Exploring the corner: Numerical evolution of Spin-2 fields at space-like infinity in Minkowski space-time.

F. Beyer, G. Doulis, J. Frauendiener and **B. Whale**

Department of Mathematics and Statistics,
University of Otago

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1. An example: Minkowski space-time
2. Numerical Results
3. Numerical replication of divergences in solutions
4. Numerical calculation to $\mathscr{I}^+$ and beyond
   - Calculation on Infinity
   - Calculation beyond infinity

Minkowski space-time

The metric is

$$\tilde{g} = (dX^0)^2 - (dX^1)^2 - (dX^2)^2 - (dX^3)^2$$

Normal coordinates based at infinity are

$$x^a = -\frac{X^a}{X^b X_b}.$$

Hence

$$\tilde{g} = \frac{1}{(x^c x_c)^2} \left((dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \right).$$

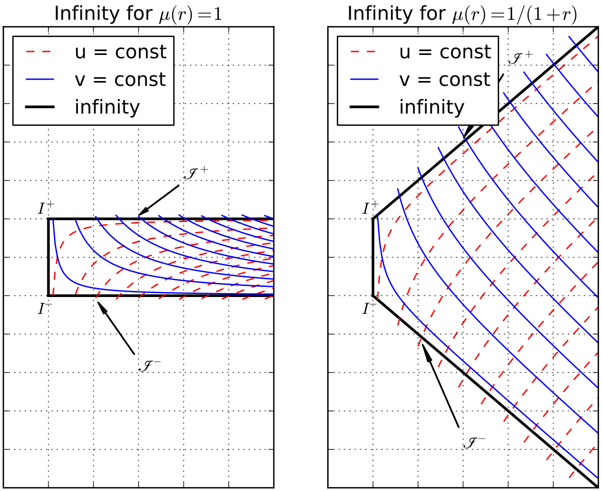
Let

1. $r^2 = (x^1)^2 + (x^2)^2 + (x^3)$,
2. $\Omega = -x^a x_a$,
3. $\mu(r)$ be a smooth function so that $\mu(0) = 1$,
4. $\kappa(r) = r\mu(r)$,
5. $t = \frac{x^0}{\kappa(r)}$.

Define

$$g = \frac{\Omega^2}{\kappa^2}\tilde{g} = dt^2 + 2t\frac{\kappa'}{\kappa}dtdr - \frac{1 - t^2\kappa'^2}{\kappa^2}dr^2 - \frac{1}{\mu^2}d\omega^2$$

Hence $J^{\pm} = \{1 \mp t\mu = 0\}$, $I = \{-1 < t < 1, r = 0\}$,
$I^{\pm} = \{t = \pm 1, r = 0\}$.


Infinity for $\mu(r) = 1$
u = const
v = const
infinity
$\mathscr{I}^+$
I^+
I^-
$\mathscr{I}^-$
Infinity for $\mu(r) = 1/(1+r)$
u = const
v = const
infinity
$\mathscr{I}^+$
I^+
I^-
$\mathscr{I}^-$

$$0 = \nabla^{h}{}_{c'}\phi_{abch}$$

Linear spin-2 zero rest mass field

F. Beyer, G. Doulis, J. Frauendiener, B. Whale (2012). 'Numerical space-times near space-like and null infinity. The spin-2 system on Minkowski space'. arXiv:1207.5854v1

To express this equation in our coordinates we choose a spin frame, o^a, ι^a, adapted to t, r and write

$$\begin{aligned}
\phi_0 &= \phi_{abcd} o^a o^b o^c o^d, \\
\phi_1 &= \phi_{abcd} o^a o^b o^c \iota^d, \\
&\vdots \\
\phi_4 &= \phi_{abcd} \iota^a \iota^b \iota^c \iota^d.
\end{aligned}$$

Then decomposing each ϕ_i in terms of spin weighted spherical harmonics we get

$$\phi_k = \sum_{l=|k-2|}^{\infty} \sum_{m=-l}^{l} \phi_k^{lm}(t, r) \; {}_sY_{lm}.$$

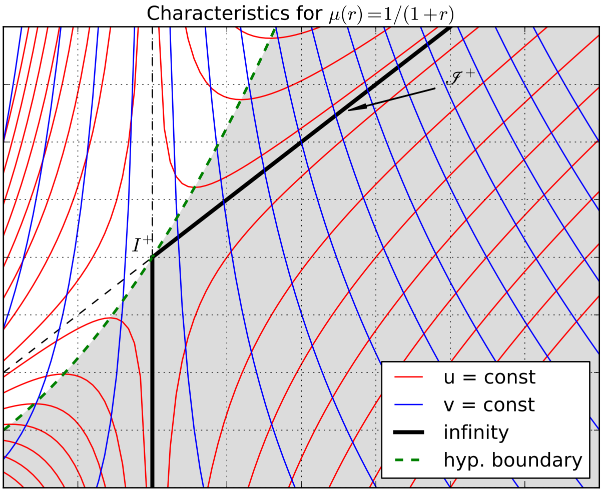
All subsequent equations are on the coefficients ϕ_i^{lm} but the l and m will be dropped to remove clutter.

$$
\begin{aligned}
(1+t\kappa')\partial_t\phi_0 - \kappa\partial_r\phi_0 &= -(3\kappa'-\mu)\phi_0 - \mu\alpha_2\phi_1 \\
\partial_t\phi_1 &= -\mu\phi_1 + \frac{1}{2}\mu\alpha_2\phi_0 - \frac{1}{2}\mu\alpha_0\phi_2 \\
\partial_t\phi_2 &= \frac{1}{2}\mu\alpha_0\phi_1 - \frac{1}{2}\mu\alpha_0\phi_3 \\
\partial_t\phi_3 &= \mu\phi_3 - \frac{1}{2}\mu\alpha_2\phi_4 + \frac{1}{2}\mu\alpha_0\phi_2 \\
(1-t\kappa')\partial_t\phi_4 + \kappa\partial_r\phi_4 &= (3\kappa'-\mu)\phi_4 + \mu\alpha_2\phi_3
\end{aligned}
$$

$$
\begin{aligned}
0 &= -2\kappa\partial_r\phi_1 + 6r\mu'\phi_1 - 2t\kappa'\mu\phi_1 + \mu\alpha_0(1-t\kappa')\phi_2 + \mu\alpha_2(1+t\kappa')\phi_0 \\
0 &= -2\kappa\partial_r\phi_2 + 6r\mu'\phi_2 + \mu\alpha_0(1-t\kappa')\phi_3 + \mu\alpha_0(1+t\kappa')\phi_1 \\
0 &= -2\kappa\partial_r\phi_3 + 6r\mu'\phi_3 + 2t\kappa'\mu\phi_3 + \mu\alpha_0(1+t\kappa')\phi_2 + \mu\alpha_2(1-t\kappa')\phi_4
\end{aligned}
$$

$$
\begin{aligned}
(1+t\kappa')\partial_t\phi_0 - \kappa\partial_r\phi_0 &= -(3\kappa' - \mu)\phi_0 - \mu\alpha_2\phi_1 \\
\partial_t\phi_1 &= -\mu\phi_1 + \frac{1}{2}\mu\alpha_2\phi_0 - \frac{1}{2}\mu\alpha_0\phi_2 \\
\partial_t\phi_2 &= \frac{1}{2}\mu\alpha_0\phi_1 - \frac{1}{2}\mu\alpha_0\phi_3 \\
\partial_t\phi_3 &= \mu\phi_3 - \frac{1}{2}\mu\alpha_2\phi_4 + \frac{1}{2}\mu\alpha_0\phi_2 \\
(1-t\kappa')\partial_t\phi_4 + \kappa\partial_r\phi_4 &= (3\kappa' - \mu)\phi_4 + \mu\alpha_2\phi_3
\end{aligned}
$$

$$
\begin{aligned}
0 &= -2\kappa\partial_r\phi_1 + 6r\mu'\phi_1 - 2t\kappa'\mu\phi_1 + \mu\alpha_0(1-t\kappa')\phi_2 + \mu\alpha_2(1+t\kappa')\phi_0 \\
0 &= -2\kappa\partial_r\phi_2 + 6r\mu'\phi_2 + \mu\alpha_0(1-t\kappa')\phi_3 + \mu\alpha_0(1+t\kappa')\phi_1 \\
0 &= -2\kappa\partial_r\phi_3 + 6r\mu'\phi_3 + 2t\kappa'\mu\phi_3 + \mu\alpha_0(1+t\kappa')\phi_2 + \mu\alpha_2(1-t\kappa')\phi_4
\end{aligned}
$$


Characteristics for $\mu(r) = 1/(1+r)$
$\mathscr{I}^+$
I^+
u = const
v = const
infinity
hyp. boundary

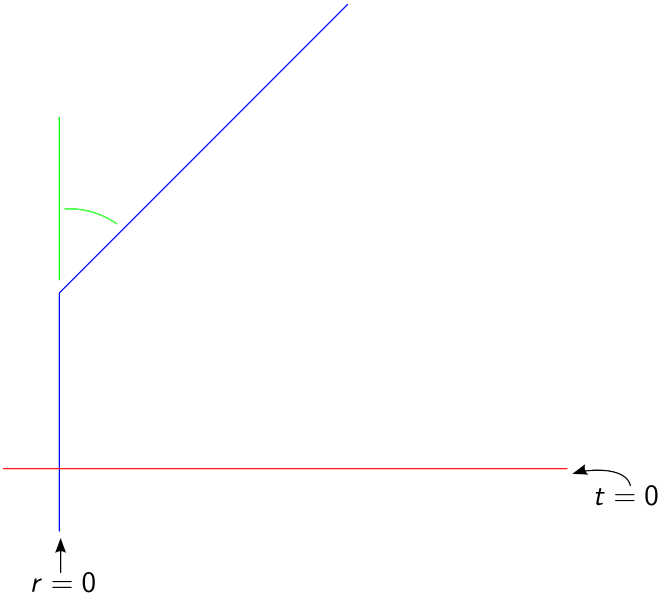
Numerical method

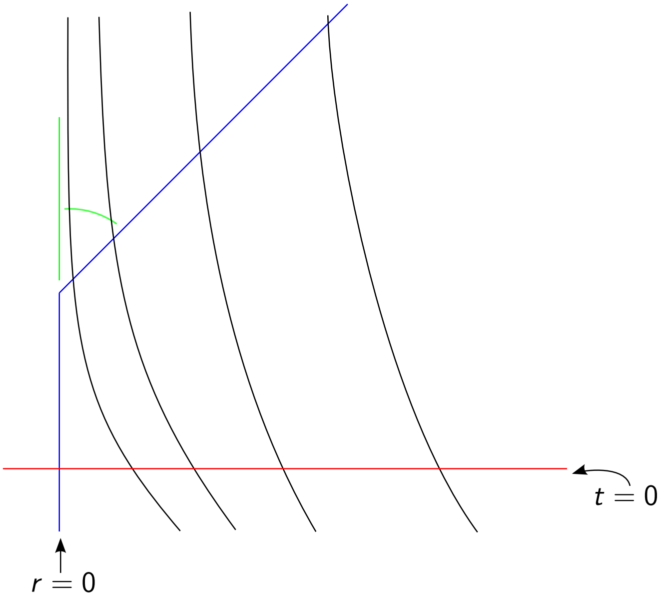
- Method of lines with standard Runge-Kutta 4 ODE solver.
- Adaptive time step based on CFL condition and characteristic speeds.
- A summation by parts finite difference operator for calculation of spatial derivatives.
- External boundaries and internal domain interfaces implemented using a penalty method (simultaneous approximation term).
- No dissipation.

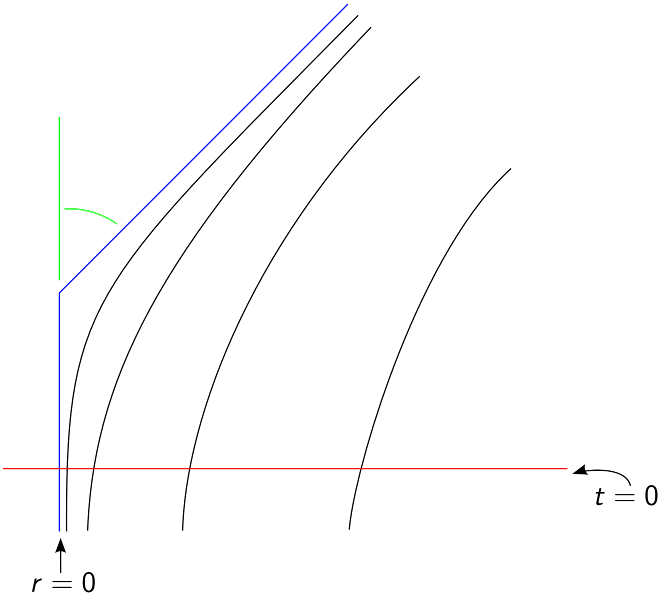
Semi-discrete formula for the boundary implementation

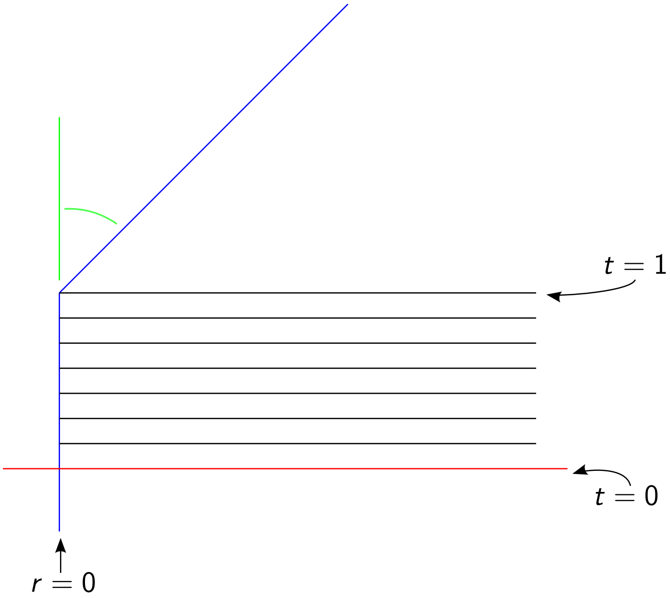
$$\partial_t \phi_0 = \frac{1}{1 + t\kappa'} \left(\kappa \partial_r \phi_0 - (3\kappa' - \mu)\phi_0 - \mu\alpha_2\phi_1 \right)$$

$$\partial_t \phi_0 = \frac{1}{1 + t\kappa'} \left(\kappa Q \phi_0 - (3\kappa' - \mu)\phi_0 - \mu\alpha_2\phi_1 \right) - \tau_1(\phi_{0,N} - b_r(t)) \frac{\kappa_N}{1 + t\kappa'_N} H^{-1}(0, 0, \ldots, 1)^T$$


t = 0
r = 0


$t = 0$
$r = 0$


$t = 0$
$r = 0$


t = 1
t = 0
r = 0

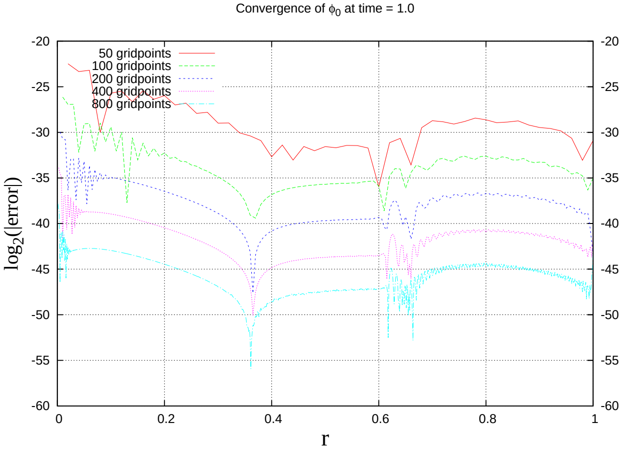


Figure: Convergence of ϕ_0 to the exact solution at time $t = 1$.

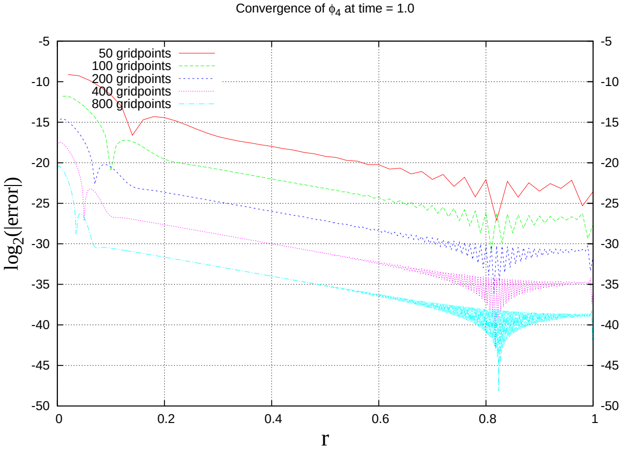


Figure: Convergence of ϕ_4 to the exact solution at time $t = 1$.

Convergence against exact solution

Grid Points	ϕ_0 $\log_2(\|\|\Delta\|\|_2)$	ϕ_0 Rate	ϕ_4 $\log_2(\|\|\Delta\|\|_2)$	ϕ_4 Rate
50	-24.90		-11.32	
100	-29.06	4.16	-14.25	2.92
200	-33.38	4.32	-17.33	3.08
400	-37.78	4.40	-20.49	3.16
800	-42.22	4.44	-23.69	3.20

Table: Absolute error Δ compared to the exact solution using the L^2 norm and convergence rates at time $t = 1$ for ϕ_0, ϕ_4. The calculation was done with a fixed time-step.

Convergence against numerical solution

	ϕ_0		ϕ_4	
Grid Points	$\log_2(\|\|\Delta\|\|_2)$	Rate	$\log_2(\|\|\Delta\|\|_2)$	Rate
50	1.45		2.23	
100	−2.51	3.96	−1.65	3.88
200	−6.48	3.97	−5.65	4.00
400	−10.56	4.08	−9.73	4.08

Table: Convergence rates of the absolute error Δ in the L^2 norm at $t = 1.0$ for compactly supported initial data. The error is computed with respect to a higher resolution run with 800 grid points.

If we expand each ϕ_k in terms of r,

$$\phi_k^{lm}(t,r) = \sum_{p=|s|}^{\infty} \sum_{l=|s|}^{p} \frac{1}{p!} \phi_{k,p}^{lm}(t) r^p, \qquad \text{with } s = 2-k,$$

then the first terms that contain divergent expressions are

$$\phi_{k,2}^{2,m}(t) = \frac{1}{2} \partial_{rr} \phi_k(t,0).$$

An exact solution exists so that

$$\phi_{4,2}^{2,m}(t) = -16 - 19t - 12t^2 - 3t^3 + \frac{3}{2}(1+t)^4 \operatorname{atanh}(t).$$

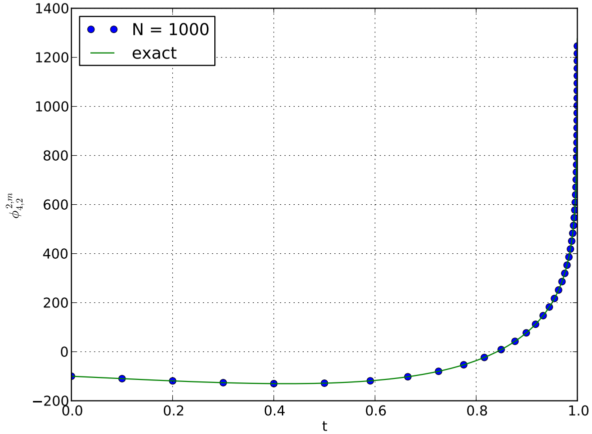


Figure: The first singular coefficient $\phi_{4,2}^{2,m}$ along $r = 0$ for times $0 \leq t \leq 1$.

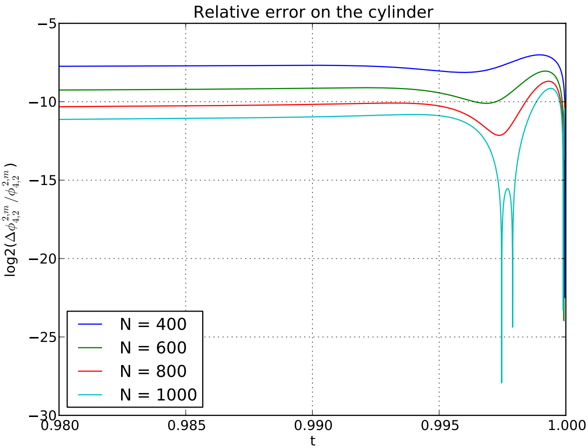
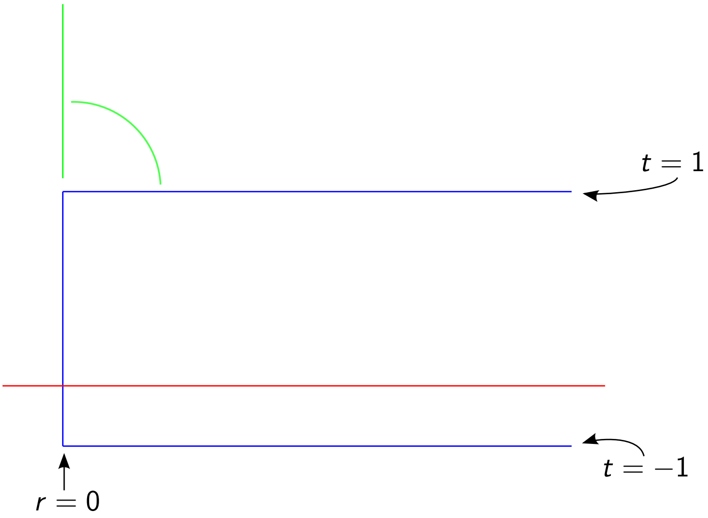
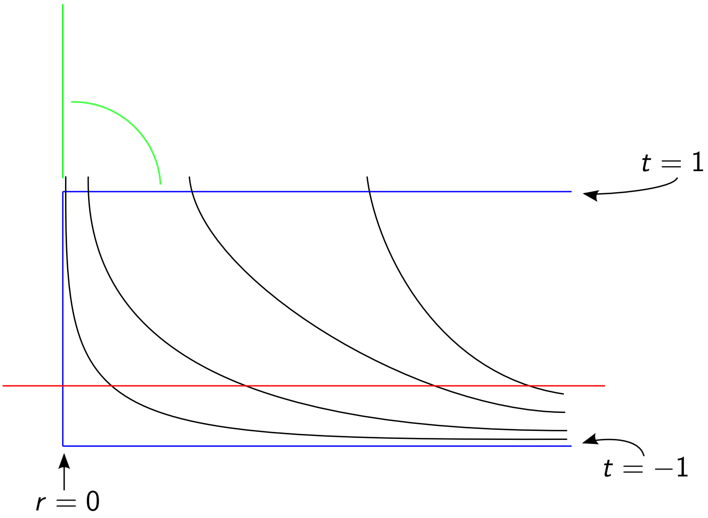
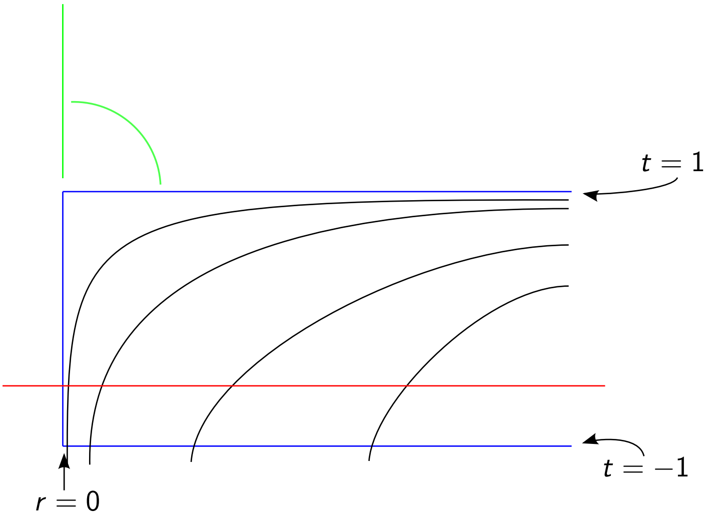
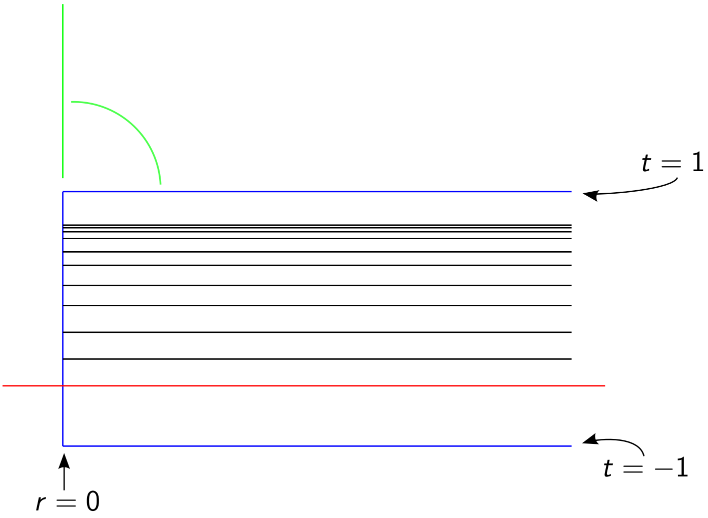


Figure: Relative error in the computation of the first singular coefficient $\phi_{4,2}^{2,m}$ along $r = 0$ for times $0.98 \leq t \leq 1$.

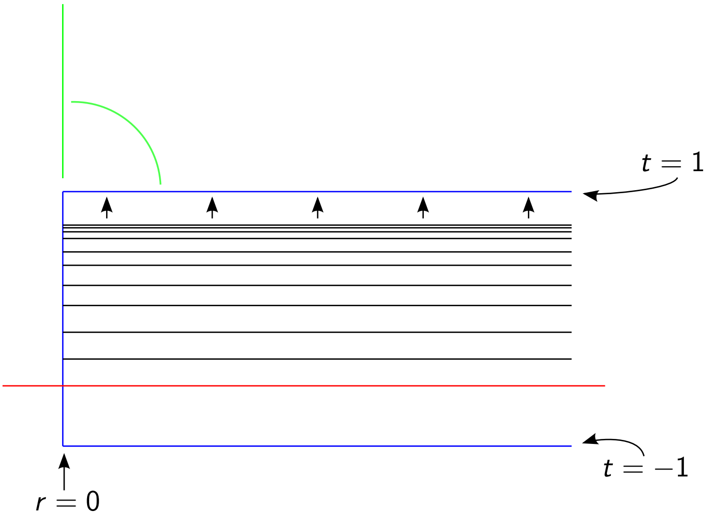

t = 1
t = −1
r = 0


$t = 1$
$t = -1$
$r = 0$


$t = 1$
$t = -1$
$r = 0$


$t = 1$
$t = -1$
$r = 0$

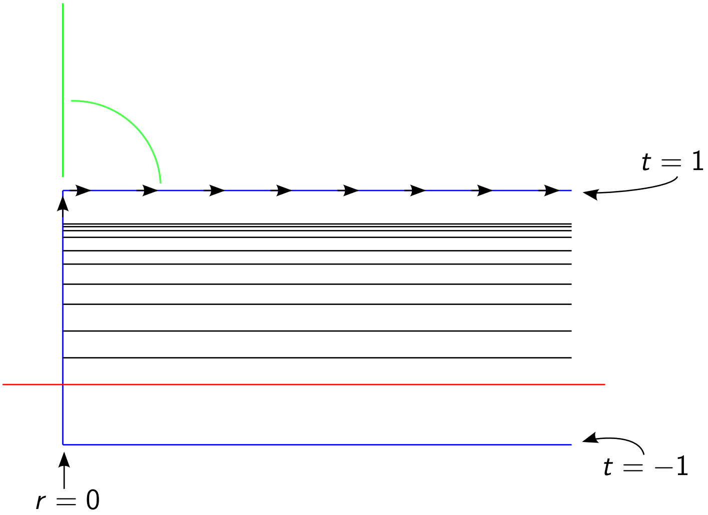
$$
\begin{aligned}
(1 + t\kappa')\partial_t\phi_0 - \kappa\partial_r\phi_0 &= -(3\kappa' - \mu)\phi_0 - \mu\alpha_2\phi_1 \\
\partial_t\phi_1 &= -\mu\phi_1 + \frac{1}{2}\mu\alpha_2\phi_0 - \frac{1}{2}\mu\alpha_0\phi_2 \\
\partial_t\phi_2 &= \frac{1}{2}\mu\alpha_0\phi_1 - \frac{1}{2}\mu\alpha_0\phi_3 \\
\partial_t\phi_3 &= \mu\phi_3 - \frac{1}{2}\mu\alpha_2\phi_4 + \frac{1}{2}\mu\alpha_0\phi_2 \\
(1 - t\kappa')\partial_t\phi_4 + \kappa\partial_r\phi_4 &= (3\kappa' - \mu)\phi_4 + \mu\alpha_2\phi_3
\end{aligned}
$$


t = 1
t = −1
r = 0

At $r = 0$: $$(1-t)\partial_t \phi_4 = 2\phi_4 + \alpha_2 \phi_3.$$

At $t = 1$: $$r\partial_r \phi_4 = 2\phi_4 + \alpha_2 \phi_3.$$

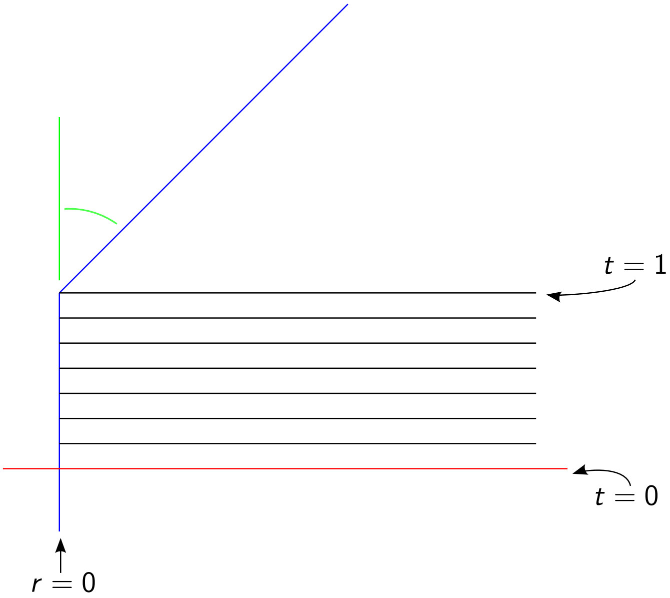
Last step for ϕ_4



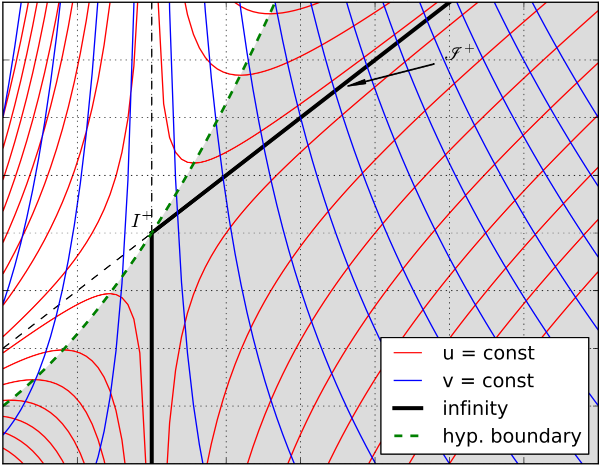
Results

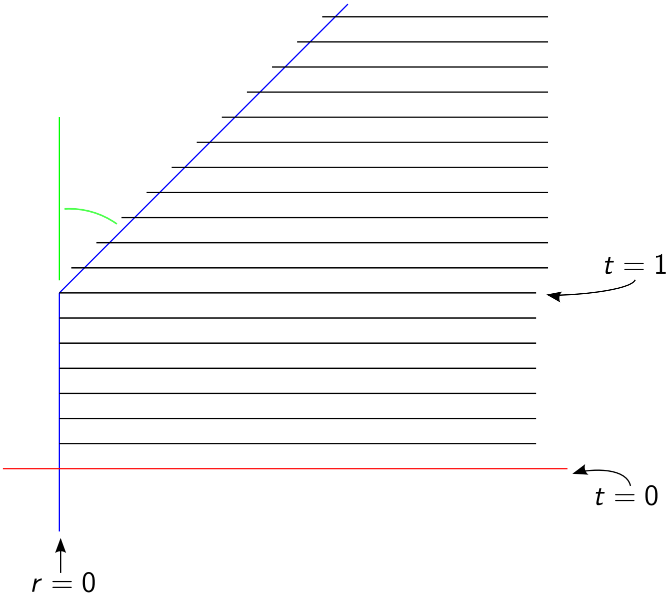
Grid Points	ϕ_0 $\log_2(\|\|\Delta\|\|_2)$	ϕ_0 Rate	ϕ_4 $\log_2(\|\|\Delta\|\|_2)$	ϕ_4 Rate
200	-7.99		0.082	
400	-11.48	3.49	-0.44	0.52
800	-14.98	3.49	-1.02	0.57
1600	-18.48	3.50	-1.73	0.71
3200	-22.07	3.58	-2.81	1.08

Table: Absolute error Δ compared to a 6400 grid point simulation using the L^2 norm and convergence rates at time $t = 1$ for ϕ_0, ϕ_4. The calculation was done with an adaptive time-step that the Euler step scheme.


t = 1
t = 0
r = 0

Characteristics for $\mu(r) = 1/(1+r)$

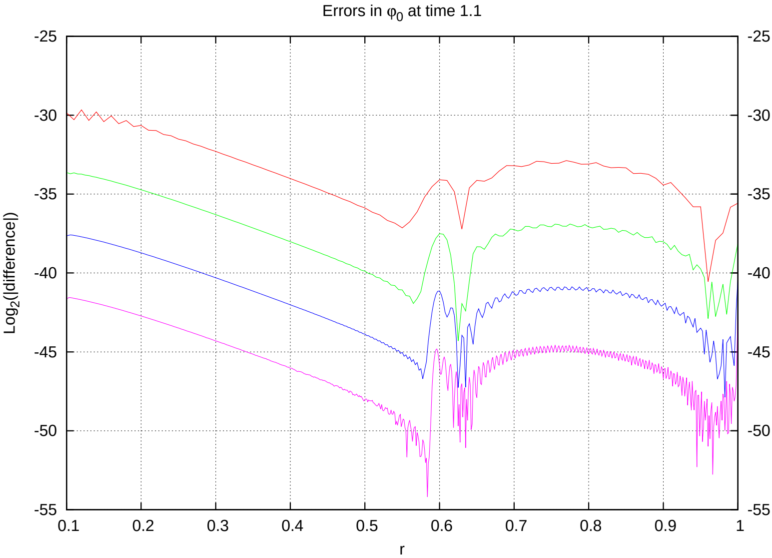


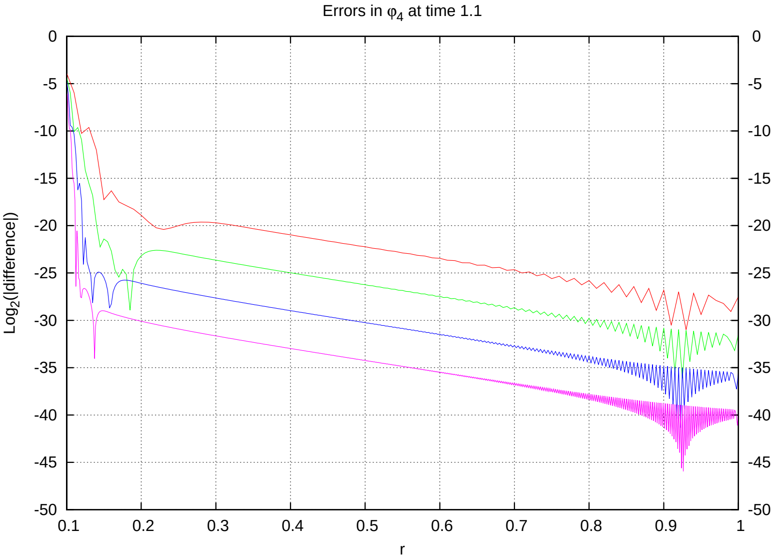

$t = 1$
$t = 0$
$r = 0$

Results

Grid Points	ϕ_0 $\log_2(\|\|\Delta\|\|_2)$	ϕ_0 Rate	ϕ_4 $\log_2(\|\|\Delta\|\|_2)$	ϕ_4 Rate
100	-28.29		-3.82	
200	-31.70	3.41	-4.19	0.38
400	-35.19	3.49	-4.42	0.22
800	-38.70	3.51	-6.05	1.63

Table: Absolute error Δ compared to an exact solution L^2 norm and convergence rates at time $t = 1.1$ for ϕ_0, ϕ_4. The calculation was done with a fixed time-step.


Errors in φ_0 at time 1.1
$\text{Log}_2(|\text{difference}|)$
r
-25
-30
-35
-40
-45
-50
-55
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Errors in φ$_4$ at time 1.1
Log$_2$(|difference|)
r
0
-5
-10
-15
-20
-25
-30
-35
-40
-45
-50
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

$$0 = \nabla^{h}{}_{c'}\phi_{abch}$$

Linear spin-2 zero rest mass field

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