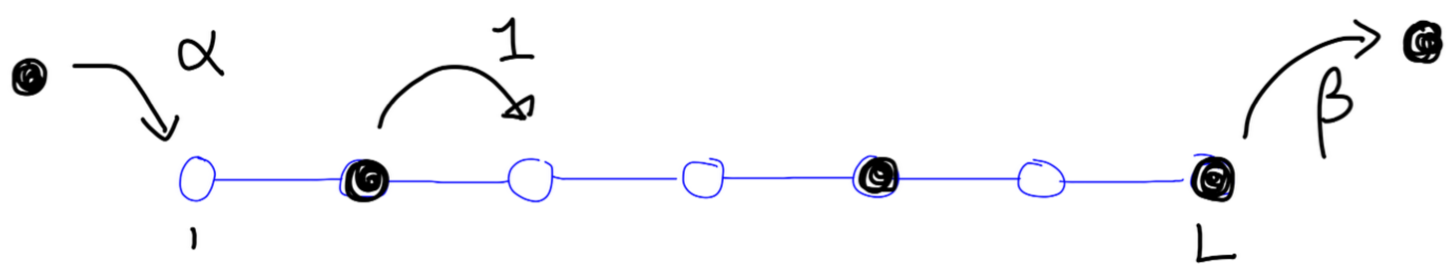
AUSTRALIAN RESEARCH COUNCIL
Centre of Excellence for Mathematics and Statistics of Complex Systems

The Asymmetric Simple Exclusion Process

- Particle hopping model

$\bar{\alpha} = \frac{1}{\alpha}$

$\bar{\beta} = \frac{1}{\beta}$



- State: $\underline{\tau} = (\tau_1, \tau_2, \dots, \tau_L)$ $\quad \tau_i = \begin{cases} 1 & \text{if particle on site } i \\ 0 & \text{otherwise} \end{cases}$

- Probability distribution: $\mathrm{Prob}(\underline{\tau}; t)$ = probability in state $\underline{\tau}$ at time t, given an initial state

- Master equation:

$$\frac{\partial}{\partial t} \mathrm{Prob}(\tau; t) = \sum_{\tau'} \mathrm{Prob}(\underline{\tau} \leftarrow \underline{\tau}'; t) - \sum_{\tau'} \mathrm{Prob}(\underline{\tau}' \leftarrow \underline{\tau}; t)$$

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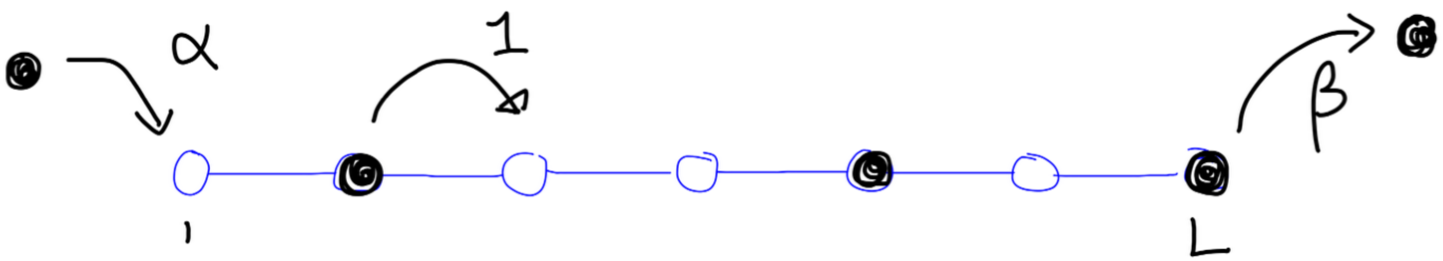

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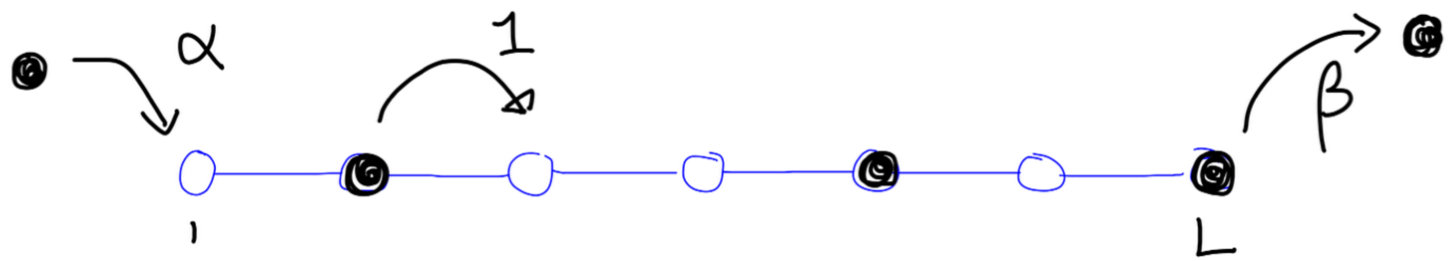

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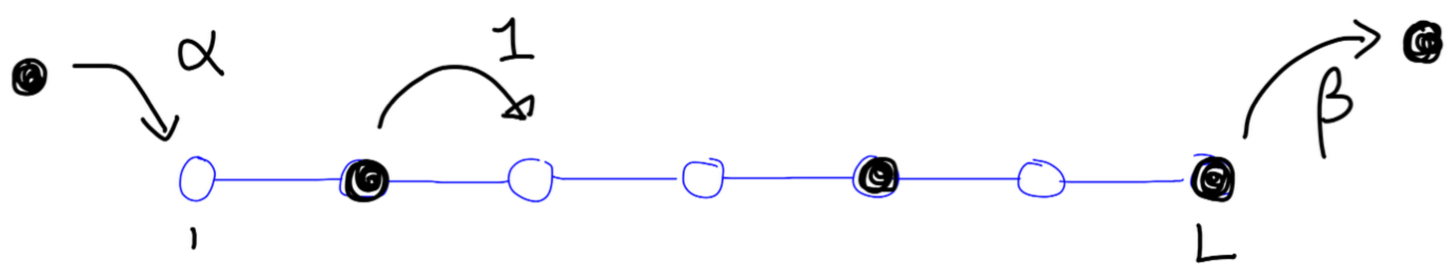

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- Stationary State : $\frac{\partial}{\partial t} P_0(\tau; t) = 0$

- Solved Derrida et. al 1993 matrix product Ansatz

$$P_0(\tau) = \langle V| \prod_{i=1}^{L} (\tau_i D + (1-\tau_i)E) |W\rangle / Z_L \qquad D, M \quad \text{matrices.}$$

Z_L = normalization.

- Represents system: $D \to$ particle on site i
  $E \to$ no particle on site i.

$$\langle V| \quad E \cdot D \cdot E \cdot E \cdot D \cdot E \cdot D \quad |W\rangle$$

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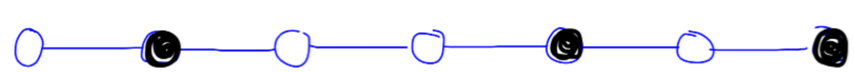

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- Stationary if: ① Matrix equation: $DE = D + E$

  ② Eigenvectors: $\langle V| E = \bar{\alpha} \langle V|$ $\quad D|W\rangle = \bar{\beta}\, |W\rangle$

- Many ways forward

1/ Find explicit representations for D, E matrices and eigenvectors

$$D_1 = \begin{bmatrix} 1 & 1 & & 0 \\ & 1 & 1 & \\ & & 1 & 1 \\ 0 & & & \ddots \end{bmatrix} \qquad E_1 = \begin{bmatrix} 1 & & & 0 \\ 1 & 1 & & \\ & 1 & 1 & \\ 0 & & 1 & \ddots \end{bmatrix}$$

$$\langle V| = \kappa \left(1, a, a^2, \dots\right) \qquad |W\rangle = \kappa \begin{pmatrix} 1 \\ b \\ b^2 \\ \vdots \end{pmatrix}$$

$$a = 1 - \bar{\alpha} \qquad b = 1 - \bar{\beta} \qquad \kappa^2 = \bar{\alpha} + \bar{\beta} - \bar{\alpha}\bar{\beta}$$

- Infinite dimensional.

- Evaluate products algebraically or combinatorially (via lattice paths)

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Ex: $Z_2 = \langle V|(D+E)^2|W\rangle$

$$(D+E)^2 = D^2 + DE + ED + E^2$$
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$$\Rightarrow \langle V|(D+E)^2|W\rangle = \bar\beta^2 + \bar\beta + \bar\alpha + \bar\alpha\bar\beta + \bar\alpha^2$$ $\langle V|W\rangle = 1$

- Independent of the order of substitutions (Blythe & Evans 2008)

Thus: $P_0(\tau)$ = rational function of α, β.

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$c_{n,m}, d_{n,m} \in \mathbb{Z}$.

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- Reformulate as a polynomial algebra:

  – Represent:

  - States by words in alphabet $A=\{d,e\}$

    $\underline{\tau} \longrightarrow de^{\underline{\tau}} = \prod_{i=1}^{L} d^{\tau_i} e^{1-\tau_i}$ eg $\longrightarrow de^{0110} = edde$.

  - Probability distribution by non-commutative polynomial ring:

    $R[d,e]$ where $R=\mathbb{Z}(\bar{\alpha},\bar{\beta})$ ie rational functions of $\bar{\alpha},\bar{\beta}$.

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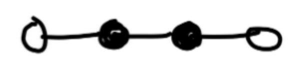
eg  $\longrightarrow de^{0110} = edde.$

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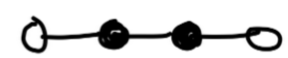
eg  $\longrightarrow de^{0110} = edde.$

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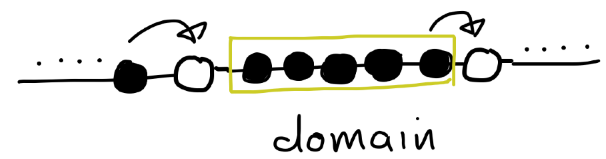

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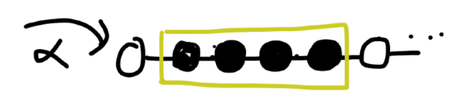

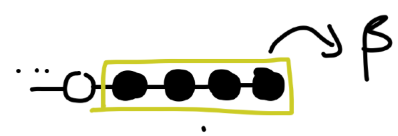

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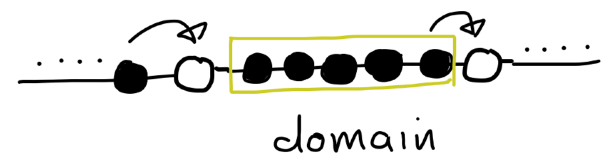

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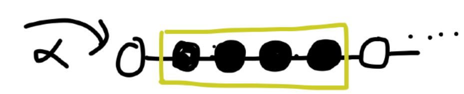

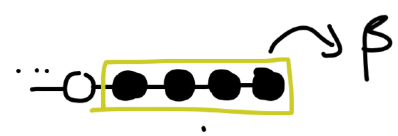

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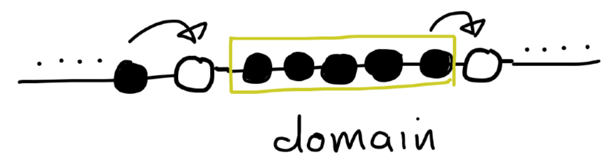
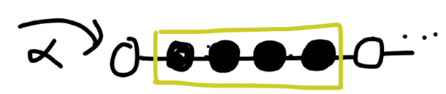
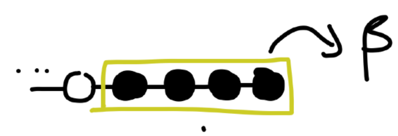
  $\hat{\mathcal{L}}_{BULK} \rightsquigarrow$ particle hopping 

  $\hat{\mathcal{L}}_{BDY} \rightsquigarrow$ particle hopping 



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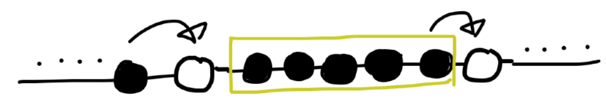
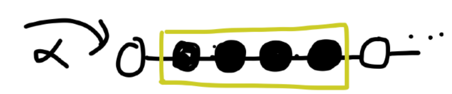
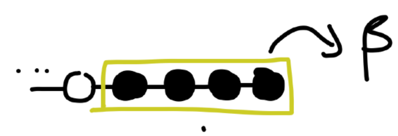
  $\hat{\mathcal{L}}_{BULK} \rightsquigarrow$ particle hopping 

  $\hat{\mathcal{L}}_{BDY} \rightsquigarrow$ particle hopping  

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• Boundary terms (ie $\mathcal{L}_{BDY}$) do <u>not</u> cancel in $\mathbb{Z}[\alpha,\beta]/I$

• Need an unusual ring: $\mathbb{Z}[\bar\alpha : \bar\beta]$ "One Transit ring"

- similar to non-commutative polynomial ring

- monomials all of the form $\bar\alpha^n \bar\beta^m$

- multiplication: $\mathbb{Z}[\bar\alpha:\bar\beta] \times \mathbb{Z}[\bar\alpha:\bar\beta] \to \mathbb{Z}[\bar\alpha:\bar\beta]$ is defined by

$$(\bar\alpha^n \bar\beta^k)\cdot(\bar\alpha^l \bar\beta^m) = \sum_{i=1}^{l} \binom{k+l-i-1}{k-1} \bar\alpha^{n+i} \bar\beta^m + \sum_{j=1}^{k} \binom{k+l-j+1}{l-1} \bar\alpha^n \bar\beta^{m+j}$$

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- The one transit ring gives boundary cancellation:

Theorem: Let $\mathbb{Z}[\bar{\alpha}:\bar{\beta}]$ be a one transit ring and $\theta: \mathbb{Z}[d,e]/I \longrightarrow \mathbb{Z}[\bar{\alpha}:\bar{\beta}]$ a ring homomorphism defined by $\theta(\overline{d^n e^m}) = \bar{\alpha}^n \bar{\beta}^m$. Then

$$\theta\left(\left(\hat{\mathcal{L}}_{BULK} + \hat{\mathcal{L}}_{BDY}\right) \cdot de^{I} \right) = 0.$$

ie $\quad \hat{\mathcal{L}} \cdot de^{I} \in \ker\theta$.

- Algebraically we get stationarity from:

$$\mathbb{Z}[d,e] \xrightarrow[\gamma]{d+e-de} \mathbb{Z}[d,e]/I \xrightarrow[\theta]{} \mathbb{Z}[\bar{\alpha}:\bar{\beta}]$$

$\Rightarrow$ Given state $\underline{I} \longrightarrow de^{I}$ $\qquad$ need to compute $\qquad \theta \circ \gamma\left(de^{I} \right)$

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<u>Theorm:</u> Let $\mathbb{Z}[\bar{\alpha}:\bar{\beta}]$ be a one transit ring and $\theta: \mathbb{Z}[d,e]/I \longrightarrow \mathbb{Z}[\bar{\alpha}:\bar{\beta}]$ a ring homomorphism defined by $\theta(\overline{d^n e^m}) = \bar{\alpha}^n \bar{\beta}^m$. Then

$$\theta\left((\hat{\mathcal{L}}_{BULK} + \hat{\mathcal{L}}_{BDY}) \cdot de^{\underline{I}} \right) = 0.$$

ie $\quad \hat{\mathcal{L}} \cdot de^{\underline{I}} \in \ker \theta$.

- Algebraically we get stationarity from:

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$\underline{\text{Th}}^{\text{m}}$(B) The set $B = \{ e^n d^m \mid n,m \in \mathbb{N}_0 \}$, $\mathbb{N}_0 = \{0,1,2,\dots\}$ is a linearly independent set which generates $\mathbb{Z}[d,e]/I$. Furthermore $\{ \bar{\alpha}^n \bar{\beta}^m \mid n,m \in \mathbb{N}_0 \}$ is a basis for $\mathbb{Z}[\bar{\alpha} : \bar{\beta}]$.

$\Rightarrow$ Any polynomial in $\mathbb{Z}[d,e]/I$ <u>uniquely</u> given by basis coeff: $c_{n,m}$

$$de^{?} = \sum_{n,m} c_{n,m} e^n d^m$$

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Alternative tableau and basis coefficients

- Planarization of quadratic algebras (Viennot)

- Combinatorial representation of substitution process

$$de = 1 \cdot d + e \cdot 1$$

- Example: $(de)^3 \equiv d^3 + 2d^2 + 2d + 2ed + 2e + 2e^2 + ed^2 + e^2d + e^3$

$$(de)^3 \longrightarrow d^2e \longrightarrow de \longrightarrow d$$

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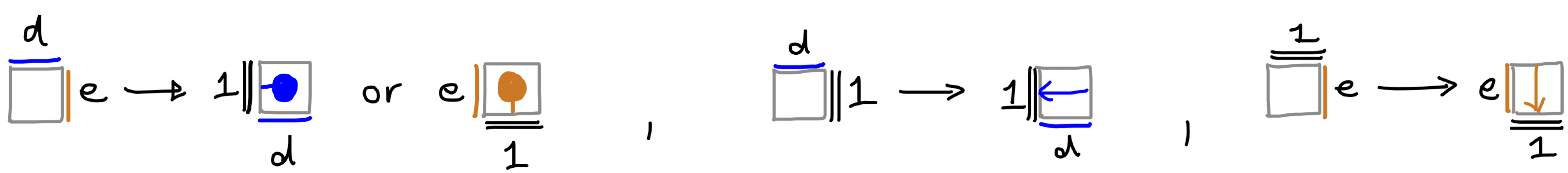

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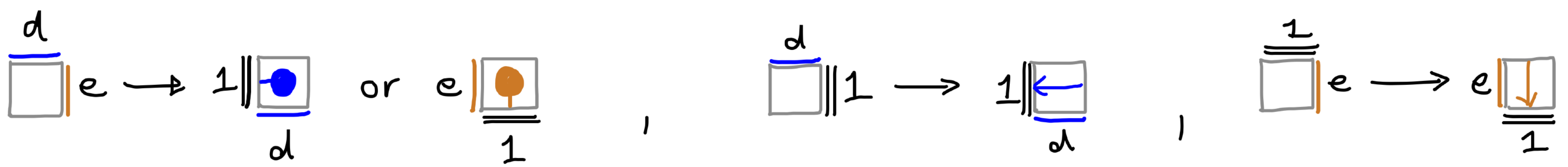

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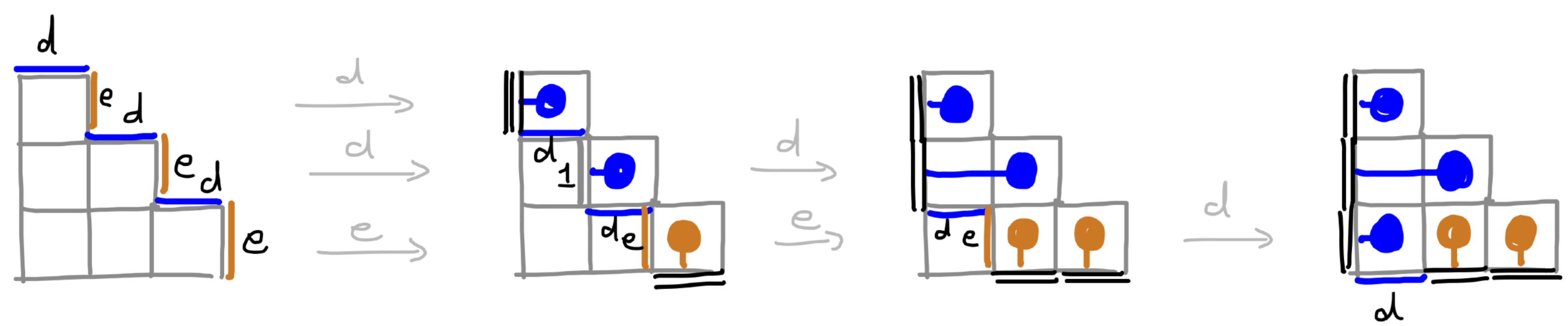

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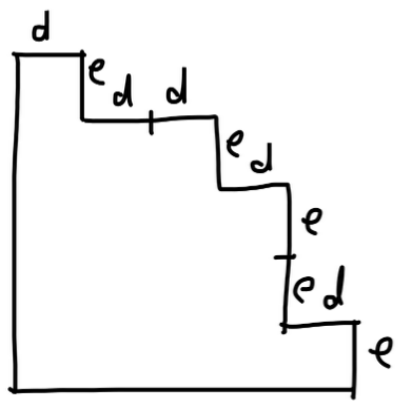
State τ $\rightarrow$ Shape of tableau $\rightarrow$ east boundary:



$\tau = 1011010010$

$$\tau_i = \begin{cases} 1 \rightarrow \text{down edge} \\ 0 \rightarrow \text{horizontal edge} \end{cases}$$

– Fill shape {blue dot, red dot} such that:

① Red and blue dots on east boundary
② No dot to west of a blue dot
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- Tableau bijects to monomial: $e^n d^m$

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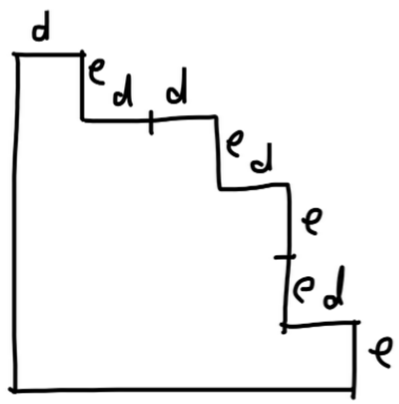

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- Several tableau $\longrightarrow$ same monomial.

Theorem (B. and Moore). The coset representing the stationary state τ, $\overline{de^{\tau}}$ contains a unique element of the form:

$$\sum_{n,m} c_{n,m} e^n d^m$$

where

$c_{n,m}$ = # tableau of shape τ with n red rows and m blue columns.

- Many combinatorial results

Theorem (B. + Moore) The set of alternative tableau of shape $(10)^L$ are in in bijection with Dyck paths of length $2L$.

Proof I: biject tableau sequence of heights

Proof II: Construct an embedded bijection.

$L=3$ $\qquad (de)^3 \equiv d^3+2d^2+2d+2ed+2e+2e^2+ed^2+e^2d+e^3$

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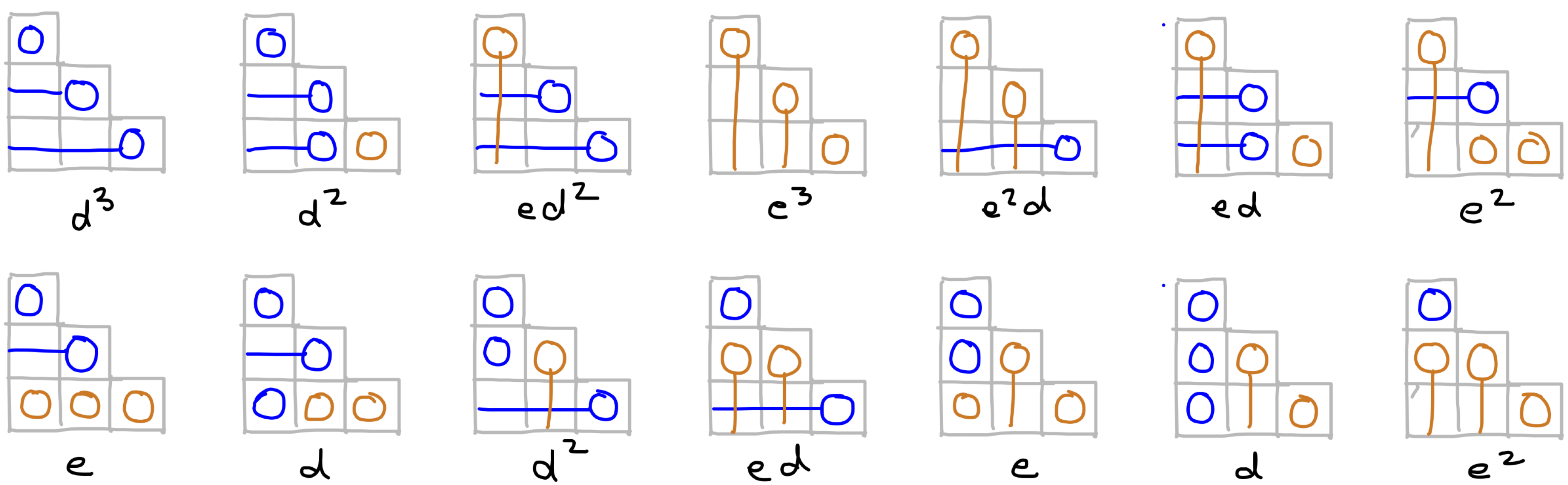

- Proof II: Construct an "embedded" bijection

- If $|A|=|B|=n \implies n!$ possible bijections: $T: A \to B$.

- Some more "natural"...

- Example: balanced brackets $\longleftrightarrow$ nested links

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$n=6$ $n=6$

Both the following are valid mappings: (there are 132! possible)

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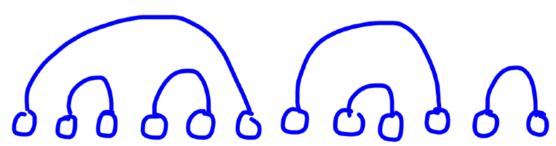

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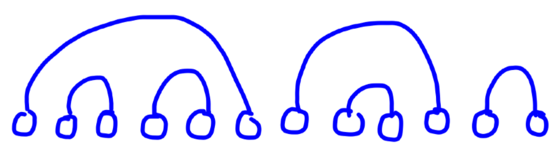

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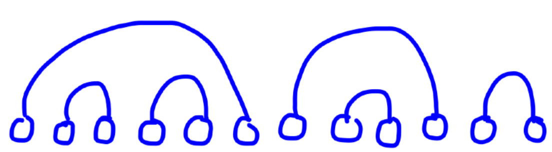

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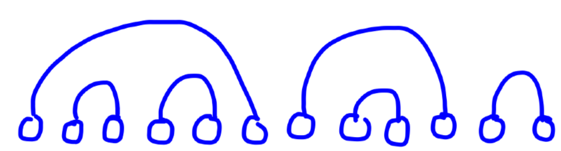

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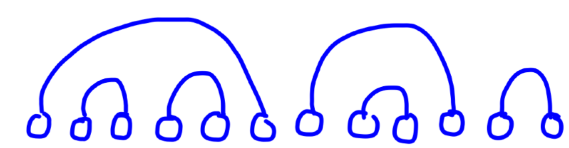

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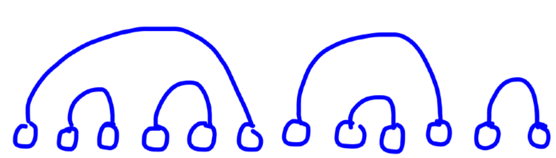

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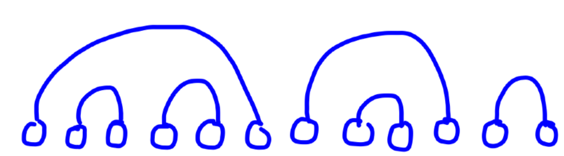

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o Define "natural" as an <u>isomorphism</u>:

- preserve Catalan product: $*_R : C_n \times C_m \longrightarrow C$

example:

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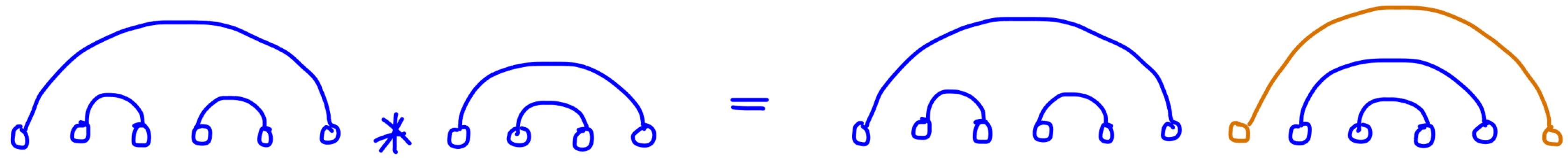

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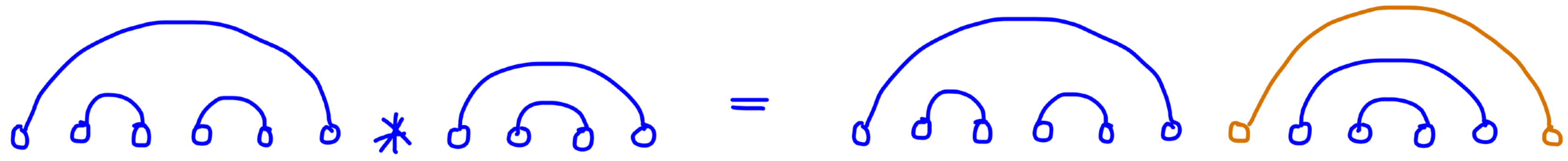

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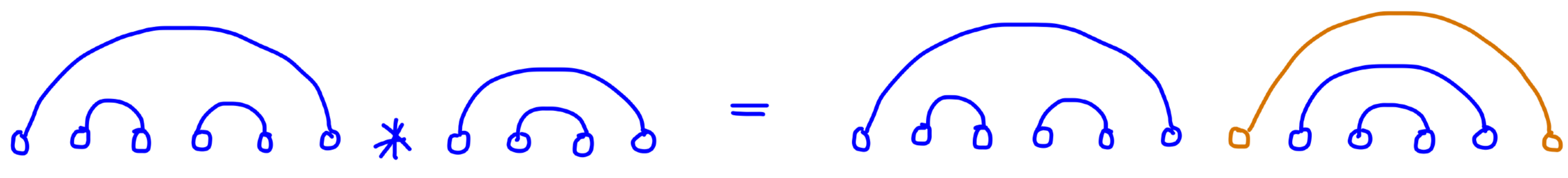

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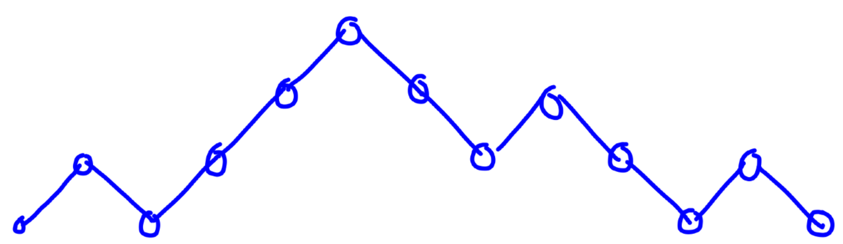

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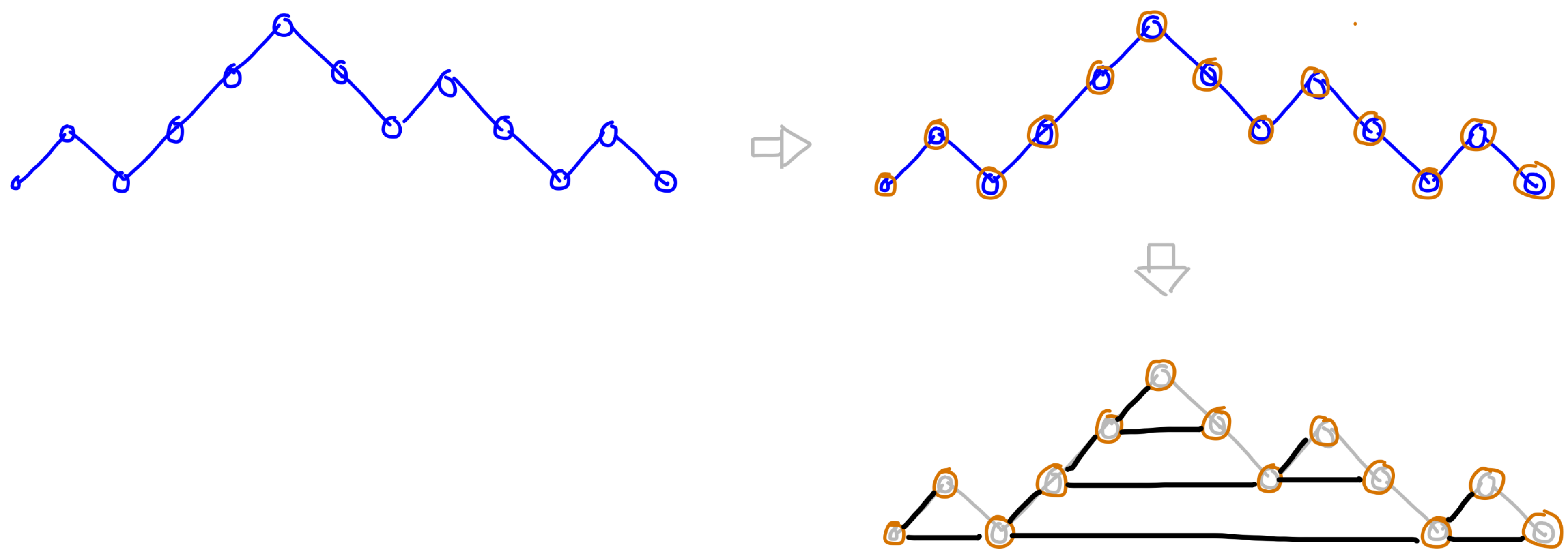

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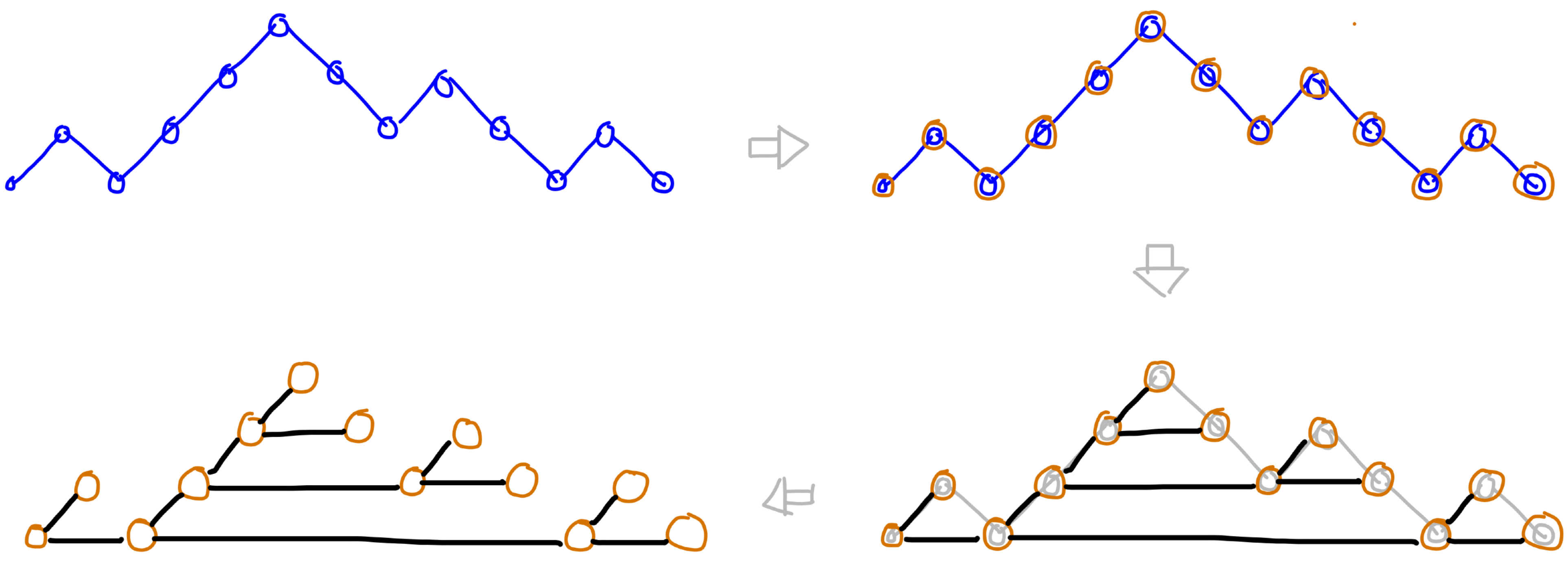

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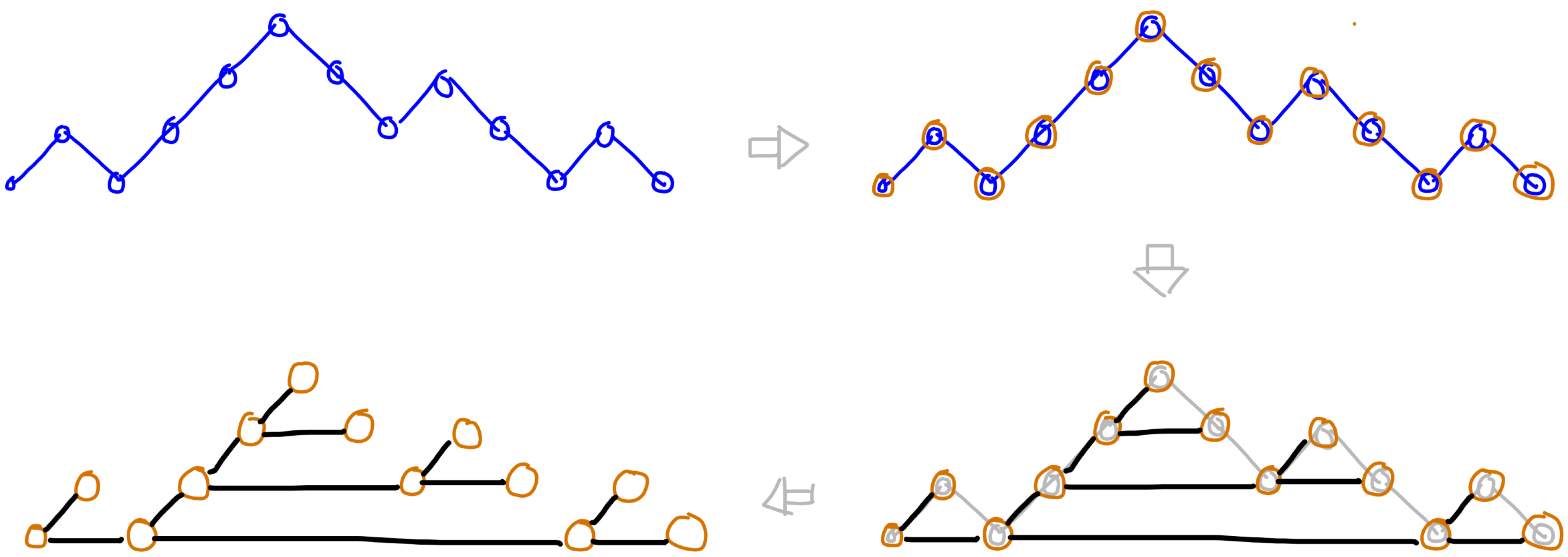

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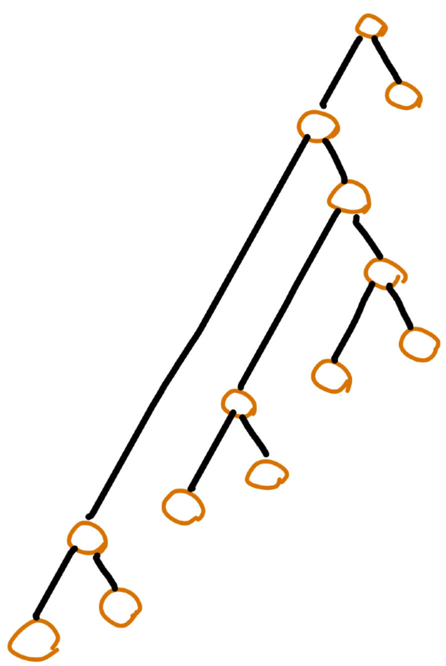
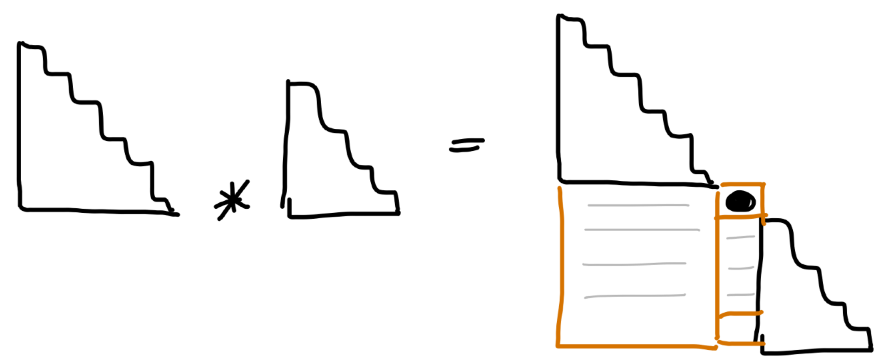


- Back to alternative tableaux
- Define tableaux product:



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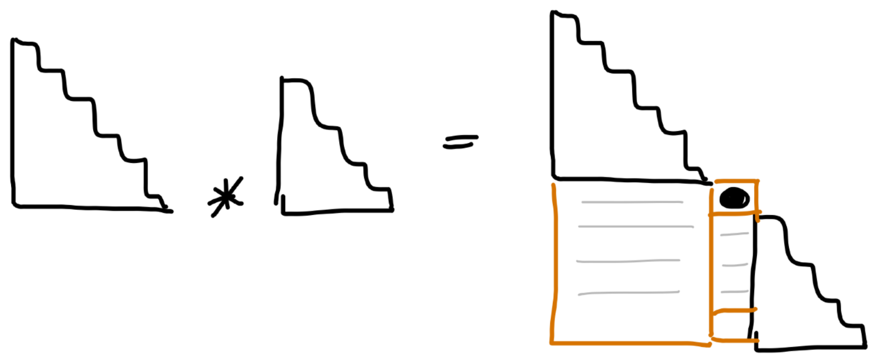

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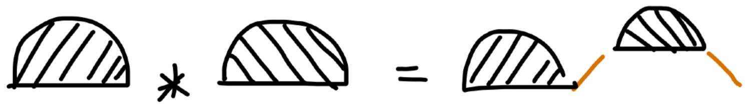

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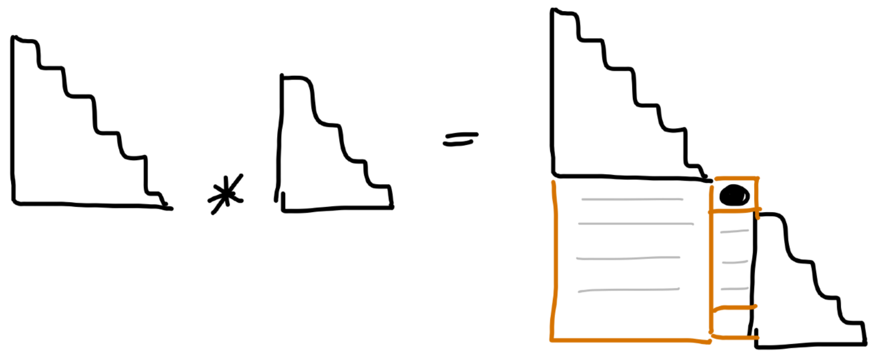

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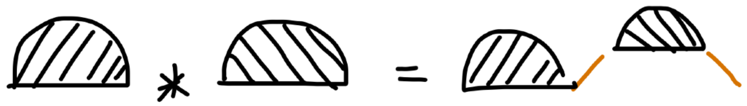

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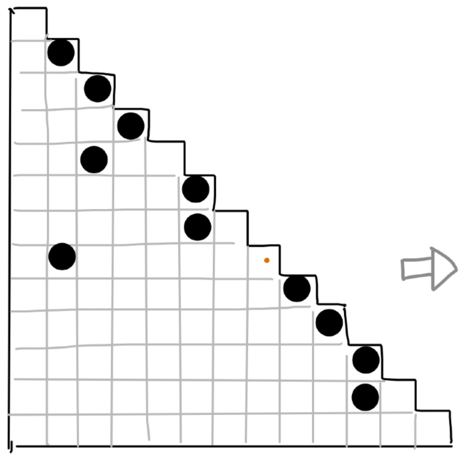

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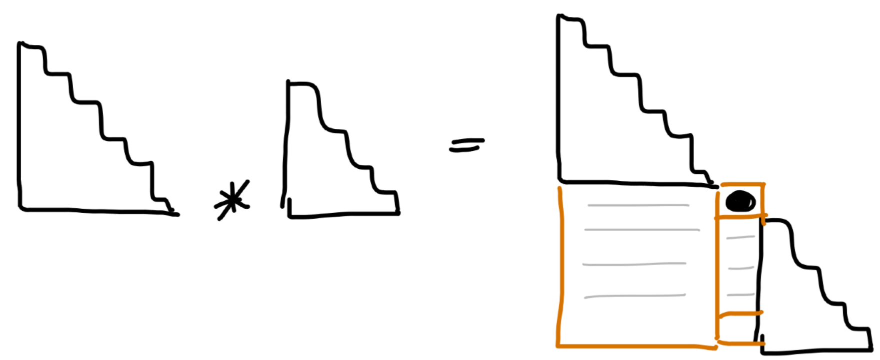

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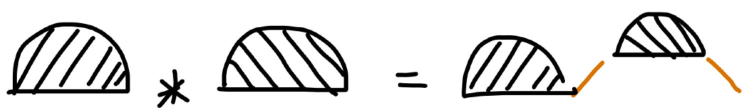

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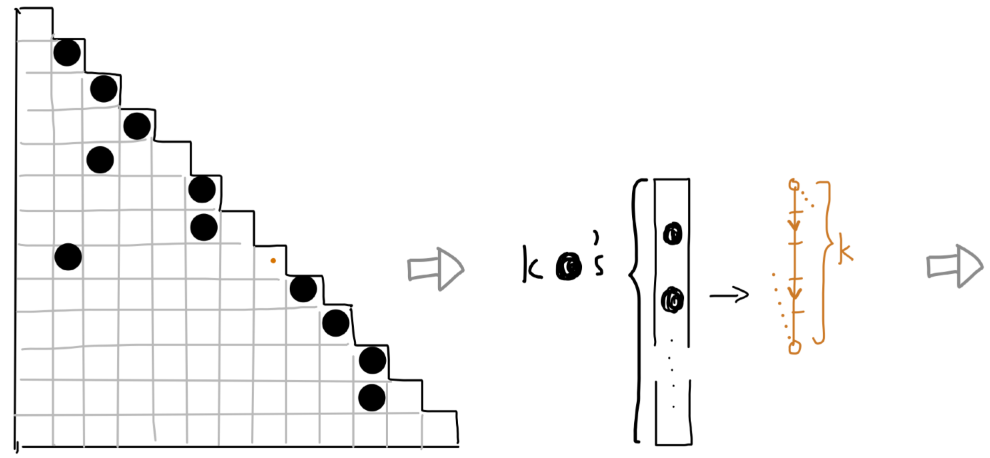

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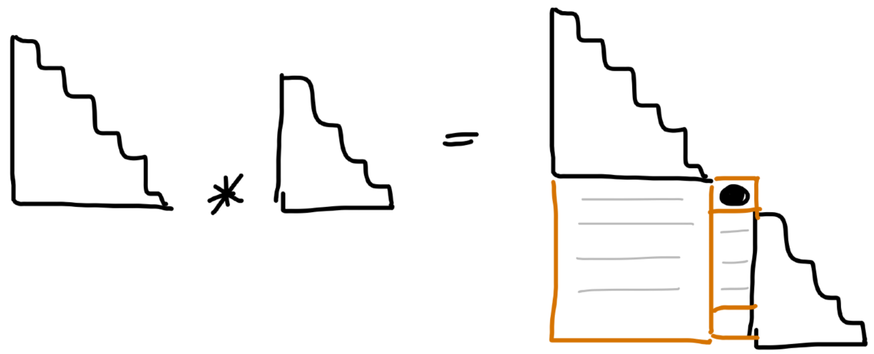

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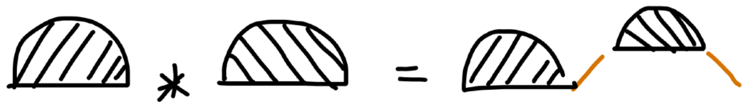

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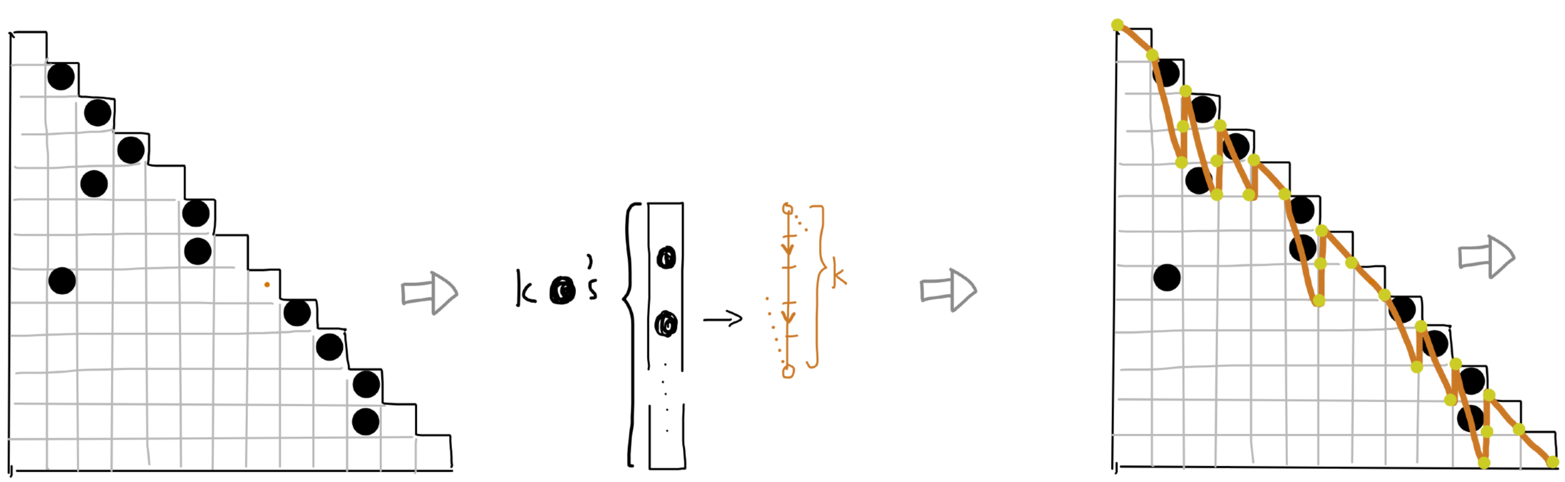
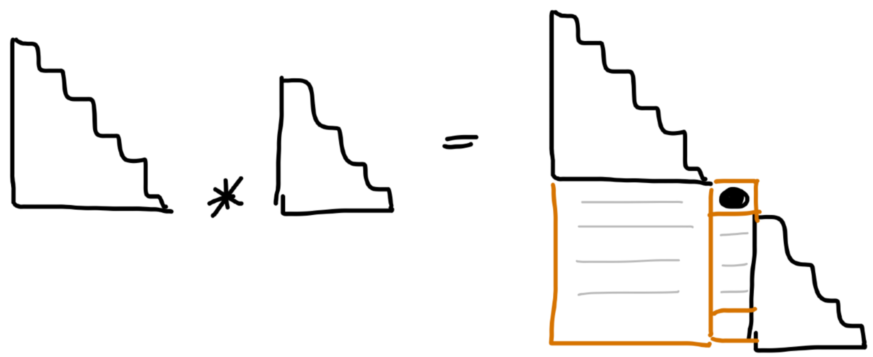
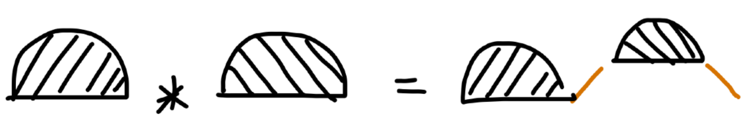


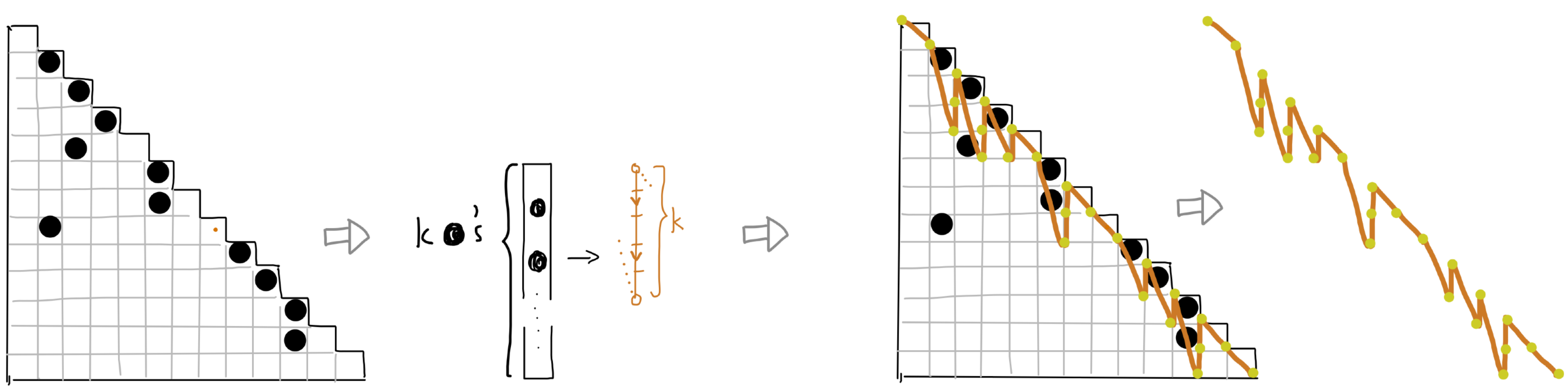
- Back to alternative tableaux
- Define tableaux product:



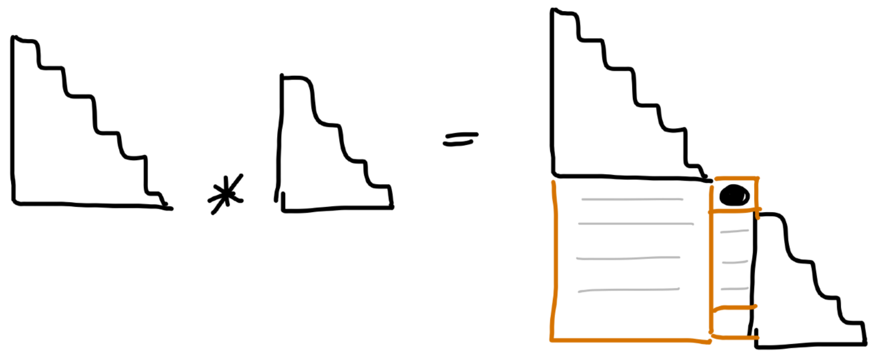
- Embedded bijection to Dyck paths:

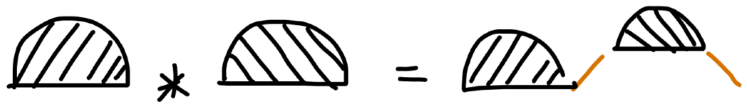


- Example:



- Back to alternative tableaux
- Define tableaux product:



- Embedded bijection to Dyck paths: 

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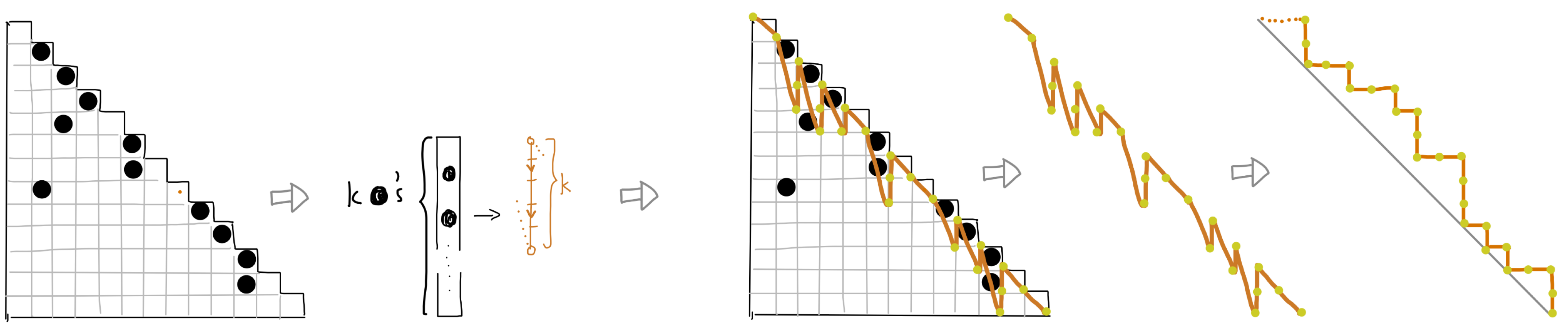

- Summary

  - Matrix product Ansatz equivalent to ring homomorphism

$$de^{\Gamma} \xrightarrow{\gamma} \overline{de^{\Gamma}} \xrightarrow{\theta} \sum_i c_{n,m}\,\bar{\alpha}^n \bar{\beta}^m.$$

  - Set $\{e^n d^m\}$ basis for $\mathbb{Z}[d,e]/I$

$$\Rightarrow de^{\Gamma} \xrightarrow{\text{unique}} \sum_i c_{n,m}\,\bar{\alpha}^n \bar{\beta}^m$$

  - Basis coefficients combinatorial $\longrightarrow$ alternative tableau of shape Γ.

  - Set of tableau from $(de)^L$ $\longrightarrow$ Catalan family (embedded)

- Summary

  - Matrix product Ansatz equivalent to ring homomorphism

$$de^{T} \xrightarrow{\gamma} \overline{de^{T}} \xrightarrow{\theta} \sum c_{n,m}\bar{\alpha}^{n}\bar{\beta}^{m}.$$

  - Set $\{e^n d^m\}$ basis for $\mathbb{Z}[d,e]/I$

$$\Rightarrow\ de^{T} \xrightarrow{\text{unique}} \sum c_{n,m}\bar{\alpha}^{n}\bar{\beta}^{m}$$

  - Basis coefficients combinatorial $\longrightarrow$ alternative tableau of shape T.

  - Set of tableau from $(de)^L$ $\longrightarrow$ Catalan family (embedded)

— THANK YOU —