Integrable structure of Quantum Field Theory: Classical flat connections versus quantum stationary states

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- **Integrable Quantum Field Theory (QFT), Integrals of Motion**
  - **Conformal Field Theory (CFT), Infinite-dimensional algebra of (extended) conformal symmetry**
  - **Bethe Ansatz, functional relations for commuting transfer matrices**
- **Theory of differential equations**
  - **Scattering problem for ODE, connection coefficients, Stocks multipliers, . . .**
  - **monodromy group, monodromy-free singular points**
  - **What's the meaning of the number 18 in the theory of the hypergeometric equation?**
  - **second order PDE, arising as "zero-curvature condition" for multivalued flat connections on the punctured Riemann sphere**
- **Space of states in QFT — Set of singular differential operators with special monodromy properties**

Local IM in CFT

(VB, Lukyanov, Zamolodchikov, 1994)

Let Vir be the Virasoro algebra generated by $L_n \in Vir$,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{m+n,0}$$

Suppose we are given a set of mutually commuting operators from the universal enveloping
of Vir:

$$\mathbb{I}_s \in U(Vir) \ \ : \quad [\mathbb{I}_s, \mathbb{I}_{s'}] = 0 \ .$$

What is the spectrum of $\mathbb{I}_s$ in the

highest weight representaion of $Vir \ \ : \quad V_{\Delta,c}$?

We are forced to make some assumptions about the Abelian subalgebra.

- It would be natural to include L_0 in the commuting set; L_0 splits $V_{\Delta,c}$ on the finite dimensional level subspaces:

$$L_0\, V^{(L)}_{\Delta,c} = (\Delta + L)\, V^{(L)}_{\Delta,c} \qquad\qquad \dim\big[\, V^{(L)}_{\Delta,c} \,\big] < \infty\ .$$

Therefore, the problem is reduced to a finite dimensional spectrum problem in $V^{(L)}_{\Delta,c}$.

- We choose the first nontrivial $\mathbb{I}_s$ in the form

$$\sum_n \alpha_n\, L_{-n} L_n + \beta\, L_0 + \gamma$$

- **Locality condition**: Let $T(x)$, $x \in S^1$ $(x \sim x + R)$ be the holomorphic component of stress-energy tensor. We assume that $\mathbb{I}_s$ are given by the integral over the **local densities** build from the field $T(x)$. For example

$$I_1 = \oint T = \frac{R}{2\pi}\,\Big[\, L_0 - \frac{c}{24}\,\Big]$$

The quadratic in L_n operator is defined up to overall normalization by our locality requirement

$$I_3 = \oint T^2 = \Big(\frac{R}{2\pi}\Big)^3\,\Big[\, 2\sum_{n=1}^{\infty} L_n L_n + L_0^2 - \frac{c+2}{12}\, L_0 + \frac{c(5c+22)}{2880}\,\Big]$$

All other operators $\mathbb{I}_s$ are defined (up to overall factor) by the commutativity condition. For example

$$I_5 = \oint \left(T^3 + \frac{c+2}{12} \, (T')^2 \right)$$

There exists an infinite set $\{\mathbb{I}_{2n-1}\}_{n=1}^{\infty}$ which first representatives are given by the above formulas. They are the so called local Integrals of Motion (IM). The odd-integers $2n-1$ stand for the values of the Lorentz spin.

We'll focus on the highest vector eigenvalues:

$$I^{(vac)}_{2n-1}(\Delta, c) \quad : \qquad \mathbb{I}_{2n-1} \, | \, \Delta \rangle = \left(\frac{R}{2\pi} \right)^{2n-1} \, I^{(vac)}_{2n-1} \, | \, \Delta \rangle \, ,$$

which are certain polynomials in Δ and c:

$$I^{(vac)}_1 = \Delta - \frac{c}{24} \, , \qquad I^{(vac)}_3 = \Delta^2 - \frac{c+2}{12} \, \Delta + \frac{c(5c+22)}{2880} \, , \quad \dots$$

CFT integrals of motion — quantization of conserved quantities

in KdV theory

$$T(x) \to -\frac{c}{6} U(x), \qquad \partial_t U = U U_x - 6 U_{xxx}, \quad c \to \infty$$

Functional relations

- Transfer matrices $\mathbb{T}_j(\mu)$ (quantum analogs of traces of monodromy matrices for mKdV) satisfy the **fusion relations**

$$\mathbb{T}_j(q\mu)\mathbb{T}_j(q^{-1}\mu) = 1 + \mathbb{T}_{j+\frac{1}{2}}(\mu)\mathbb{T}_{j-\frac{1}{2}}(\mu)\ , \quad \big(\, q = \mathrm{e}^{\mathrm{i}\pi\beta^2}\ , \quad c = 1 - 6\,(\beta - \beta^{-1})^2\,\big)$$

- $\mathbb{T}_j$ can be regarded as generating function for the local IM

$$\log \mathbb{T}_j \sim \sum_{n=0}^{\infty} c_n^{(j)}\ \mathbb{I}_{2n-1}\ \kappa^{1-2n} \qquad\qquad \kappa = \mu^{\frac{1}{2(1-\beta^2)}}$$

- As $\beta^2 = \frac{p}{p'}$ the functional relations are truncated. In this case the vacuum eigenvalues,

$$\mathbb{T}_j(\mu)|\Delta\rangle = t_j(\mu)\,|\Delta\rangle$$

satisfy a finite set of integral equations (TBA equations). Numerical values of the vacuum eigenvalues $I_{2n-1}^{(vac)}$ can be extracted from the solutions of the TBA equations.

- The TBA equations are expecially simple in the case

$$\beta^2 = \frac{1}{N+1}\ , \quad N = 1,\, 2, \ldots \qquad \Delta = \frac{1-4N^2}{6(N+1)}\ .$$

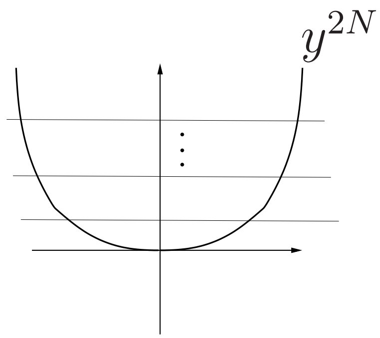
Related to RSOS models (Andrews-Baxter-Forrester '84, Baxter-Pearce '87).

ODE/IM correspondence

Let us consider the anharmonic potential

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}y^2} + y^{2N} - E \right) \Psi = 0 \ .$$

The WKB spectrum can be determined by means of the WKB approximation.



WKB spectra $\{E_n\}_{n=1}^{\infty} \Longrightarrow \oint \mathrm{d}y \sqrt{E_n - U(y)} = 2\pi(n + ...)$

E_2

E_1

- **Voros (1992)** derived the exact Exact Bohr-Sommerfeld quantization condition.
- **Dorey-Tateo (1998)** observed that TBA for $\beta^2 = \frac{1}{N+1}$ are exactly the same as the Voros one.
- The observation was immediately generalized and proven **BLZ (1998)**
- ODE/IM correspondence for the excited states **BLZ (2003)**

According to **BLZ (1998)** the vacuum eigenvalues of $\mathbb{T}_j(\mu)$, i.e., $t_j(\mu)$, $(j=\frac{1}{2},1,\ldots)$ coincide with certain monodromy coefficients for the ODE

$$\Big(-\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{l(l+1)}{z^2}+\kappa^2\,p(z)\Big)\,\Psi=0\ ,\qquad p(z)=z^{2\alpha}-1\ .$$

One can reformulate this result in terms of the vacuum eigenvalues $I^{(vac)}_{2n-1}$;

$$w=\int\mathrm{d}z\sqrt{p(z)}\qquad:\qquad\Big(-\frac{\mathrm{d}^2}{\mathrm{d}w^2}+\hat{u}(w)+\kappa^2\Big)\,\tilde{\Psi}=0$$

$$c_n=\frac{(2n-3)!!}{2^n n!}$$

$$\tilde{\Psi}(w)\sim\mathrm{e}^{F(w)}\exp\Big(-\kappa w+\textstyle\sum_{n=1}^{\infty}\kappa^{1-2n}\ c_n\ \int^{w}\mathrm{d}w\ U_n[\hat{u}]\Big)$$

$$F(w)=\sum_{n=1}^{\infty}\kappa^{-2n}F_n[\hat{u}(w)]\qquad F_n[\hat{u}]-\text{differential polinomials in}\ \ \hat{u}\ .$$

Also $U_n[\,\hat{u}\,]$ are homogeneous $(\mathrm{grade}(\hat{u})=2,\ \mathrm{grade}(\partial)=1,\ \mathrm{grade}(U_n)=2n)$ differential polynomials in $\hat{u}$ of degree n (known as the Gel'fand-Dikii polynomials):

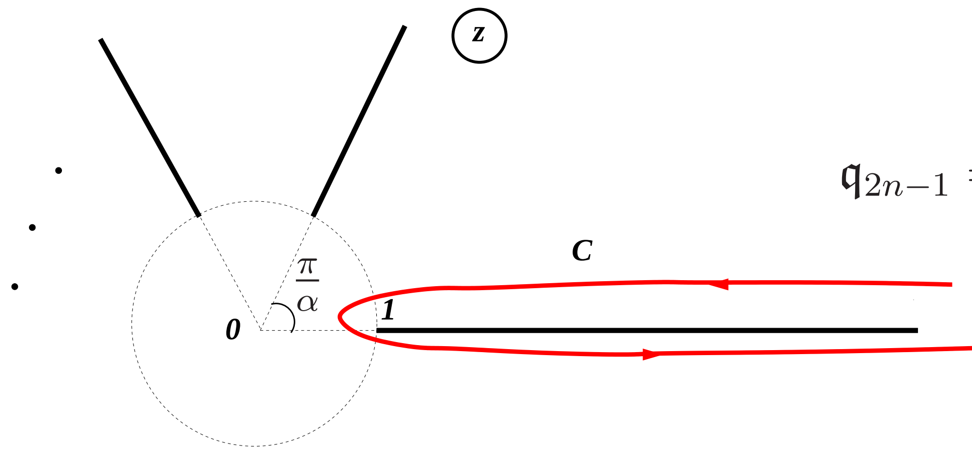
$$U_1=\hat{u}\,,\qquad U_2=\hat{u}^2-\frac{1}{3}\,\hat{u}''\ \ldots$$

Hence the monodromy coefficients are given by

$$\log t_{\frac{1}{2}}(\mu) \sim \sum_n c_n\ \kappa^{1-2n}\ \mathfrak{q}_{2n-1}\ , \qquad \mathfrak{q}_{2n-1} = \oint_{C_w} \mathrm{d}w\, U_n[\hat{u}(w)]$$

We may now return to the original variable z

$$w \to z\ , \quad U_n[\hat{u}(w)] \to \tilde{U}_n(z)$$



$$\mathfrak{q}_{2n-1} = \oint_C \mathrm{d}z\ \tilde{U}_n(z) \qquad\qquad (p(z) = z^{2\alpha} - 1)$$

$$\text{The ODE/IM correspondence :} \qquad \boldsymbol{I^{(vac)}_{2n-1} = d_n\ \mathfrak{q}_{2n-1}}$$

Here d_n are some (known) constants which depend on normalization conventions for $\mathfrak{q}_{2n-1}$ and $\mathbb{I}_{2n-1}$, whereas the parameters are identified as follows:

$$c = 1 - \frac{6\alpha^2}{\alpha+1}\ , \qquad \Delta = \frac{(2l+1)^2 - 4\alpha^2}{16(\alpha+1)}\ .$$

Excited states

$$\big(-\partial_z^2+T_L(z)\big)\,\psi=0,\qquad T_L(z)=-\sum_{i=1}^{L+3}\Big(\frac{\delta_i}{(z-z_i)^2}+\frac{c_i}{z-z_i}\Big)$$

with $\{z_i\}=\{z_1,z_2,z_3,x_1,\dots,x_L\}$ and

$$\delta_i=\frac{1}{4}-p_i^2,\quad i=1,2,3;\qquad\qquad \delta_{a+3}=-2,\quad a=1,2,\dots,L$$

Monodromy group

$$\boldsymbol{M}\ :\ \pi_1\big(\mathbb{CP}^1\backslash\{z_i\}\big)\mapsto\mathbb{SL}(2,\mathbb{C})\,,\qquad\qquad \mathrm{Tr}\big(\boldsymbol{M}^{(i)}\big)=-2\,\cos(2\pi p_i)\ .$$

Condition: points $x_1,\dots,x_L$ are monodromy-free

$$T_L(z)=-\frac{l_a(l_a+1)}{(z-x_a)^2}-\frac{c_{a+3}}{z-x_a}-\sum_{k=0}^{+\infty}t_k^{(a)}\,(z-x_a)^k,\qquad a=1,\dots,L$$

$$(c_{a+3})^3-4\,c_{a+3}\,t_0^{(a)}+4\,t_1^{(a)}=0\ .$$

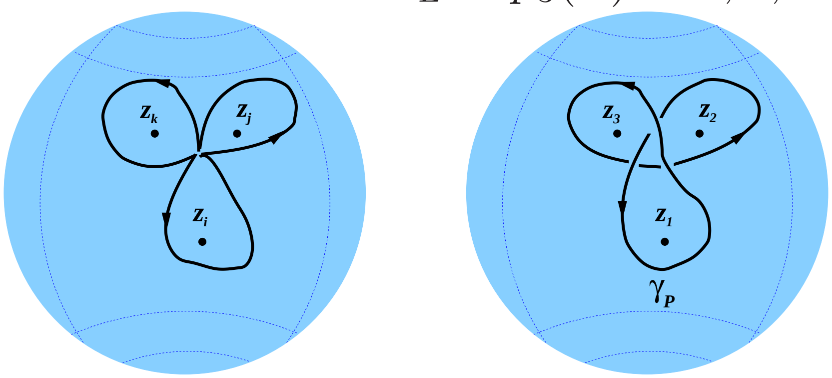
For fixed p_i, the only free parameters are the positions $x_1,\dots,x_L$.

$$\mathcal{D}(\lambda) = -\frac{\mathrm{d}^2}{\mathrm{d}z^2} + T_L(z) + \lambda^2\ \mathcal{P}(z)\ , \quad \mathcal{P}(z) = \frac{(z_3 - z_2)^{a_1}\,(z_1 - z_3)^{a_2}\,(z_2 - z_1)^{a_3}}{(z - z_1)^{2-a_1}(z - z_2)^{2-a_2}(z - z_3)^{2-a_3}}$$

and parameters $0 < a_i < 2$ satisfy the constraint $a_1 + a_2 + a_3 = 2$. Monodromy free conditions give additional L equations

$$c_{a+3} = -\partial_z \log \mathcal{P}(z)\big|_{z=x_a} = \sum_{i=1}^{3} \frac{2 - a_i}{x_a - z_i}, \qquad a = 1, \ldots L\,.$$

number of solutions $\mathcal{N}_L = p_3(L) = 3, 9, 22, \ldots$. (stationary states in CFT for Fateev model).



Monodromy matrix for the Pochhammer loop ($c(x) = \cos(\pi x)$)

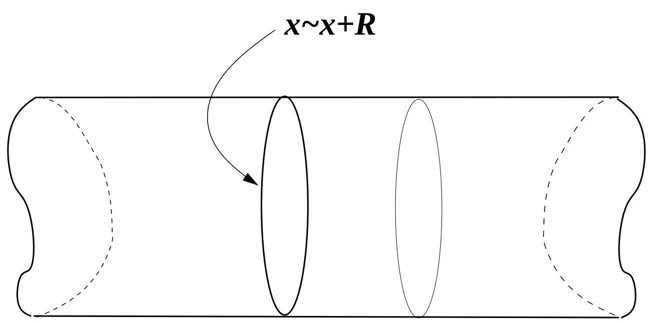
$$\mathcal{W}(\lambda) = \mathrm{Tr}\,\boldsymbol{M}(\gamma_P) = 2\,\big(2 + c(4p_1) + c(4p_2) + c(4p_3) + c(2p_1 + 2p_2 + 2p_3)$$
$$+c(2p_1 + 2p_2 - 2p_3) + c(2p_1 - 2p_2 + 2p_3) + c(-2p_1 + 2p_2 + 2p_3)\big) + O(\lambda^2)$$

For $p_i = 0$ the constant term equals 18.

WHY? What does it mean for the hypergeometric equation?

ODE/IM correspondence for massive integrable QFT

Now we consider the CFT perturbed by a relevant operator in the bulk



$$\mathcal{A}_\mu = \mathcal{A}_{CFT} + \mu \int \mathrm{d}^2 x \; \Phi \qquad (d_\Phi = 2\Delta_\Phi < 2)$$

In general one expects that the perturbation leads to the massive QFT

$$M_a \sim \mu^{\frac{1}{2-d_\Phi}}$$

In the case of integrable perturbation the theory possesses an infinite set of local IM

$$\mathbb{I}_s\big|_{\mu\to 0} = \mathbb{I}_s^{(CFT)} \;, \qquad \bar{\mathbb{I}}_s\big|_{\mu\to 0} = \bar{\mathbb{I}}_s^{(CFT)}$$

Let $I_{2n-1} = \bar{I}_{2n-1}$ be the vacuum eigenvalues of $\mathbb{I}_s$ and $\bar{\mathbb{I}}_s$.

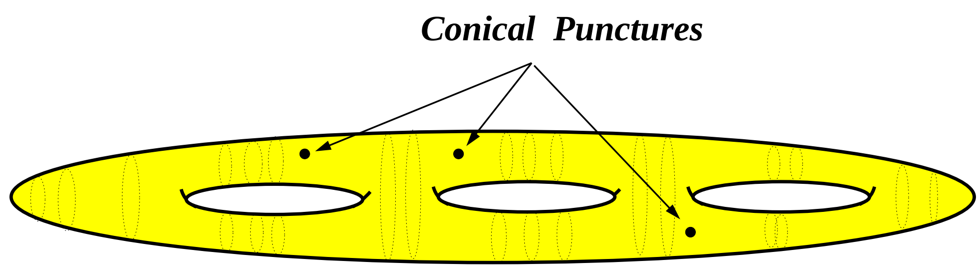
Is it possible to relate $I_{2n-1}(\mu)$ to monodromic characteristics of some ODE?

During the decade 1998-2008, all attempts to incorporate massive integrable QFT in the ODE/IM correspondence have failed.

- **Gaiotto, Moore and Neitzke (2008)**: TBA-like equations for the Hitchin systems
- **Alday, Maldacena (2009)**: Strong coupling amplitudes in ADS/CFT
- **Zamolodchikov, Lukyanov (2010)**: ODE/IM for the sin(h)-Gordon model

CMC embedding of a 3-punctured sphere in AdS_3

Let $\Sigma_{g,n}$ be a compact Riemann surface with n marked points ("punctures") and $a_1,\, a_2, \ldots a_n$ be positive numbers such that $\boldsymbol{2\,\chi(\Sigma_g) + \sum_{i=1}^n (a_i - 2) = 0}$. Then there exists a **flat** metric on $\Sigma_{g,n}$ with conical singularities of angle πa_i at the i^{th} puncture. The metric is unique up to homothety.



In the case $\Sigma_{0,3} = \mathbb{S}^2/\{P_1, P_2, P_3\}$: $a_1 + a_2 + a_3 = 2$

Introduce a complex coordinate z and define a holomorphic differential $p(z)\,(\mathrm{d}z)^2$ on the universal cover of $\Sigma_{0,3}$:

$$p(z) = \rho^2\ \frac{(z_3 - z_2)^{a_1}\,(z_1 - z_3)^{a_2}\,(z_2 - z_1)^{a_3}}{(z - z_1)^{2-a_1}(z - z_2)^{2-a_2}(z - z_3)^{2-a_3}} \quad : \quad (\mathrm{d}s)_0^2 = \sqrt{p(z)\bar{p}(\bar{z})}\ \mathrm{d}z\mathrm{d}\bar{z}$$

Here ρ stands for the homothety parameter and z_i labels the punctures.

Consider now the problem of constant mean curvature embedding of $\Sigma_{0,3}$ into AdS_3. In this case, the Gauss-Peterson-Codazzi equation can be brought to the form of the **modified Sinh-Gordon** (MShG) equation

$$\partial_z \partial_{\bar{z}} \eta - \mathrm{e}^{2\eta} + p(z)\bar{p}(\bar{z})\,\mathrm{e}^{-2\eta} = 0 \ ,$$

where the field η defines the induced metric

$$(\mathrm{d}s)^2_{\mathrm{cmc}} = \frac{4}{1+H^2}\ \frac{\mathrm{e}^{2\eta}}{\sqrt{p(z)\bar{p}(\bar{z})}}\ (\mathrm{d}s)^2_0$$

and $H = const$ stands for the mean curvature. A suitable solution should be real and smooth as $z \neq z_i$, and, if we want to preserve the amount of the Gaussian curvature localized at the punctures, it should satisfy the conditions

$$\eta - \tfrac{1}{4}\ \log\big(\, p(z)\bar{p}(\bar{z})\,\big) = O(1) \qquad \text{at} \quad z \to z_i \ \ (i = 1,2,3) \ \ \text{and} \ \infty \ .$$

$$\textbf{Generalized problem}: \ \ \eta = \begin{cases} -2\,\log|z| + O(1) & \text{at} \qquad z \to \infty \\ 2m_i\,\log|z - z_i| + O(1) & \text{at} \qquad z \to z_i \end{cases}$$

$$\text{If} \ \ 0 < a_i < 2 \ \ \text{and} \ \ -\frac{1}{2} < m_i \leq -\frac{1}{4}\,(2 - a_i)$$

then the solution of the generalized problem exists and is unique.

The MShG equation is the compatibility condition of the linear problem

$$\boldsymbol{D}(\lambda)\,\boldsymbol{\Psi}=0\ ,\qquad \bar{\boldsymbol{D}}(\bar\lambda)\,\boldsymbol{\Psi}=0\ .$$

$$\boldsymbol{D}(\lambda)=\partial_z-\boldsymbol{A}_z\ ,\qquad \bar{\boldsymbol{D}}(\bar\lambda)=\partial_{\bar z}-\boldsymbol{A}_{\bar z}\,,\qquad \lambda=\rho\mathrm{e}^{\theta},\qquad \bar\lambda=\rho\mathrm{e}^{-\theta}$$

$$\begin{aligned}\boldsymbol{A}_z &= -\tfrac{1}{2}\,\partial_z\eta\,\sigma_3+\lambda\,\big(\sigma_+\,\mathrm{e}^{\eta}+\sigma_-\,\mathcal{P}(z)\,\mathrm{e}^{-\eta}\big)\\ \boldsymbol{A}_{\bar z} &= \tfrac{1}{2}\,\partial_{\bar z}\eta\,\sigma_3+\bar\lambda\,\big(\sigma_-\,\mathrm{e}^{-\eta}+\sigma_+\,\bar{\mathcal{P}}(\bar z)\,\mathrm{e}^{\eta}\big)\ .\end{aligned}$$

Additional monodromy-free punctures

$$\mathrm{e}^{-\eta}\sim\frac{\bar z-\bar x_a}{z-x_a},\quad (a=1,\dots L),\qquad \mathrm{e}^{-\eta}\sim\frac{z-y_b}{\bar z-\bar y_b},\quad (b=1,\dots\bar L)\ .$$

satisfy the conditions

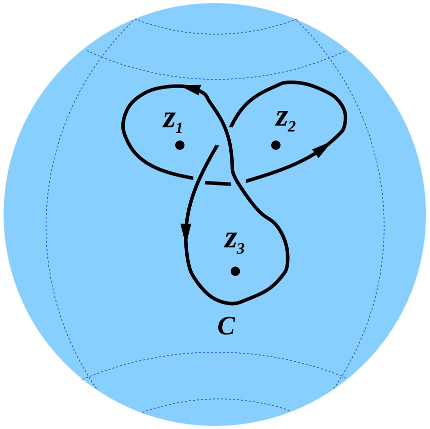
$$\partial_z\eta=\frac{1}{z-x_a}+\frac{1}{2}\,\gamma_a+o(1)\,,\qquad \partial_{\bar z}\eta=-\frac{1}{\bar z-\bar x_a}+o(1)\,,\quad a=1,\dots L$$

and

$$\gamma_a=\partial_z\log\mathcal{P}(z)|_{z=x_a}$$

and similarly for y_b.

The MShG equation is a flatness condition for $sl(2)$-valued connection $\boldsymbol{A} = \boldsymbol{A}_z dz + \bar{\boldsymbol{A}}_{\bar{z}} d\bar{z}$. The connection is not single-valued on the punctured sphere. However, it does return to the original branch after a continuation along the non-contractible loop C



Therefore the Wilson loop

$$W(\theta) = \mathrm{Tr}\Big[\mathcal{P}\exp\Big(\oint_C \boldsymbol{A}\Big)\Big]$$

does not depend on the precise shape of the cycle used. It can be regarded as generating functions for the conserved charges

$$\log W(\theta) \sim -\mathfrak{q}_0\, \mathrm{e}^{\theta} + \sum_{n=1}^{\infty} c_n\ \mathfrak{q}_{2n-1}\ \mathrm{e}^{-(2n-1)\theta} \quad \text{as} \quad \Re e(\theta) \to +\infty, \quad |\Im m(\theta)| < \frac{\pi}{2}$$

here $c_n = \frac{(-1)^n}{2n!}\,\frac{\Gamma(n-\frac{1}{2})}{\sqrt{\pi}}$.

Fateev model (1996)

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{16\pi} \sum_{i=1}^{3} \big((\partial_t \varphi_i)^2 - (\partial_x \varphi_i)^2 \big) \\
&+ \; 2\mu \Big[\mathrm{e}^{\mathrm{i}\,\alpha_3 \varphi_3} \; \cos(\alpha_1 \varphi_1 + \alpha_2 \varphi_2) + \mathrm{e}^{-\mathrm{i}\alpha_3 \varphi_3} \; \cos(\alpha_1 \varphi_1 - \alpha_2 \varphi_2) \Big]
\end{aligned}
$$

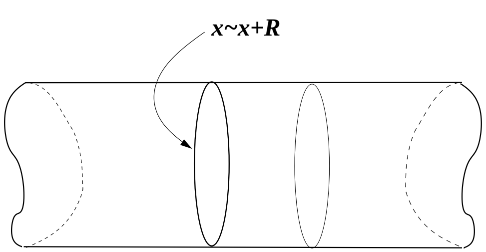
Here α_i are coupling constants subject to a single constraint

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \frac{1}{2} \; .$$

$$\alpha_1^2 > 0\,, \qquad \alpha_2^2 > 0\,, \qquad \alpha_3^2 > 0 \; .$$

The parameter μ in the Lagrangian sets the mass scale, $\mu \sim [\,\text{mass}\,]$. We shall consider the theory in finite-size geometry, with the spatial coordinate x in $\varphi_i = \varphi_i(x,t)$ compactified on a circle of circumference R, with the periodic boundary conditions

$$\varphi_i(x+R,t) = \varphi_i(x,t) \; .$$



$$\mathcal{A}_\mu = \mathcal{A}_{CFT} + \mu \int \mathrm{d}^2 x \; \Phi \qquad (d=1) \; .$$

Due to the periodicity of the potential term in φ_i,

$$\begin{aligned}\mathcal{L} &= \frac{1}{16\pi}\sum_{i=1}^{3}\Big(\,(\partial_t\varphi_i)^2-(\partial_x\varphi_i)^2\,\Big)\\ &+ 2\mu\,\Big[\,\mathrm{e}^{\mathrm{i}\,\alpha_3\varphi_3}\,\cos(\alpha_1\varphi_1+\alpha_2\varphi_2)+\mathrm{e}^{-\mathrm{i}\alpha_3\varphi_3}\,\cos(\alpha_1\varphi_1-\alpha_2\varphi_2)\Big]\end{aligned}$$

the space of states $\mathcal{H}$ splits on the orthogonal subspaces $\mathcal{H}_{k_1,k_2,k_3}$ characterized by the three "quasimomentums" k_i:

$$\varphi_i\to\varphi_i+2\pi/\alpha_i\ \ :\quad \mid\Psi_{k_1,k_2,k_3}\,\rangle\to\mathrm{e}^{2\pi\mathrm{i}k_i}\,\mid\Psi_{k_1,k_2,k_3}\,\rangle\ .$$

The Fateev model is integrable, in particular it has infinite set of commuting local IM $\mathbb{I}^{(+)}_{2n-1}$, $\mathbb{I}^{(-)}_{2n-1}$, $2n = 2, 4, 6, \ldots$ being the Lorentz spins of the associated local densities

$$\mathbb{I}^{(\pm)}_{2n-1} = \int_0^R \frac{\mathrm{d}x}{2\pi} \ \Big[\sum_{i+j+k=n} C^{(n)}_{ijk} \ (\partial_\pm \varphi_1)^{2i} \, (\partial_\pm \varphi_2)^{2j} \, (\partial_\pm \varphi_3)^{2k} + \ldots \Big]$$

where $\partial_\pm = \frac{1}{2}(\partial_x \mp \partial_t)$ and $\ldots$ stand for the terms involving higher derivatives of φ_i, as well as the terms proportional to powers of μ. The constant $C^{(n)}_{ijk}$ is known (**Zamolodchikov, Lukyanov, 2012**)

$$C^{(n)}_{ijk} = \frac{n!}{i! \ j! \ k!} \ \frac{\big(2\alpha_1^2(1-2n)\big)_{n-i}\big(2\alpha_2^2\,(1-2n)\big)_{n-j}\,\big(2\alpha_3^2\,(1-2n)\big)_{n-k}}{(2n-1)^3 \ (4\alpha_1^2)^{1-i} \ (4\alpha_2^2)^{1-j} \ (4\alpha_3^2)^{1-k}} \ ,$$

where $(x)_n$ is the Pochhammer symbol. The displayed terms with the given $C^{(n)}_{ijk}$ set the normalization of $\mathbb{I}^{(\pm)}_{2n-1}$ unambiguously.

Of primary interest are the eigenvalues

$$I_{2n-1} = I^{(+)}_{2n-1}(\{k_i\}\,|\,R) = I^{(-)}_{2n-1}(\{k_i\}\,|\,R)$$

especially the k-vacuum energy

$$E = 2\ I_1\ .$$

In the large-R limit all vacuum eigenvalues I_{2n-1} vanish except I_1. The vacuum energy is composed of an extensive part proportional to the length of the system,

$$E = R\,\mathcal{E}_0 + o(1) \qquad\quad \text{at} \quad R \to \infty$$

Specific bulk energy (**Fateev, 1996**)

$$\mathcal{E}_0 = -\pi\mu^2\ \prod_{i=1}^{3} \frac{\Gamma(2\alpha_i^2)}{\Gamma(1-2\alpha_i^2)}\ .$$

ODE/IM correspondence

The eigenvalues of the local IM in the Fateev model can be expressed in terms of the classical conserved charges $\mathfrak{q}_{2n-1}$:

$$\begin{aligned} \mu^{-1}\,\big(\,I_1 - \tfrac{1}{2}\,R\,\mathcal{E}_0\big) &= d_1\;\mathfrak{q}_1 \\ \mu^{1-2n}\;I_{2n-1} &= d_n\;\mathfrak{q}_{2n-1} \qquad\qquad (n = 2,\,3,\,\ldots)\;. \end{aligned}$$

Here d_n are constants, independent of k_i and R. With the normalization conditions for $\mathfrak{q}_{2n-1}$ and $\mathbb{I}^{(\pm)}_{2n-1}$ described above, d_n reads explicitly as

$$d_n = (2\pi)^{2n-1}\;\frac{(-1)^{n-1}}{16\,\pi^2}\;\prod_{i=1}^{3}\Gamma\big(\,2\,(2n-1)\,\alpha_i^2\,\big)\;.$$

The parameters of the quantum and classical problems are identified as follows:

$$\begin{aligned} \alpha_i^2 &= \frac{a_i}{4} \qquad\qquad (i = 1, 2, 3) \\ |k_i| &= \frac{1}{a_i}\,(2m_i + 1) \\ \mu R &= 2\rho \end{aligned}$$

Conclusion

- There a connection between the theory of **Integrable Models** in two dimensions and the spectral theory of **Ordinary Differential Equations**.

- Classical conserved charges = Eigenvalues of IM in the integrable QFT

- Eigenvalues of transfer matrices = connection coefficients between different bases solutions of ODE.

- We considered a class of "Perturbed Fuchsian differential equations"

- Multivalued flat connections on the punctured sphere and monodromy properties of associated singular differential operators

- Future tasks: Apply "Quantum Inverse Problem Method" to the Fateev model (Yang-Baxter structure, lattice regularization, etc.)

- What is 18? (Mininal dimension of representation of the quantized exceptional affine superalgebra $U_q(\widehat{D}(2,1,;\alpha)))$