Spectrum of states in gauge-string duality

Arkady Tseytlin

- Review of duality between N=4 supersymmetric planar 4d gauge theory and superstring theory in $AdS_5 \times S^5$

- Some recent progress
  Beccaria, Roiban, Giombi, Macorini, AT
  arXiv:1203.5710, arXiv: 1205.3656

- Last 10 years: enormous progress of understanding gauge theory - string theory duality based on integrability

- Promise of first exact solution of a 4d QFT as well as string theory in curved background

- Remarkable connections with different areas of mathematical physics: integrable spin chains, integrable 2d sigma models on supercosets, 2d CFT's, 4d CFT's stimulates research in related areas

- Maximally symmetric example:
N=4 super Yang-Mills theory dual to
superstring theory in $AdS_5 \times S^5$

- N=4 SYM is a 4d Conformal Field Theory
plus it is integrable in planar limit: spectrum of
dimensions from integrable system
(integrability rare in 4d QFT: at 1-loop only even
in N=2 SYM)

- string theory is a based on a 2d CFT: $AdS_5 \times S^5$
integrable conformal sigma model (integrability
rare for 2d s-models: G/H cosets, gauged WZW,
few pp-waves, not much more)

Review of AdS/CFT Integrability: An Overview

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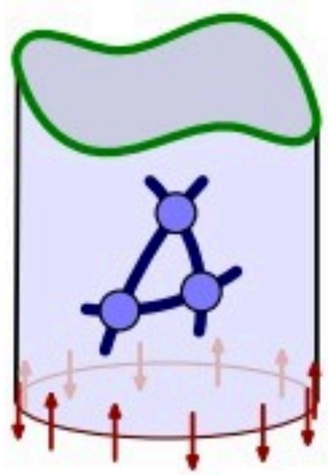

Abstract: This is the introductory chapter of a review collection on integrability in the context of the AdS/CFT correspondence. In the collection we present an overview of the achievements and the status of this subject as of the year 2010.

INTEGRABILITY IN GAUGE AND STRING THEORY

ETH Zurich, 20 – 24 August 2012

Speakers include
Nima Arkani-Hamed
Gleb Arutyunov
Benjamin Basso
Nikolay Gromov
Ben Hoare
Tomasz Łukowski
Vasily Pestun
Agustin Sabio Vera
Emery Sokatchev
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Arkady Tseytlin
Pedro Vieira

N=4 SYM as "harmonic oscillator" of 4d QFT

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Conference "$\mathcal{N}$=4 Super Yang-Mills Theory, 35 Years After"

Ooguri's conference photos

The conference is held at California Institute of Technology in Pasadena, CA, March 29-31, 2012.

...is section we follow the same procedure as in the ... of the 10-Dimensional Theory
with the action

$$S = \int d^{10}x \left\{ -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \frac{1}{2} \bar{\lambda}^a \Gamma \cdot D \lambda^a \right\}$$

satisfies both the Majorana (2.4) and the Weyl ...
...mmetry transformations are, as before,

$$\delta A^a_\mu = i \bar{\alpha} \Gamma_\mu \lambda^a$$

We invite you to celebrate the developments in maximally supersymmetric gauge theories over the past 35 years and the completion of renovation on the fourth floor of Lauritsen-Downs.

Invited speakers include:

Ofer Aharony	Lars Brink	Joseph Polchinski	Maria Spiropulu
Nima Arkani-Hamed	Clifford Cheung	Alexander Polyakov	Matthias Staudacher
Niklas Beisert	Henriette Elvang	Lisa Randall	Arkady Tseytlin
Nathan Berkovits	Juan Maldacena	Ashoke Sen	Anastasia Volovich
Zvi Bern	Gregory Moore	David Skinner	Edward Witten

Link to all lecture titles.

Announcement: Prof. David Gross will give a public lecture at 8:00 PM on Wednesday March 28.

We look forward to seeing you all at Caltech!

The organizing committee:

Lars Brink
Sergei Gukov
Anton Kapustin
John Schwarz

$\mathcal{N} = 4$ Gauge Theory

$\mathrm{U}(N)$ gauge field $\mathcal{A}_\mu$, 4 adjoint fermions Ψ^a_α, 6 adjoint scalars Φ_m

$$\lambda\, S_{\mathcal{N}=4} = N \int \frac{d^4x}{4\pi^2}\, \mathrm{Tr}\Big(\tfrac{1}{4}(\mathcal{F}_{\mu\nu})^2 + \tfrac{1}{2}(\mathcal{D}_\mu \Phi_m)^2 - \tfrac{1}{4}[\Phi_m, \Phi_n]^2 + \ldots\Big)$$

Some remarkable properties:

- Unique action due to maximal supersymmetry,
- single coupling constant $g \sim \sqrt{\lambda} \sim g_{\mathrm{YM}}\sqrt{N}$ (plus top. θ-angle),
- all fields adjoints: $N \times N$ matrices for $\mathrm{U}(N)$ gauge group,
- all fields massless (pure gauge),
- "finite" theory: beta-function exactly zero, no running coupling,
- unbroken conformal symmetry,
- superconformal symmetry $\mathrm{PSU}(2,2|4)$.

conformal SO(2,4), global SO(6), Q- and S-supersymmetry

AdS/CFT Correspondence

Conjectured exact duality of [Maldacena hep-th/9711200][Gubser Klebanov Polyakov][Witten hep-th/9802150]

- IIB string theory on $AdS_5 \times S^5$ and
- $\mathcal{N} = 4$ gauge theory (CFT).

Symmetry groups match: $\mathrm{PSU}(2,2|4)$.

Holography: Boundary of AdS_5 is conformal $\mathbb{R}^4$.

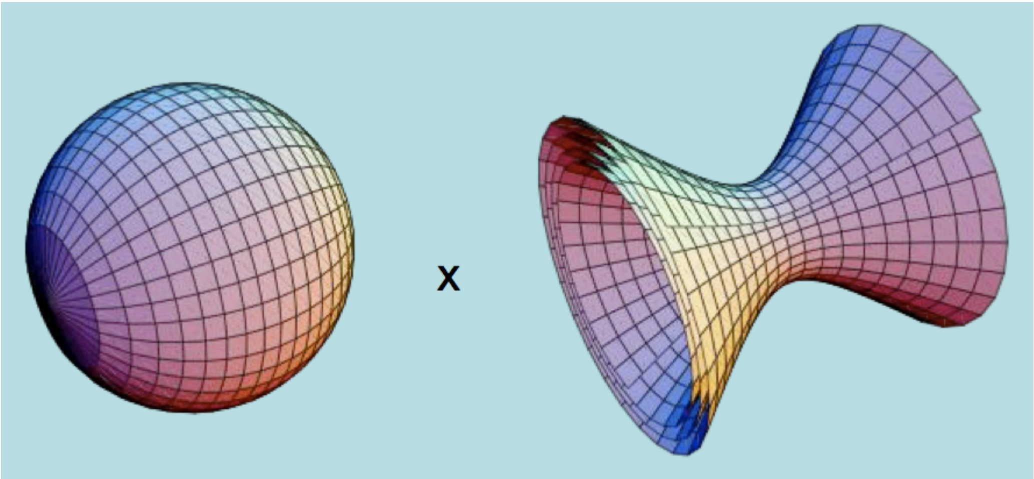
Prospects:

- More general AdS/CFT may explain aspects of QCD strings.
- Study aspects of quantum gravity with QFT means.

No proof yet!

(Very basic) AdS/CFT duality

- $\boxed{\text{IIB superstring on } AdS_5 \times S^5 \quad \leftrightarrow \quad \mathcal{N} = 4 \text{ SYM in } d = 4}$
- **Kinematics $\equiv$ symmetry = $\surd$**
- Isometries of $AdS_5 \times S^5$: $SO(4,2) \times SO(6)$



- **and** in $\mathcal{N} = 4$ SYM

$$SO(4,2) \times SO(6) \overset{bosonic}{\subset} PSU(2,2|4) \qquad !$$

- **What about dynamics ?**

What does it mean to solve 4d CFT:

basic CFT data: $\Delta_i,\ C_{ijk}$

- last few years: scaling dimensions Δ at any coupling

$$\langle \mathcal{O}(x^{(1)})\mathcal{O}(x^{(2)})\rangle = \frac{1}{|x^{(12)}|^{2\Delta(\lambda)}}$$

$$x^{(ij)} = x^{(i)} - x^{(j)}$$

- recent progress: computing some 3-point functions

$$\langle \mathcal{O}_1(x^{(1)})\mathcal{O}_2(x^{(2)})\mathcal{O}_3(x^{(3)})\rangle$$
$$= \frac{C_{123}(\lambda)}{|x^{(12)}|^{\Delta_1+\Delta_2-\Delta_3}|x^{(23)}|^{\Delta_2+\Delta_3-\Delta_1}|x^{(31)}|^{\Delta_3+\Delta_1-\Delta_2}}$$

- $\Delta_i,\ C_{ijk}$ determine higher correlators via OPE:

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k |x|^{\Delta_k-\Delta_i-\Delta_j} C_{ij}^k \mathcal{O}_k(0)$$

Planar theory: $SU(N),\quad N\to\infty,\quad \lambda = g_{\rm YM}^2 N$=fixed

What does it mean to solve string theory:

Compute spectrum of energies of string states

find corresponding vertex operators and their correlations functions (scattering amplitudes)

String in $AdS_5 \times S^5$:

tension $T = \frac{R^2}{4\pi\alpha'} = \frac{\sqrt{\lambda}}{4\pi}$

Spectrum:

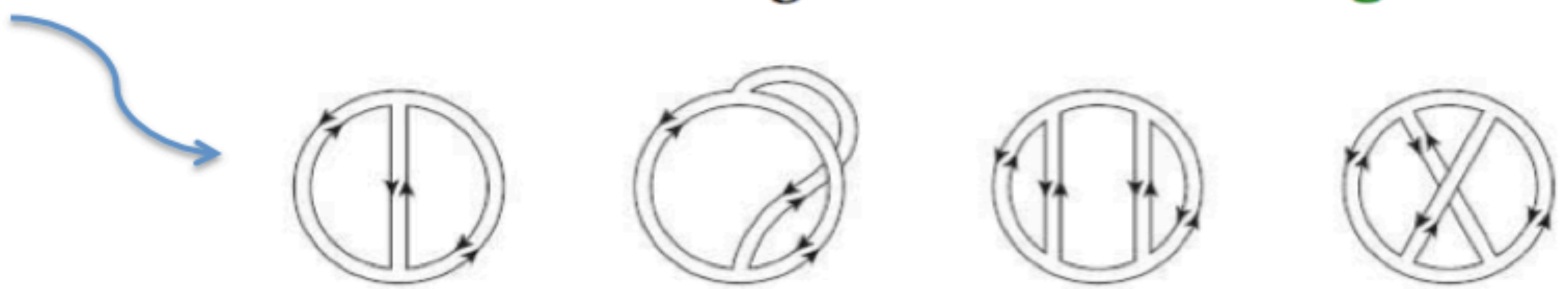
energy as function E of tension or λ

and conserved charges (mode numbers)

- Gauge/string coupling relations

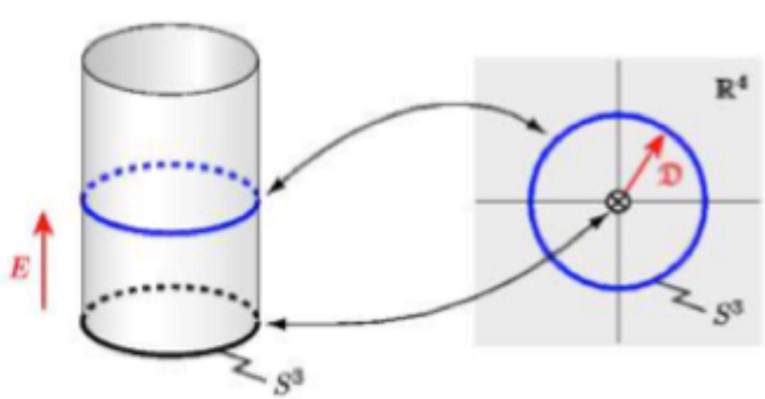
$$\underbrace{\frac{4\pi\lambda}{N_c}}_{gauge} = \underbrace{g_s}_{string}, \qquad (\lambda = N_c\, g_{YM}^2)$$

- Planar limit $N_c \to \infty \Longrightarrow g_s \to 0$, **free string**



- **Weak - Strong** duality

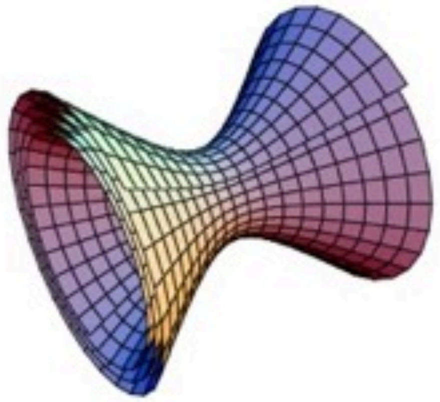
$$\boxed{\sqrt{\lambda} = \frac{R^2}{\alpha'} \equiv (\text{non-linear } \sigma\text{-model coupling})^{-1}}$$



$$\boxed{\textit{basic prediction:} \qquad E_{\text{string}} = \Delta_{\text{CFT}}}$$

Spectra of String and Gauge Theory

String Theory:



States: Solutions of classical equations of motion
plus quantum corrections.

Energy: Charge for translation along AdS-time
(rotations along unwound circle in figure)

Gauge Theory:

States: Local operators. Local combinations of the fields, e.g.

$$\mathcal{O} = \mathrm{Tr}\, \Phi_1 \Phi_2 (\mathcal{D}_1 \mathcal{D}_2 \Phi_2)(\mathcal{D}_1 \mathcal{F}_{24}) + \ldots$$

Energy: Scaling dimensions, e.g. two-point function in conformal theory

$$\langle \mathcal{O}(x)\, \mathcal{O}(y) \rangle = C|x-y|^{-2D(\lambda)}$$

Matching: String energies and gauge dimensions match, $E(\lambda) = D(\lambda)$.

Notation: $D = \Delta$

Strong/Weak Duality

Problem: Strong/weak duality.

- Perturbative regime of strings at $\lambda \to \infty$

$$E(\lambda) = \sqrt{\lambda}\, E_0 + E_1 + E_2/\sqrt{\lambda} + \dots$$

E_ℓ: Contribution at ℓ (world-sheet) loops. Limit: 1 or 2 loops.

- Perturbative regime of gauge theory at $\lambda \approx 0$.

$$D(\lambda) = D_0 + \lambda D_1 + \lambda D_2 + \dots$$

D_ℓ: Contribution at ℓ (gauge) loops. Limit: 3 or 4 loops.

Tests impossible unless quantities are known at finite λ.

Cannot compare, not even approximately. [a priori]

Integrability may help.

How can we test AdS/CFT ?

- The two sides of the correspondence **can be worked** out in **opposite regimes**

string theory $\sqrt{\lambda} = \frac{R^2}{\alpha'} \gg 1.$

conformal theory $\lambda = g_{YM}^2 N_c \ll 1.$

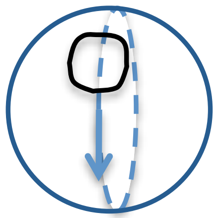
- **BPS** states $\mathrm{Tr}(Z^L)$ have a **trivial** λ dependence : Equality easily checked

- **Near-BPS** states are also **~ok** since σ model corrections can be suppressed

$$\mathrm{Tr}(\underbrace{ZZZZ\ldots ZZZXZZZZ\ldots ZZ}_{J}),$$

$J \gg 1$, dual to

(dilute limit)



What about **far-from-BPS** states ?

***Simple* gauge theory operators / *Simple* classical string solutions**

To sum up:

- gauge-string duality: spectrum of gauge dimensions = spectrum of string energies but perturbative expansions are opposite

- how to compute dimensions of gauge theory operators exactly?

- how to compute string energies exactly?

- Answer: exact description by common integrable 2d system

Gauge theory anomalous dimensions:

- Protected operators (conserved currents, BPS, ...)

$$\begin{aligned} \mathcal{O}_\mu^\nu &= \mathrm{Tr}\left(F_{\mu\lambda}\,F^{\lambda\nu} - \frac{1}{4}\,\delta_\mu^\nu\,F^2 + \text{scalars} + \text{fermions}\right) \\ \Delta &= 4. \end{aligned}$$

- Non degenerate operators without mixing (Konishi)

$$\begin{aligned} \mathcal{O} &= \mathrm{Tr}\,\Phi^a\,\Phi^a \\ \Delta &= 2 + \frac{3\,\lambda}{4\,\pi^2} - \frac{3\,\lambda^2}{16\,\pi^4} + \frac{21\,\lambda^3}{256\,\pi^6} + \cdots \end{aligned}$$

- The calculation of $\Delta(\lambda)$ for unprotected operators $\equiv$ difficult mixing problem (esp. for large charges...)

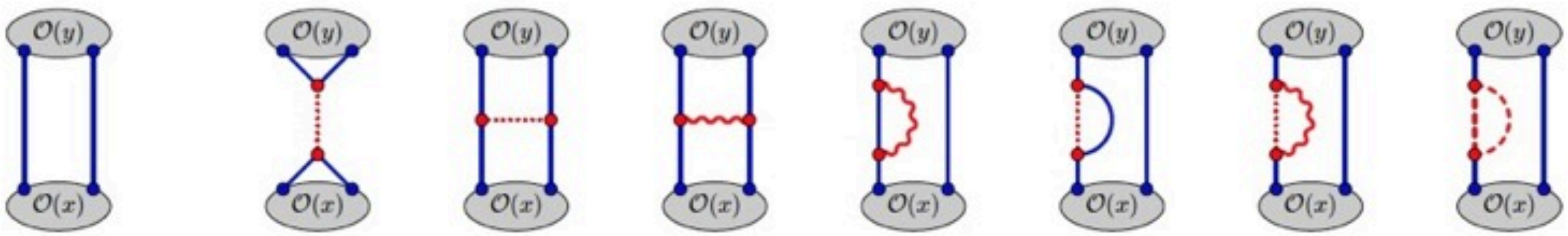
$$\begin{aligned} \mathcal{O} &= \mathrm{Tr}\left[\Phi^a\,\Phi^a\,\Phi^b\,\Phi^b + \mathcal{O}(\lambda)\,\Phi^a\,\Phi^b\,\Phi^a\,\Phi^b + \cdots\right] \\ \Delta &= 4 + \mathcal{O}(\lambda) \end{aligned}$$

A Sample Operator

Local, gauge invariant combination of the fields, e.g.

$$\mathcal{O}_{kl}^{\mathrm{bare}}(x) = \mathrm{Tr}\, \Phi_k(x)\, \Phi_l(x).$$

Two-point function at one loop. Diagrams:



Correlator (in dimensional reduction scheme) [Bianchi, Kovacs Rossi, Stanev]

$$\left\langle \mathcal{O}_{kl}^{\mathrm{bare}}(x)\, \mathcal{O}_{mn}^{\mathrm{bare}}(y) \right\rangle = \frac{2(1-1/N^2)}{|x-y|^{4-4\epsilon}} \left(\delta_{k\{m}\delta_{n\}l} - \frac{6g^2\, \delta_{kl}\delta_{mn}}{\epsilon |x-y|^{-2\epsilon}} + \dots \right).$$

Renormalisation and Mixing

Correlator

$$\left\langle \mathcal{O}_{kl}^{\mathrm{bare}}(x)\,\mathcal{O}_{mn}^{\mathrm{bare}}(y)\right\rangle = \frac{2(1-1/N^2)}{|x-y|^{4-4\epsilon}}\left(\delta_{k\{m}\delta_{n\}l} - \frac{6g^2\,\delta_{kl}\delta_{mn}}{\epsilon|x-y|^{-2\epsilon}} + \ldots\right).$$

Renormalisation: coefficients divergent & unphysical

$$\mathcal{O}_{kl} = \mathcal{O}_{kl}^{\mathrm{bare}} + \frac{g^2}{2\epsilon}\,\delta_{kl}\delta_{mn}\mathcal{O}_{mn}^{\mathrm{bare}} + \ldots$$

Mixing: Correlotor is non-diagonal $\langle \mathcal{O}_{11}(x)\,\mathcal{O}_{22}(y)\rangle \neq 0$

$$\mathcal{Q}_{kl} = \mathcal{O}_{kl} - \tfrac{1}{6}\delta_{kl}\delta_{mn}\mathcal{O}_{mn}, \qquad \mathcal{K} = \delta_{mn}\mathcal{O}_{mn}.$$

Here: Mixing resolved by representation of $\mathfrak{so}(6)$; 20-plet $\mathcal{Q}_{kl}$, singlet $\mathcal{K}$.

Usually: Mixing among many states with equal quantum numbers.

The dilatation operator

(γ from d log$Z^A{}_B$/dlogΛ
plus symmetry...)

- **The dilatation operator** $\mathfrak{D} \in \mathfrak{psu}(2,2|4)$
  (dual to t-isometry on the string side)

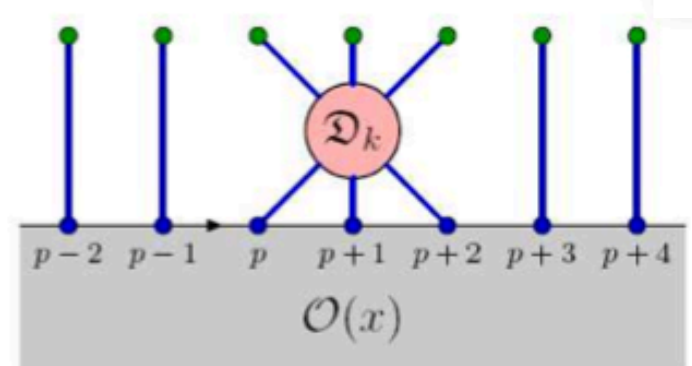
- In the **planar** limit,
  $\mathfrak{D} \longrightarrow$ integrable Hamiltonians

[Beisert et al, 03]

$$\mathfrak{D} = \sum_{\ell \geq 1} \lambda^{\ell} \mathcal{H}^{(\ell)}_{\text{integrable}}.$$

Local interactions

$$\mathfrak{D}_k = \sum_{p=1}^{L}$$



- $\mathcal{H}^{(\ell)}_{\text{integrable}} \longrightarrow$ spin chain with range $\sim \ell \longrightarrow$ **wrapping**

Dilatation Generator

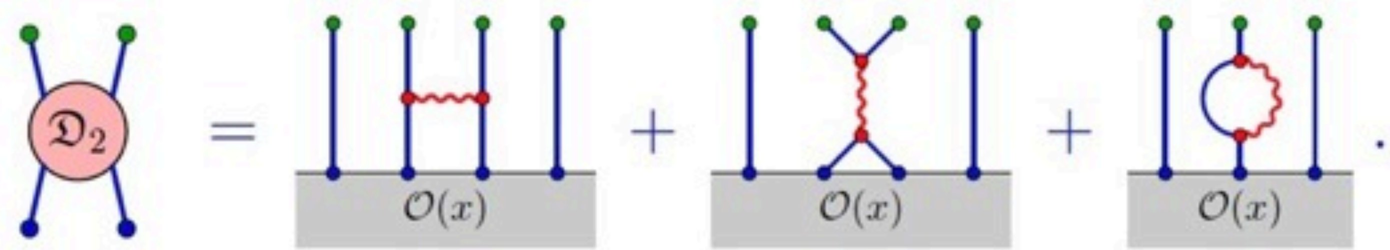
Scaling dimensions $D_{\mathcal{O}}(g)$ as eigenvalues of the dilatation generator $\mathfrak{D}(g)$

$$\mathfrak{D}(g)\,\mathcal{O} = D_{\mathcal{O}}(g)\,\mathcal{O}.$$

Spin chain picture: Hamiltonian $\delta\mathfrak{D} = g^2\mathcal{H}$ & energies $\delta D = g^2 E$.

At **leading order** (one loop): Interactions of nearest-neighbours

$\mathcal{H}_0 =$



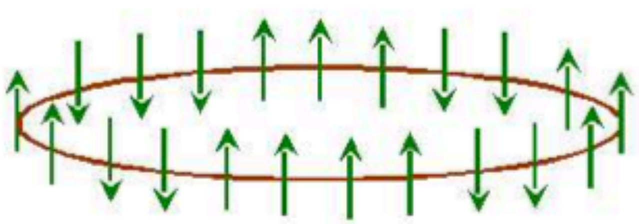
Regularised action of $\delta\mathfrak{D}$ in $\mathfrak{su}(2)$ sector: Heisenberg $\mathrm{XXX}_{1/2}$ chain [Minahan Zarembo]

$$\mathcal{H} = \sum_{p=1}^{L}\left(\mathcal{I}_{p,p+1} - \mathcal{P}_{p,p+1}\right) = \sum_{p=1}^{L}\tfrac{1}{2}\left(1 - \vec{\sigma}_p\cdot\vec{\sigma}_{p+1}\right).$$

An explicit example: the $\mathfrak{su}(2)$ sector at one loop

In $\mathfrak{su}(2)$, $Z = \varphi_1 + i\,\varphi_2$, $W = \varphi_3 + i\,\varphi_4$,

$$\mathcal{O}_\alpha^{J_1,J_2} = \mathrm{Tr}\left(\underbrace{Z\cdots Z}_{J_1}\,\underbrace{W\cdots W}_{J_2} + \text{permutations}\right).$$



- At **tree level**, $\mathfrak{D}^{(0)} = J_1 + J_2 =$ classical dimension.
- At 1 loop, in the **planar limit** [Minahan, Zarembo, 02]

$$\mathfrak{D}^{(1)} = \frac{\lambda}{8\pi^2}\sum_{i=1}^{L}(1 - P_{i,i+1}) = \frac{\lambda}{16\pi^2}\sum_{i=1}^{L}(1 - \vec{\sigma}_i\cdot\vec{\sigma}_{i+1}) = \frac{\lambda}{8\pi^2}\,\mathcal{H}_{XXX}$$

One-loop Bethe Ansatz

$$1 = \frac{x(u_k - \frac{i}{2})^L}{x(u_k + \frac{i}{2})^L} \prod_{\substack{j=1 \\ j \neq k}}^{K} \frac{u_k - u_j + i}{u_k - u_j - i}, \qquad x(u) = \tfrac{1}{2}u + \tfrac{1}{2}u\sqrt{1 - 2g^2/u^2}\,.$$

Momentum constraint (cyclity of trace) and higher-loop scaling dimension:

$$\prod_{j=1}^{K} \frac{x(u_j - \frac{i}{2})}{x(u_j + \frac{i}{2})} = 1, \qquad D = L + g^2 \sum_{j=1}^{K} \left(\frac{i}{x(u_j + \frac{i}{2})} - \frac{i}{x(u_j - \frac{i}{2})} \right).$$

Beyond one loop ?

- Dilatation operator not known explicitly beyond 4 loops -- but can make assumption of all-order integrability and then verify its consistency
- Bethe Ansatz generalized to higher loops for ``long'' operators [Asymptotic Bethe Ansatz]
- Checked against available perturbative data and general principles (crossing of magnon S-matrix, etc.)

Higher order integrability ? The $\mathfrak{su}(2)$ sector

- Loop expansion of $\mathfrak{D}$

$$\mathfrak{D} = \sum_{\ell=1}^{L} \left(1 + g^2 H_1 + g^4 H_2 + g^6 H_3 + \cdots\right)$$

- H_i are **integrable** spin chains with increasing range (hopping expansion of Hubbard model ?)

$$\begin{aligned}
H_1 &= \frac{1}{2}(1 - \sigma_\ell \cdot \sigma_{\ell+1}) \\
H_2 &= -(1 - \sigma_\ell \cdot \sigma_{\ell+1}) + \frac{1}{4}(1 - \sigma_\ell \cdot \sigma_{\ell+2}) \\
H_3 &= \frac{15}{4}(1 - \sigma_\ell \cdot \sigma_{\ell+1}) - \frac{3}{2}(1 - \sigma_\ell \cdot \sigma_{\ell+2}) + \frac{1}{4}(1 - \sigma_\ell \cdot \sigma_{\ell+3}) + \\
&\quad -\frac{1}{8}(1 - \sigma_\ell \cdot \sigma_{\ell+3})(1 - \sigma_{\ell+1} \cdot \sigma_{\ell+2}) + \\
&\quad +\frac{1}{8}(1 - \sigma_\ell \cdot \sigma_{\ell+2})(1 - \sigma_{\ell+1} \cdot \sigma_{\ell+3})
\end{aligned}$$

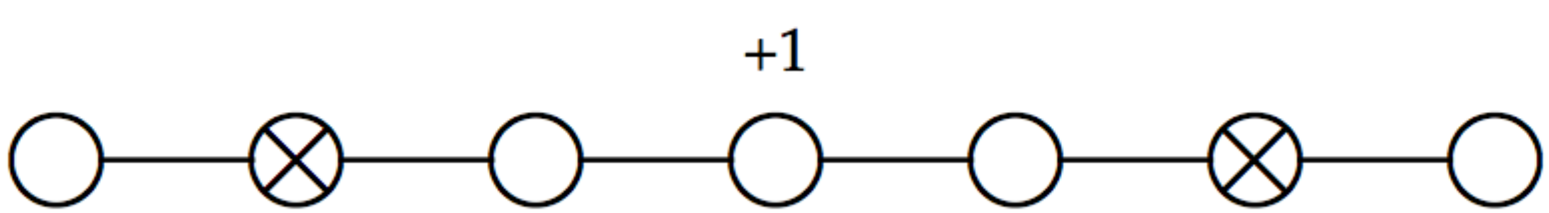
Extension to full psu(2,2|4) spin chain

One-loop general $\mathfrak{g}$-invariant Bethe Ansatz

- **minimal** integrable chain with (super) algebra $\mathfrak{g}$
- rank r, state with $K = K_1 + \cdots + K_r$ Bethe roots $u_i, i = 1, \ldots, K$.
- $k_j = 1, \ldots, r$ labels which simple roots is associated with u_j

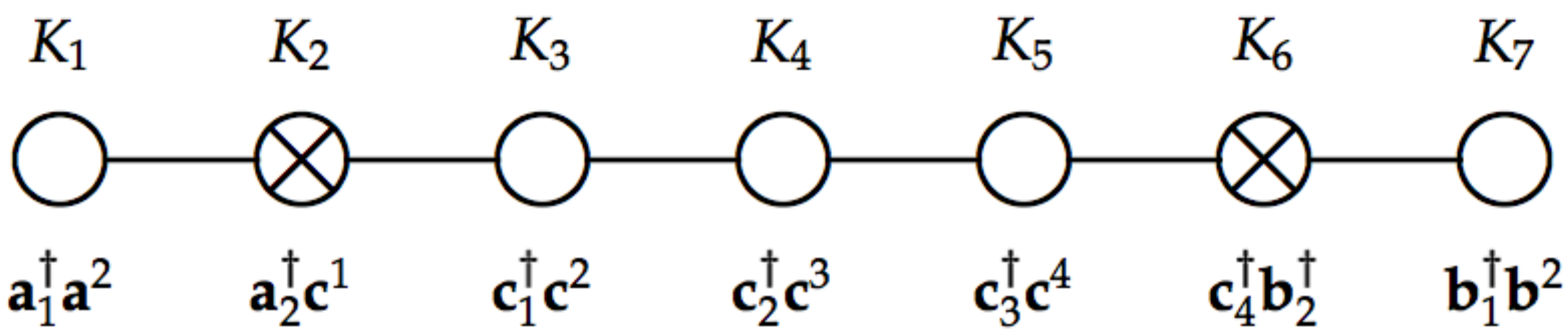
Bethe equations [Ogievetsky, Wiegmann, 86]

$$\left(\frac{u_j + \frac{i}{2}V_{k_j}}{u_j - \frac{i}{2}V_{k_j}}\right)^L = \prod_{\substack{\ell=1 \\ \ell\neq j}}^{K} \frac{u_j - u_\ell + \frac{i}{2}M_{k_j,k_\ell}}{u_j - u_\ell - \frac{i}{2}M_{k_j,k_\ell}}.$$



$\mathfrak{psu}(2,2|4)$ is no exception

- Favourite Dynkin diagram for $\mathcal{N} = 4$ SYM



- Cartan matrix and **singleton** representation on $D^n\,(\varphi, \lambda, A)$
- For any particular (highest weight) state

$$w = [\lambda_1, \lambda_2, \lambda_3]^{\Delta_0}_{(j,\bar{j})}.$$

- We compute the excitations $K_1, \ldots, K_7$ over the BPS vacuum

$$\bigotimes^{L} |Z\rangle, \qquad |Z\rangle = \Phi_{34} = \mathbf{c}_3^\dagger\, \mathbf{c}_4^\dagger\, |0\rangle.$$

and solve (numerically ?!) **the Bethe equations !**

Asymptotic Bethe Ansatz equations

[Beisert,Eden, Staudacher 2006]

coupling constant

$$g^2 = \frac{\lambda}{8\pi^2}$$

transformation between u and x

$$x(u) = \tfrac{1}{2}u + \tfrac{1}{2}u\sqrt{1-2g^2/u^2}\,, \qquad u(x) = x + \frac{g^2}{2x}$$

$x^\pm$ parameters

$$x^\pm = x(u \pm \tfrac{i}{2})$$

spin chain momentum

$$e^{ip} = \frac{x^+}{x^-}$$

local charges

$$q_r(x) = \frac{1}{r-1}\left(\frac{i}{(x^+)^{r-1}} - \frac{i}{(x^-)^{r-1}}\right), \qquad Q_r = \sum_{j=1}^{K_4} q_r(x_{4,j})$$

anomalous dimension

$$\delta D = g^2 Q_2 = g^2 \sum_{j=1}^{K_4}\left(\frac{i}{x^+_{4,j}} - \frac{i}{x^-_{4,j}}\right)$$

Bethe equations

$$1 = \prod_{j=1}^{K_4} \frac{x^+_{4,j}}{x^-_{4,j}}$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{u_{2,k}-u_{2,j}-i}{u_{2,k}-u_{2,j}+i} \prod_{j=1}^{K_3} \frac{u_{2,k}-u_{3,j}+\frac{i}{2}}{u_{2,k}-u_{3,j}-\frac{i}{2}}$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}+\frac{i}{2}}{u_{3,k}-u_{2,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k}-x^+_{4,j}}{x_{3,k}-x^-_{4,j}}$$

$$1 = \left(\frac{x^-_{4,k}}{x^+_{4,k}}\right)^{K_0} \prod_{\substack{j=1\\ j\neq k}}^{K_4} \left(\frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i}\,\sigma^2(x_{4,k},x_{4,j})\right) \prod_{j=1}^{K_3} \frac{x^-_{4,k}-x_{3,j}}{x^+_{4,k}-x_{3,j}} \prod_{j=1}^{K_5} \frac{x^-_{4,k}-x_{5,j}}{x^+_{4,k}-x_{5,j}}$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k}-u_{6,j}+\frac{i}{2}}{u_{5,k}-u_{6,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5,k}-x^+_{4,j}}{x_{5,k}-x^-_{4,j}}$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_6} \frac{u_{6,k}-u_{6,j}-i}{u_{6,k}-u_{6,j}+i} \prod_{j=1}^{K_5} \frac{u_{6,k}-u_{5,j}+\frac{i}{2}}{u_{6,k}-u_{5,j}-\frac{i}{2}}$$

function $\sigma(x_1,x_2)$ for quantum strings, coefficients: $c_{r,s} = \delta_{r+1,s} + \mathcal{O}(1/g)$

$$\sigma(x_1,x_2) = \exp\left(i \sum_{r<s=2}^{\infty} (\tfrac{1}{2}g^2)^{(r+s-1)/2}\, c_{r,s}(g)\, \big(q_r(x_1)\,q_s(x_2) - q_r(x_2)\,q_s(x_1)\big)\right)$$

BES dressing phase

- Non-trivial phase fixed using additional assumptions (crossing, etc.) -- existence of underlying integrable 2d system

- Generalization to ``short`` operators of any length: include finite-size effects (wrapping contributions)

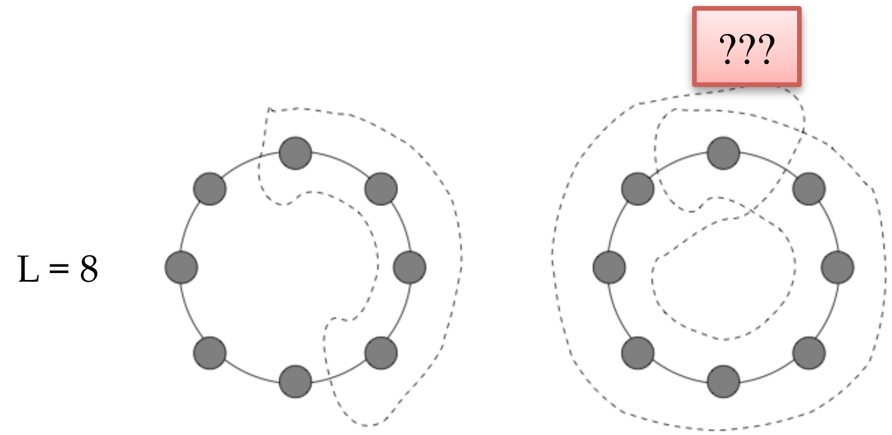
- Hint from string side - analogy with 2d models - generalize ABA to TBA Thermodynamic Bethe Ansatz (Y-system, etc)

- Use of string sigma-model picture: $R^2 \to R \times S^1$

Bethe Ansatz misses wrapping corrections !

Gauge theory/discrete chain, **only asymptotic** Bethe Ansatz

AdS/CFT



(L=2 in the previous example)

The interaction range increases with loop order

Wrapping corrections are very difficult (exponentially suppr.)

String theory/continuum σ model, **thermodynamical** Bethe Ansatz

Mirror theory (S-matrix, ABA, ...)

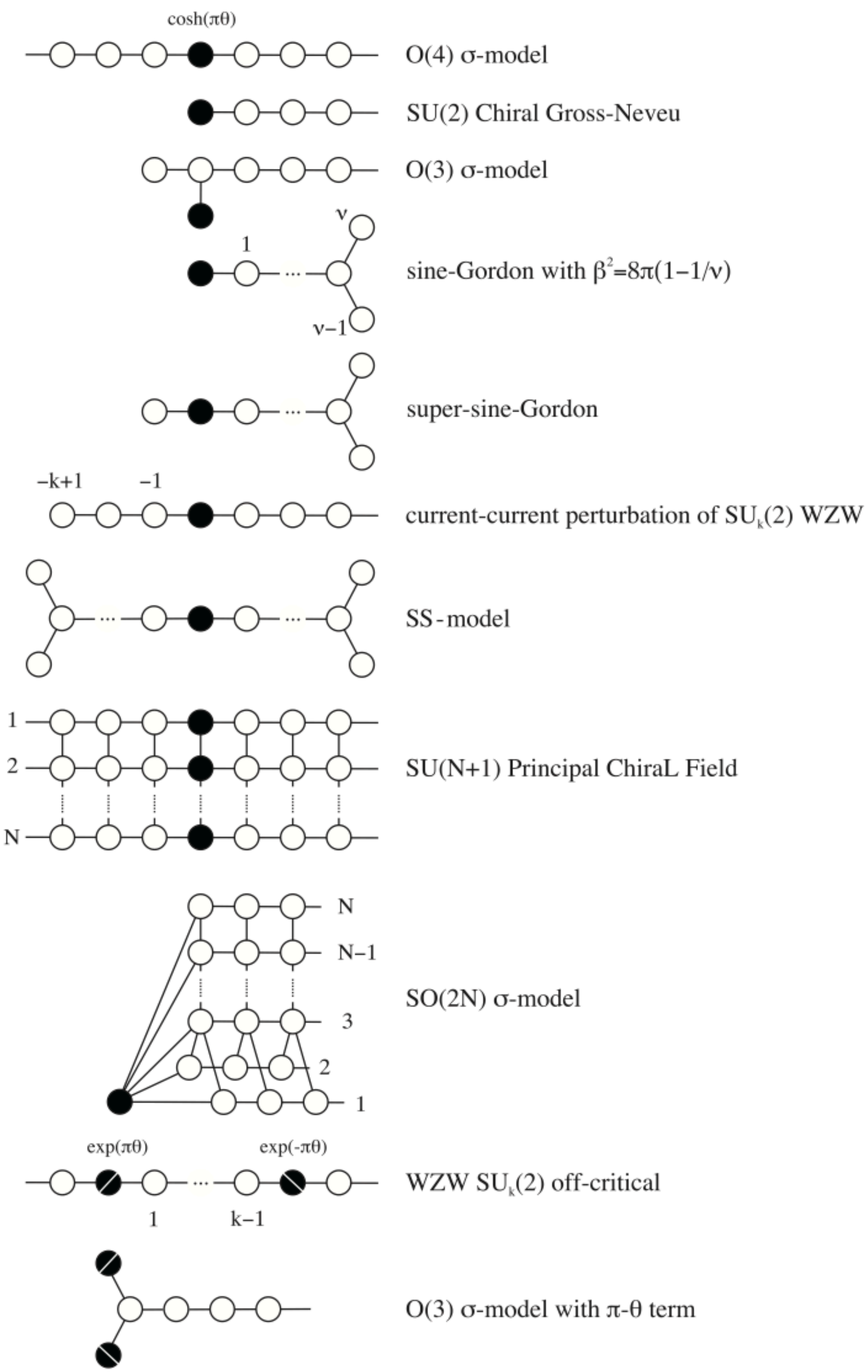
Idea: lenght $\leftrightarrow$ temperature

(Zamolodchikov)

For integrable models TBA equations are related to universal Y-systems

TBA, Y-system and Hirota equation (just a glance)

[Symmetry as input !]



$$\frac{Y^+_{a,s} Y^-_{a,s}}{Y_{a+1,s} Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

- $Y_{a,s}$ are related to the densities of particular ***clustered*** solutions of BA equations in the continuum limit **(need a string hypothesis)**

- Can be put in Hirota form

$$Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}}$$

$$T^+_{a,s} T^-_{a,s} = T_{a,s+1} T_{a,s-1} + T_{a+1,s} T_{a-1,s}$$

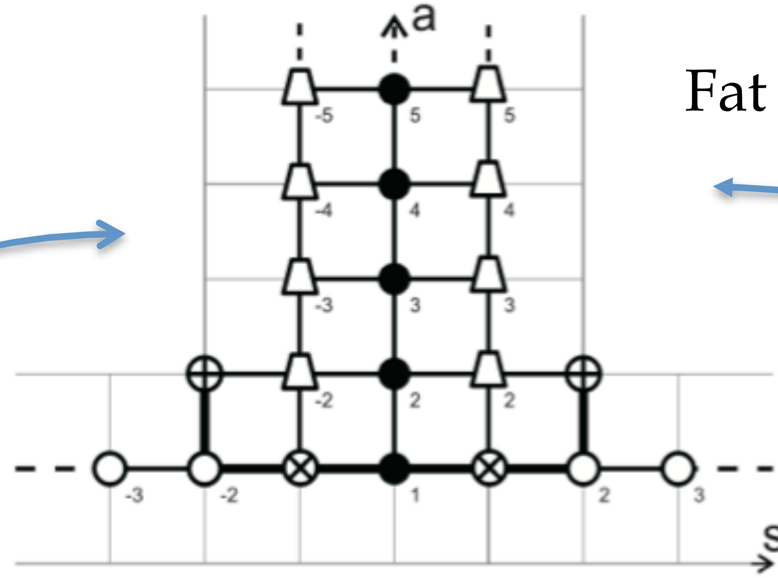
[Zamolodchikov , Krichever, Zabrodin, ...]

The PSU(2, 2 | 4) case (very non trivial)

[need ABA and extra analyticity constraints as input]

Fat hook diagram for psu(2,2|4)

SU(2|2)^2 wings



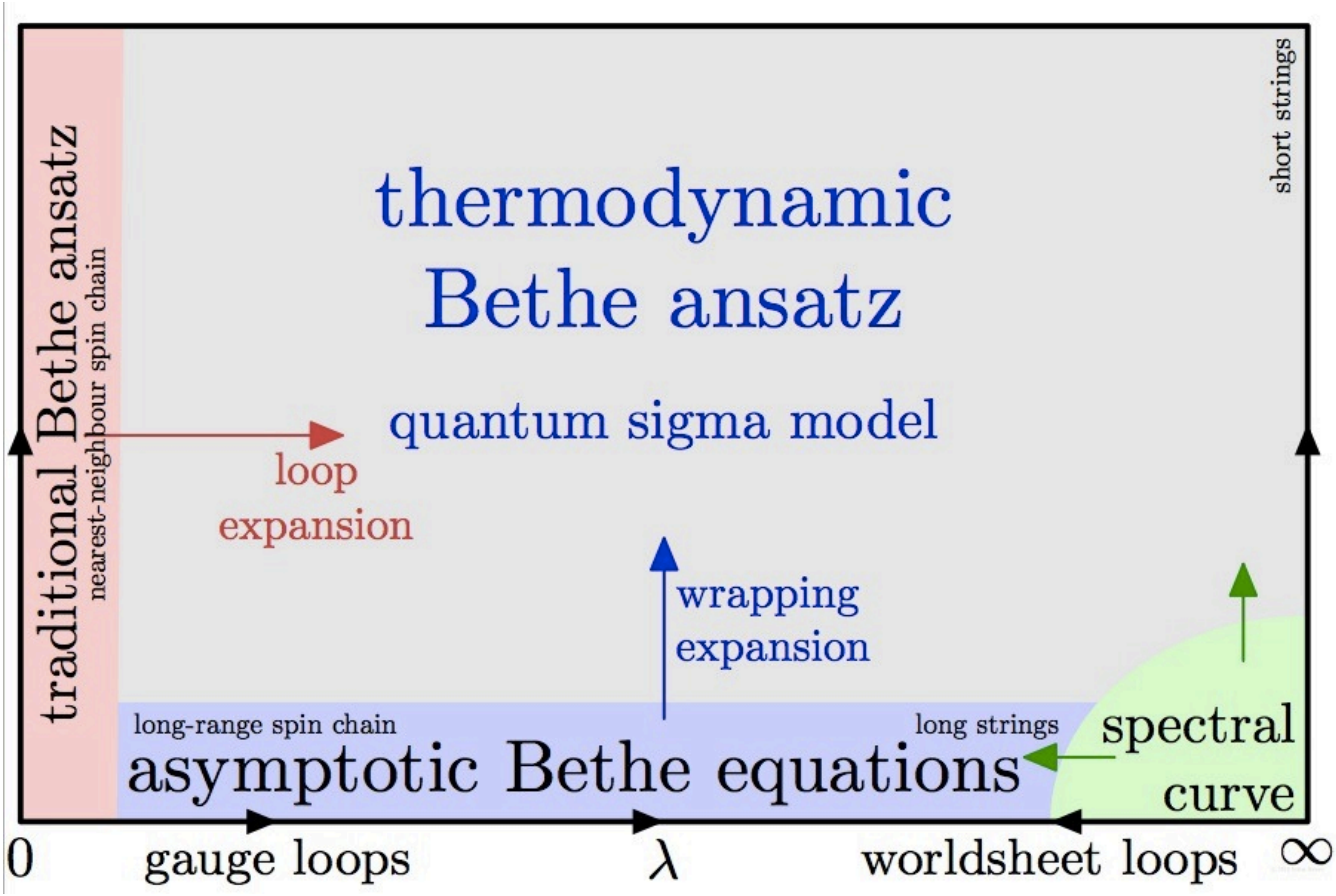
$$Y_{as}(z+\tfrac{i}{4g})Y_{as}(z-\tfrac{i}{4g}) = \frac{(1+Y_{a,s+1}(z))(1+Y_{a,s-1}(z))}{(1+1/Y_{a+1,s}(z))(1+1/Y_{a-1,s}(z))},$$

State $\leftrightarrow$ boundary conditions

The **solutions** determine the spectrum

$$E = \sum_{j=1}^{M} \varepsilon_1^{\mathrm{ph}}(z_j) + \sum_{a=1}^{\infty} \int_{\mathbb{R}} \frac{dz}{2\pi i} \frac{\partial \varepsilon_a^{\mathrm{mir}}(z)}{\partial z} \log(1+Y_{a,0}^{\mathrm{mir}})$$

Gromov, Kazakov, Vieira, ... (2009-2012)

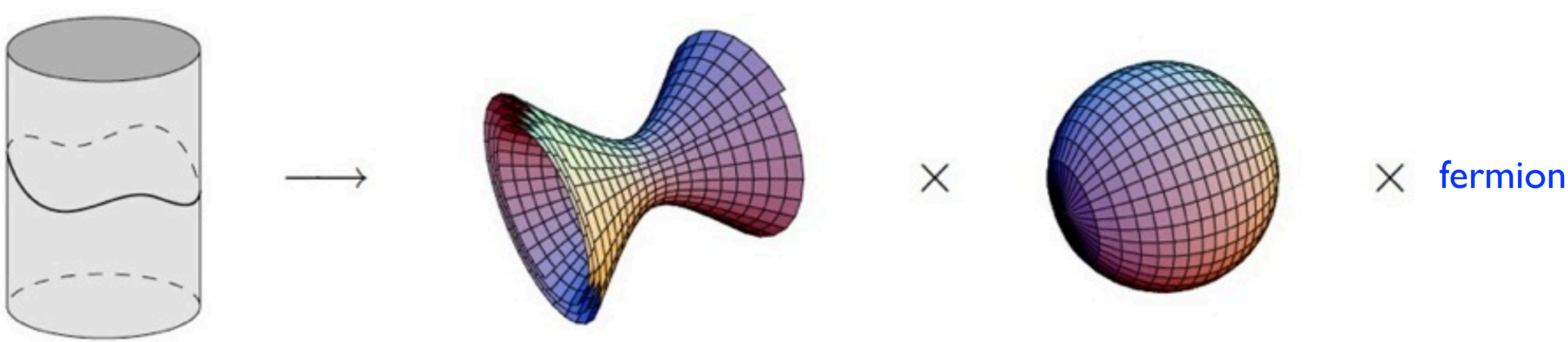

thermodynamic
Bethe ansatz
quantum sigma model
traditional Bethe ansatz
nearest-neighbour spin chain
loop
expansion
wrapping
expansion
short strings
long-range spin chain
long strings
asymptotic Bethe equations
spectral
curve
0
gauge loops
λ
worldsheet loops
∞

- Exact ABA results so far:
  (i) BES equation for scaling function (2006)
  (ii) exact slope function (2011)

- TBA results so far:
  Konishi dimension to 7-th (8-th ?) loop, numerical computation starting from weak coupling $\rightarrow$ match with strong coupling expansion from string theory side (2009-2012)

Strings on $AdS_5 \times S^5$

IIB superstrings on the curved $AdS_5 \times S^5$ superspace

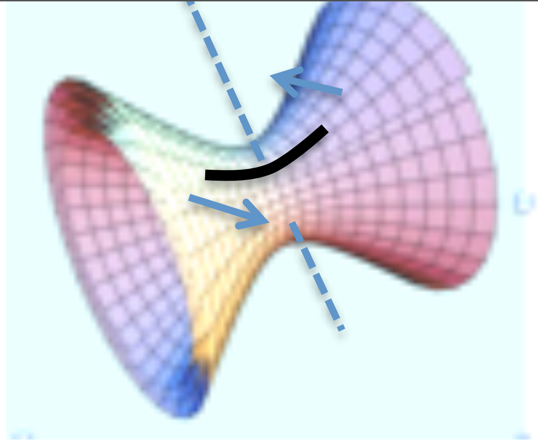


Coset space

$$AdS_5 \times S^5 \times \text{fermi} = \frac{\mathrm{PSU}(2,2|4)}{\mathrm{Sp}(1,1) \times \mathrm{Sp}(2)}\,.$$

$$I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \Big[\partial Y_p \bar{\partial} Y^p + \partial X_k \bar{\partial} X_k + \text{fermions} \Big]$$

$$-Y_0^2 - Y_5^2 + Y_1^2 + \ldots + Y_4^2 = -1\,, \qquad X_1^2 + \ldots + X_6^2 = 1$$



$$I=-\frac{\sqrt{\lambda}}{2\pi}\int d^2\xi\,\left[L_B(x,y)+L_F(x,y,\theta)\right]\,,\qquad \sqrt{\lambda}\equiv\frac{R^2}{\alpha'}$$

$$L_B=\frac{1}{2}\sqrt{-g}g^{ab}\left[G^{(AdS_5)}_{mn}(x)\partial_a x^m\partial_b x^n+G^{(S^5)}_{m'n'}(y)\partial_a y^{m'}\partial_b y^{n'}\right]$$

- Very **non-trivial** L_F (already in flat space). **Quadratic part**

$$\boxed{L_F=i(\sqrt{-g}g^{ab}\delta^{IJ}-\epsilon^{ab}s^{IJ})\bar{\theta}^I\varrho_a D_b\theta^J+O(\theta^4)}$$

$$\varrho_a\equiv\Gamma_A E^A_M\partial_a X^M=(\Gamma_p E^p_M+\Gamma_{p'}E^{p'}_M)\partial_a X^M$$

RR five form

$$D_a=\partial_a X^M\,D_M$$

$$D^{IJ}_M=(\partial_M+\tfrac{1}{4}\omega^{AB}_M\Gamma_{AB})\delta^{IJ}-\tfrac{1}{8\cdot 5!}F_{A_1\ldots A_5}\Gamma^{A_1\ldots A_5}\Gamma_M\epsilon^{IJ}$$

classical integrability

- Coset σ model

symmetry
$$\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}.$$

bosonic subalgebra
$$PSU(2,2|4)|_B = SU(2,2)\times SU(4) \simeq SO(4,2)\times SO(6)$$

$$\frac{SO(4,2)}{SO(4,1)}\times\frac{SO(6)}{SO(5)} = AdS_5\times S^5.$$

- Internal 4° order automorphism $\mathfrak{g}=\mathfrak{su}(2,2|4)=\bigoplus_{k=0}^{3}\mathfrak{g}^{(k)}, \qquad \Omega(\mathfrak{g}^{(k)})=i^k\,\mathfrak{g}^{(k)}$

$$A=-g^{-1}dg=\sum_{k=0}^{3}A^{(k)}, \qquad \partial_a A_b-\partial_b A_a-[A_a,A_b]=0, \qquad a,b=\sigma,\tau.$$

=1

$$\mathscr{L}=-\frac{\sqrt{\lambda}}{4\pi}\left[\sqrt{-h}\,h^{ab}\,\mathrm{Str}(A_a^{(2)}A_b^{(2)})+\kappa\,\epsilon^{ab}\,\mathrm{Str}(A_a^{(1)}A_b^{(3)})\right],$$

Metsaev, AT 1998

• The string equations of motion can be put in Lax form

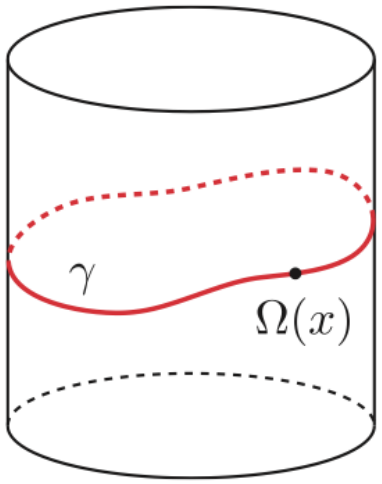
$$\begin{aligned}\partial_\sigma \Psi &= L_\sigma(\sigma,\tau,z)\,\Psi,\\ \partial_\tau \Psi &= L_\tau(\sigma,\tau,z)\,\Psi,\end{aligned}$$

z : **arbitrary** spectral parameter

• As usual, compatibility requires **flatness** $\partial_a L_b - \partial_b L_a - [L_a, L_b] = 0, \qquad a,b=\sigma,\tau.$

• The trace of the monodromy matrix is τ **independent**

$$T(z) = P\,\exp\int_0^{2\pi} d\sigma\, L_\sigma(\sigma,\tau,z),$$



infinite set of conserved charges

• Such a Lax connection indeed exists

$$L_a(z) = c_0(z)\, A_a^{(0)} + c_1(z)\, A_a^{(2)} + c_2(z)\, \gamma_{ab}\epsilon^{bc} A_c^{(2)} + c_3(z)\, A_a^{(1)} + c_4(z)\, A_a^{(3)},$$

where (in addition to the κ-symmetry condition $\kappa^2 = 1$ we must impose

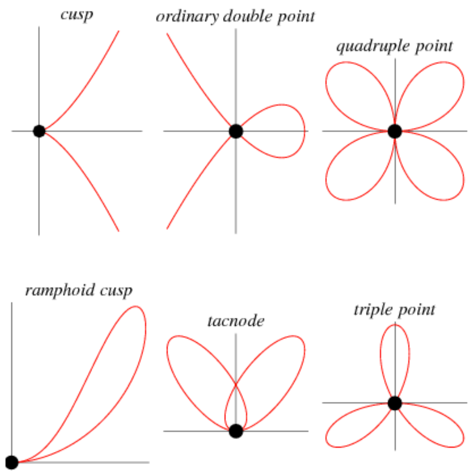
$$c_0 = 1, \quad c_1 = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right), \quad c_2 = -\frac{1}{2\kappa}\left(z^2 - \frac{1}{z^2}\right), \quad c_3 = \frac{1}{c_4} = z.$$

• We have 4+4 **gauge invariant** eigenvalues (bosonic+fermionic)

$$UTU^{-1} = \text{diag}(e^{i\widetilde{p}_1(z)}, \ldots, e^{i\widetilde{p}_4(z)} \,|\, e^{i\widehat{p}_1(z)}, \ldots, e^{i\widehat{p}_4(z)}).$$

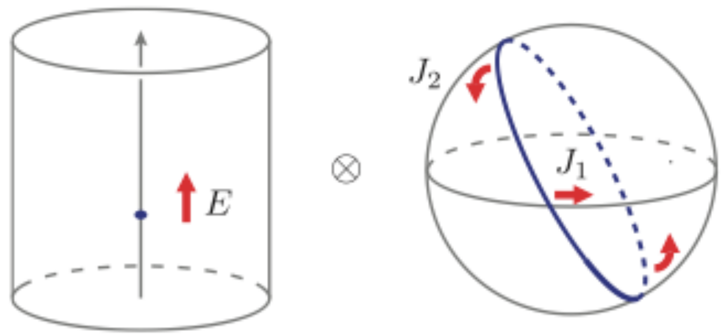
• The eigenvalues of $\quad Y(z) = -i\, z\, \frac{\partial}{\partial z} \log T(z).$

lie on an algebraic curve with only poles or branch points in z



Classical solutions in $\mathbb{R} \times S^3$ vs finite cut Bethe states in CFT

$\Phi_1, \ldots, \Phi_4$ on S^3, with $\Phi^2 = 1$, $\quad \Phi_0$ on $\mathbb{R}$.



$$g = \begin{pmatrix} \Phi_1 + i\,\Phi_2 & \Phi_3 + i\,\Phi_4 \\ -\Phi_3 + i\,\Phi_4 & \Phi_1 - i\,\Phi_2 \end{pmatrix} = \begin{pmatrix} Z & X \\ -\bar{X} & \bar{Z} \end{pmatrix} \in SU(2).$$

$$\begin{aligned} S &= \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \left[(\partial_a \Phi_i)^2 - (\partial_a \Phi_0)^2\right] = \\ &= -\frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \left[\frac{1}{2}\mathrm{Tr}(g^{-1}\partial_a g)^2 + (\partial_a \Phi_0)^2\right]. \end{aligned}$$

local symmetries and currents

$$SU(2)_L \times SU(2)_R \simeq SO(4)$$

$$\ell_a = \partial_a g\, g^{-1}, \qquad j_a = g^{-1}\partial_a g.$$

In the gauge $\Phi_0 = \kappa\tau$ we compute the energy

$$\Delta = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma\, \dot{\Phi}_0 = \sqrt{\lambda}\,\kappa.$$

Finite gap solutions vs scaling limit of BA equations

- The resolvent is **analytic** with **possible cuts**

$$G(x) = p(x) + \frac{\pi\kappa}{x-1} + \frac{\pi\kappa}{x+1},$$

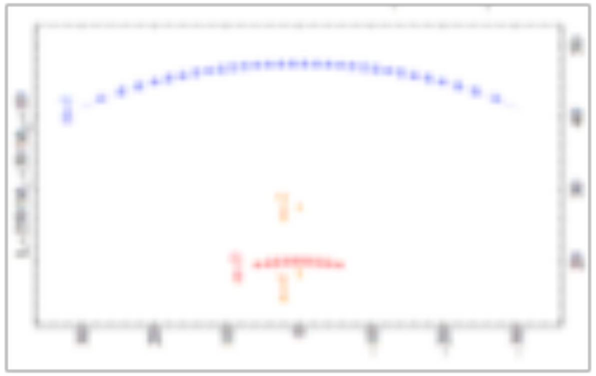
- It can be represented as a density supported on the cuts

$$G(x) = \int dy \frac{\rho(y)}{x-y}.$$

Riemann-Hilbert problem

$$G(x - i\,0) + G(x + i\,0) = 2 ⨍ dy \frac{\rho(y)}{x-y} = 2\pi\, n_C + \frac{4\pi\kappa\, x}{x^2 - 1}$$

- Nothing but the **continuous limit** of $XXX_{-1/2}$ Bethe equations



$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{\ell\neq k=1}^{M} \frac{u_k - u_i + i}{u_k - u_i - i}.$$

$$\xrightarrow[L \to \infty]{u \to L\,u.}$$

$$\frac{1}{u} = 2\pi\tilde{n}_C + 2 ⨍ dv \frac{\tilde{\rho}(v)}{u-v}.$$

$$x \to xL/g\,,$$

$$4\pi\kappa = \Delta/g$$

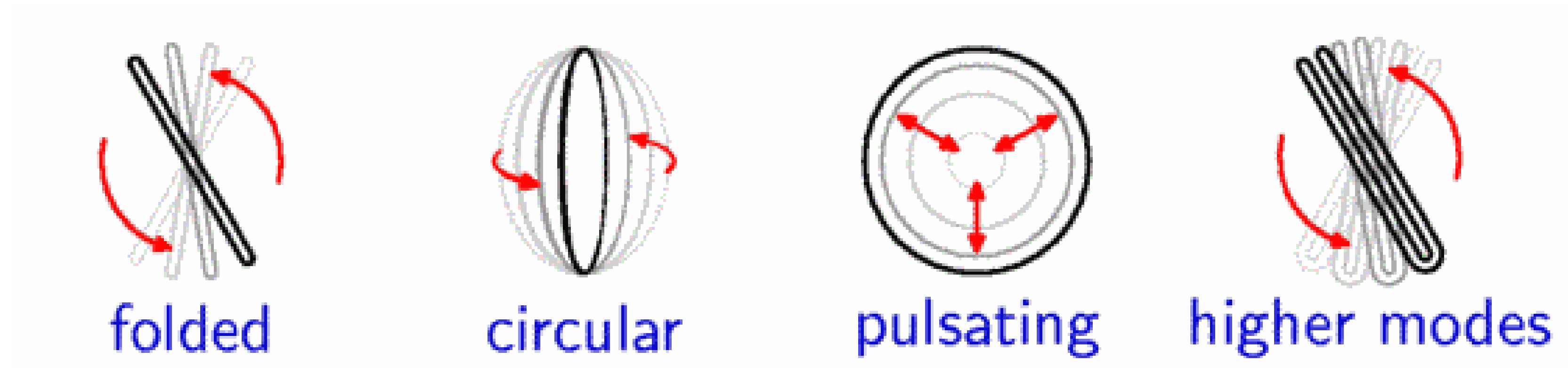
AdS/CFT at work !!!

$$\frac{x\,\Delta/L}{x^2 - g^2/L} + 2\pi n_C = 2 ⨍ dy \frac{\rho(y)}{x-y}.$$

$$(\Delta = L + \cdots)$$

Spinning strings and semiclassical corrections

classical solutions with simple geometry



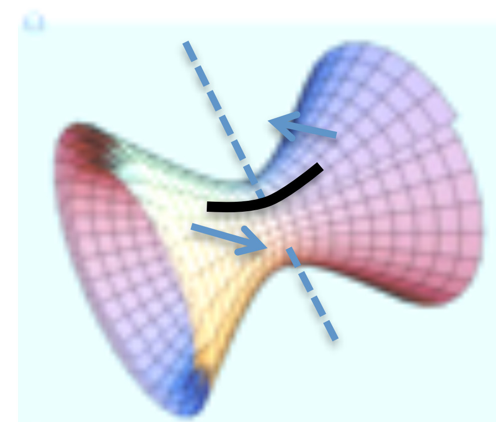
• Smooth scaling limit of multi-spins, energy, etc

$$\mathcal{E} = \frac{E}{\sqrt{\lambda}} = \text{fixed}, \qquad \mathcal{J} = \frac{J}{\sqrt{\lambda}} = \text{fixed}\ldots \qquad \text{with} \quad \lambda \to \infty$$

σ-model energy expansion: $$E_{tot} = \sqrt{\lambda}\mathcal{E}(\mathcal{J}) + \mathcal{E}_1(\mathcal{J}) + \frac{1}{\sqrt{\lambda}}\mathcal{E}_2(\mathcal{J}) + \ldots$$

• Example of gauge/gravity correspondence: Twist operators/folded string in AdS_3

$$\mathrm{Tr}(\Phi^{L-1}\mathcal{D}_+^S\Phi) \quad \xleftrightarrow{0 \longrightarrow \lambda \longrightarrow \infty}$$



Classical folded string rotating in AdS_3

- Folded spinning string in AdS_3

  classical solution of the form

$$ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\phi^2$$

$$t = \kappa\,\tau, \qquad \phi = \omega\,\tau, \qquad \rho = \rho(\sigma) = \rho(\sigma + 2\pi),$$

- Equations of motion compatible with Virasoro constraints and solution

solution of 1d sinh-Gordon equation

$$\begin{aligned} \rho'^2 &= \kappa^2\cosh^2\rho - w^2\sinh^2\rho, \\ \sinh\rho(\sigma) &= \frac{k}{\sqrt{1-k^2}}\,\mathrm{cn}(\omega\,\sigma + \mathbb{K}, k^2)\ , \end{aligned}$$

$$\rho'(\sigma) = \kappa\,\mathrm{sn}(\omega\,\sigma + \mathbb{K}\,|\,k^2)\ ,$$

$$k = \frac{\kappa}{\omega}$$

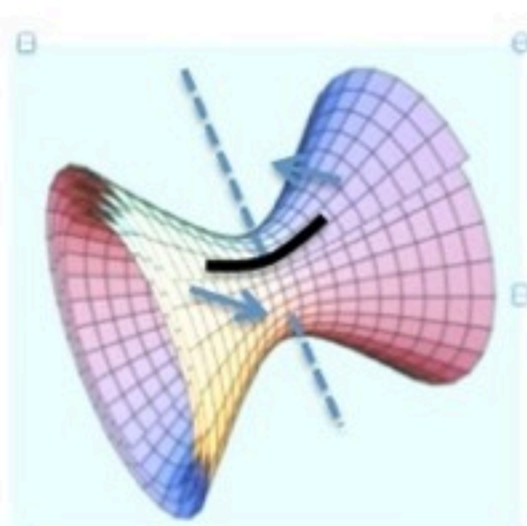
- Maximal extension is the only free parameter

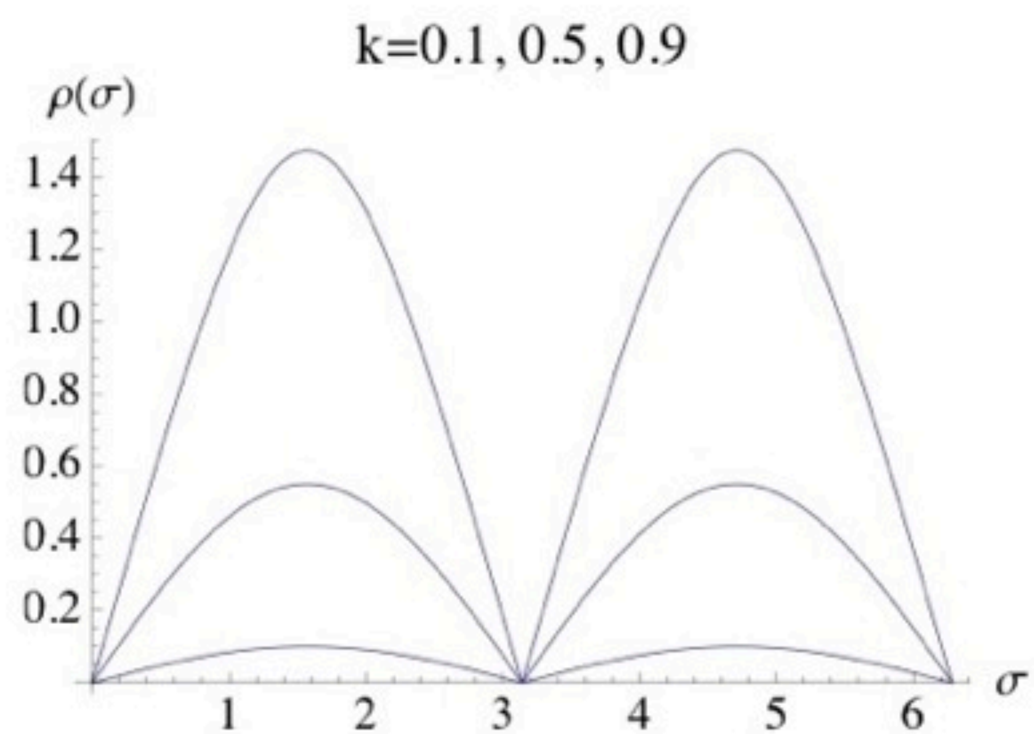
$$\coth^2\rho_0 = \frac{\omega^2}{\kappa^2} \equiv 1 + \eta \equiv \frac{1}{k^2}$$

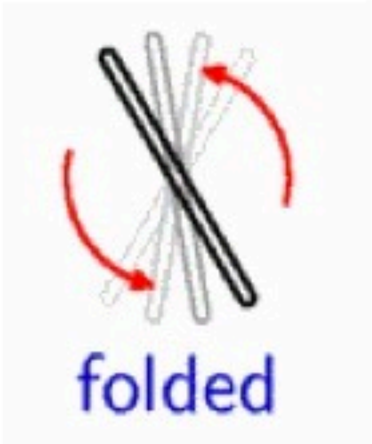
$$\kappa = \frac{2\,k}{\pi}\,\mathbb{K},$$

$$\mathbb{K} \equiv \mathbb{K}(k^2)$$

$$\omega = \frac{2}{\pi}\,\mathbb{K}.$$





example: 

dual to $\mathrm{Tr}(\Phi D_+^S \Phi)$.

- **Large spin** behaviour is

$$E = S + f(\lambda)\,\log S + \cdots\,,$$

(cusp anomalous dimension...)

$$\begin{aligned}\text{gauge} &: \quad f(\lambda) = a_1\lambda + a_2\lambda^2 + \ldots\,,\\ \text{string} &: \quad f(\lambda) = b_0\sqrt{\lambda} + b_1 + \frac{b_2}{\sqrt{\lambda}} + \ldots\,.\end{aligned}$$

Quantum string corrections: start from string action and expand near solitonic string solution

Semiclassical corrections to the energy

- **Easy case**, fluctuations around an *almost static* solution, usual Euclidean trick

$$E_1 = \frac{\Gamma}{\kappa \mathcal{T}}, \qquad \mathcal{T} \equiv \int d\tau \to \infty, \qquad \Gamma = -\ln Z$$

ratio of functional determinants

- The effective action is reduced to (coupled) Schrodinger functional determinants.

Simple 1d example:

e.g. single gap Lame,etc

$$\ln\det[-\partial_\sigma^2 - \partial_\tau^2 + M^2(\sigma)] = \mathcal{T}\int_{-\infty}^{+\infty}\frac{d\Omega}{2\pi}\ \ln\det[-\partial_\sigma^2 + \Omega^2 + M^2(\sigma)]$$

Key example: universal scaling dimension

(or "cusp anomalous dimension": controls IR singular part of planar gluon scattering amplitudes)

Dimension of $\mathcal{O} = \mathrm{Tr}(\phi D^S \phi)$, for $S \gg 1$

$$\Delta = S + f(\lambda) \ln S + ...$$

Asymptotic Bethe Ansatz → BES integral equation for $f(\lambda)$ determines coefficients in expansion at $\lambda \ll 1$ or $\lambda \gg 1$

$$f_{\lambda \ll 1} = c_1 \lambda + c_2 \lambda^2 + c_3 \lambda^3 + c_4 \lambda^4 + ...$$
$$f_{\lambda \gg 1} = \sqrt{\lambda} a_0 + a_1 + \frac{a_2}{\sqrt{\lambda}} + ...$$

Compare to gauge theory:

c_n from Feynman graphs of 4d CFT – $\mathcal{N} = 4$ SYM

$$f_{\lambda \ll 1} = \frac{1}{2\pi^2} \left[\lambda - \frac{\lambda^2}{48} + \frac{11 \lambda^3}{2^8 \times 45} - \left(\frac{73}{630} + \frac{4 \zeta^2(3)}{\pi^6} \right) \frac{\lambda^4}{2^7} + ... \right]$$

3-loop: Kotikov, Lipatov et al 03; 4-loop: Bern, Dixon, et al 06

Beisert-Eden-Staudacher equation:

(derived from full set of ABA equations)

$$\sigma(t,g) = \frac{t}{e^t - 1}\left[K(2gt,0) - 4g^2 \int_0^\infty dt' K(2gt,2gt')\sigma(t',g)\right]$$

$$K(t,t') = \frac{1}{tt'} \sum_{n,m=1}^{\infty} z_{nm}(g) J_n(t) J_m(t')$$

$$f(g) = \sigma(0,g) \ , \qquad g = \frac{\sqrt{\lambda}}{4\pi}$$

J_n – Bessel functions

z_{nm} from coefficients in the phase θ $\qquad (\sigma = e^{i\theta})$

strong-coupling expansion $\qquad$ Basso, Korchemsky, Kotanski 07

Summary

I. Spectrum of "long" operators / "semiclassical" string states determined by Asymptotic Bethe Ansatz (2002-2007)

• its final [Beisert-Eden-Staudacher] form found by intricate superposition of data from $\lambda \ll 1$ gauge theory (spin chain, BA,...) and perturbative string theory (classical and 1-loop phase, BMN), symmetries (S-matrix), assumption of exact integrability

• consequences checked against available gauge and string data

Key example: cusp anomalous dimension – dim of $\mathrm{Tr}(\Phi D^S \Phi)$

$$\Delta = S + 2 + f(\lambda) \ln S + \dots , \qquad S \gg 1$$

$$f_{\lambda \ll 1} = \frac{\lambda}{2\pi^2} \Big[1 - \frac{\lambda}{48} + \frac{11\lambda^2}{45 \cdot 2^8} - \big(\frac{73}{630} + \frac{4\zeta_3^2}{\pi^6} \big) \frac{\lambda^3}{2^7} + \dots \Big]$$

$$f_{\lambda \gg 1} = \frac{\sqrt{\lambda}}{\pi} \Big[1 - \frac{3 \ln 2}{\sqrt{\lambda}} - \frac{K}{(\sqrt{\lambda})^2} - \dots \Big] + O(e^{-\frac{1}{4}\sqrt{\lambda}})$$

$\zeta_k = \zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}, \quad K = \beta(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915\dots$

from 2-loop string sigma-model integrals [Roiban, Tirziu, AT]

exact integral eq. [Basso, Korchemsky, Kotanski]: any order term

Another exact result - slope function - coefficient in short string limit

$$E^2 = J^2 + h_1(\lambda, J)\, N + h_2(\lambda, J)\, N^2 + h_3(\lambda, J)\, N^3 + \dots$$

$$h_1 = 2\sqrt{\lambda} + n_{11} + \frac{n_{21}}{\sqrt{\lambda}} + \frac{n_{31}}{(\sqrt{\lambda})^2} + \dots + J^2\left(\frac{n_{01}}{\sqrt{\lambda}} + \frac{\tilde{n}_{11}}{(\sqrt{\lambda})^2} + \dots\right)$$

exact "slope" h_1 for sl(2) sector operator $\mathrm{Tr}(D^S \Phi^J)$
dual to AdS_5 folded spinning string ($N = S$)
from BA (I_J - modif. Bessel of 1st type) [Basso 11,12;Gromov 12]

$$\begin{aligned} h_1(\lambda, J) &= 2J + 2\sqrt{\lambda}\, \frac{I_{J+1}(\sqrt{\lambda})}{I_J(\sqrt{\lambda})} \\ &= 2\sqrt{\lambda}\sqrt{1+\mathcal{J}^2} - \frac{1}{1+\mathcal{J}^2} - \frac{\frac{1}{4} - \mathcal{J}^2}{\sqrt{\lambda}(1+\mathcal{J}^2)^{5/2}} + \dots \\ &= 2\sqrt{\lambda + J^2} - \frac{\lambda}{\lambda + J^2} - \frac{\lambda(\frac{1}{4}\lambda - J^2)}{(\lambda + J^2)^{5/2}} + \dots \end{aligned}$$

[Similar exact expressions found for some BPS Wilson Loops by Feynman graph summation or localization]

II. Spectrum of "short" operators = quantum string states

Thermodynamic Bethe Ansatz (2005-...)

- reconstructed from ABA using solely methods/intuition of 2-d integrable QFT, i.e. inspired by string-theory side
- highly non-trivial construction – lack of 2-d Lorentz invariance in standard BMN-vacuum-adapted l.c. gauge
- in few cases ABA "improved" by Luscher corrections is enough: 4- and 5-loop Konishi dim, 4-loop dim. of twist 2 operator
- complicated set of integral equations in need of simplification; so far predictions extracted only numerically starting from weak coupling and interpolating to larger λ
- need more data to check predictions at $\lambda \ll 1$ and $\lambda \gg 1$ – against perturbative gauge-theory and string-theory data

Key example:

dimension $\Delta = 2 + \gamma(\lambda)$ of Konishi operator $\mathrm{Tr}(\bar{\Phi}_i \Phi_i)$

$g^2 = \frac{\lambda}{(4\pi)^2} \ll 1$

7-loop result:

$$\begin{aligned}
\Delta = 4 + 12g^2 - 48g^4 + 336g^6 \\
&+96\Big[-26 + 6\,\zeta_3 - 15\,\zeta_5\Big]g^8 \\
&-96\Big[-158 - 72\,\zeta_3 + 54\,\zeta_3^2 + 90\,\zeta_5 - 315\,\zeta_7\Big]g^{10} \\
&-48\Big[160 + 432\,\zeta_3^2 - 2340\,\zeta_5 \\
&\qquad -72\,\zeta_3(-76 + 45\,\zeta_5) - 1575\,\zeta_7 + 10206\,\zeta_9\Big]g^{12} \\
&+48\Big[-44480 - 8784\,\zeta_3^2 + 2592\,\zeta_3^3 - 4776\,\zeta_5 - 20700\,\zeta_5^2 \\
&\qquad +24\,\zeta_3(4540 + 357\,\zeta_5 - 1680\,\zeta_7) \\
&\qquad -26145\,\zeta_7 - 17406\,\zeta_9 + 152460\,\zeta_{11}\Big]g^{14} + \ldots
\end{aligned}$$

all coefficients in γ are integer, divisible by 12 ....

new (multiple zeta?) numbers at 8 loops ? exact expression ?

5-loop results first found using integrability
[Banjok, Janik 11]
confirmed later by more standard QFT methods
[Velizhanin; Eden et al 12]
very recent progress:
6-loop term: derivation from TBA [Leurent, Serban, Volin 12]
6- and 7-loop terms: from Luscher corrections approach
[Banjok, Janik 12] (8-loop result is to appear....)

Suppose one can sum up (convergent) $\lambda \ll 1$ expansion
and then re-expand at $\lambda \gg 1$

What one should expect to get for $\gamma(\lambda \gg 1)$?

Duality to string theory predicts the structure
of strong-coupling expansion:
leading term – near-flat-space expansion for fixed quant. numbers
[Gubser, Klebanov, Polyakov 98]

$$\Delta = \sqrt{2N\sqrt{\lambda}} + \ldots \quad = 2 + \gamma(\lambda)$$

Subleading terms: $\alpha' = \frac{1}{\sqrt{\lambda}}$ expansion of 2d anom. dimensions
of corresponding vertex operators [Roiban, AT 09] $(N = 2)$

$$\gamma(\lambda \gg 1) \;=\; 2\sqrt[4]{\lambda} + \frac{b_1}{\sqrt[4]{\lambda}} + \frac{b_2}{(\sqrt[4]{\lambda})^3} + \frac{b_3}{(\sqrt[4]{\lambda})^{5/2}} + \ldots$$

$$\;=\; 2\sqrt[4]{\lambda}\Big[1 + \frac{b_1}{2\sqrt{\lambda}} + \frac{b_2}{2(\sqrt{\lambda})^2} + \frac{b_3}{2(\sqrt{\lambda})^3} + \ldots\Big]$$

Values of b_k from string theory? From TBA? they match?

Dimensions of “short” SYM operators

= energies of quantum string states

find leading $\alpha' = \frac{1}{\sqrt{\lambda}}$ corrections to energy of

“lightest” massive string states on first massive string level

dual to operators in Konishi multiplet in SYM theory

– compare with predictions of TBA approach

important to check integrability-based approach

which involves subtle assumptions

directly against perturbative string sigma model

$$\gamma(\lambda \gg 1) \;\; = \;\; 2\sqrt[4]{\lambda} + \frac{b_1}{\sqrt[4]{\lambda}} + \frac{b_2}{(\sqrt[4]{\lambda})^3} + \frac{b_3}{(\sqrt[4]{\lambda})^{5/2}} + ...$$

TBA results:

start at weak coupling for sl(2) Konishi descendant $\mathrm{Tr}(\Phi D^2 \Phi)$
use TBA to find $\Delta(\lambda)$ numerically;
match to expected form of strong-coupling expansion to extract b_k
[Gromov, Kazakov, Vieira 09; Frolov 10, 12]

$$b_1 \approx 1.988 \,, \qquad b_2 \approx -3.07$$

Compare to string theory:

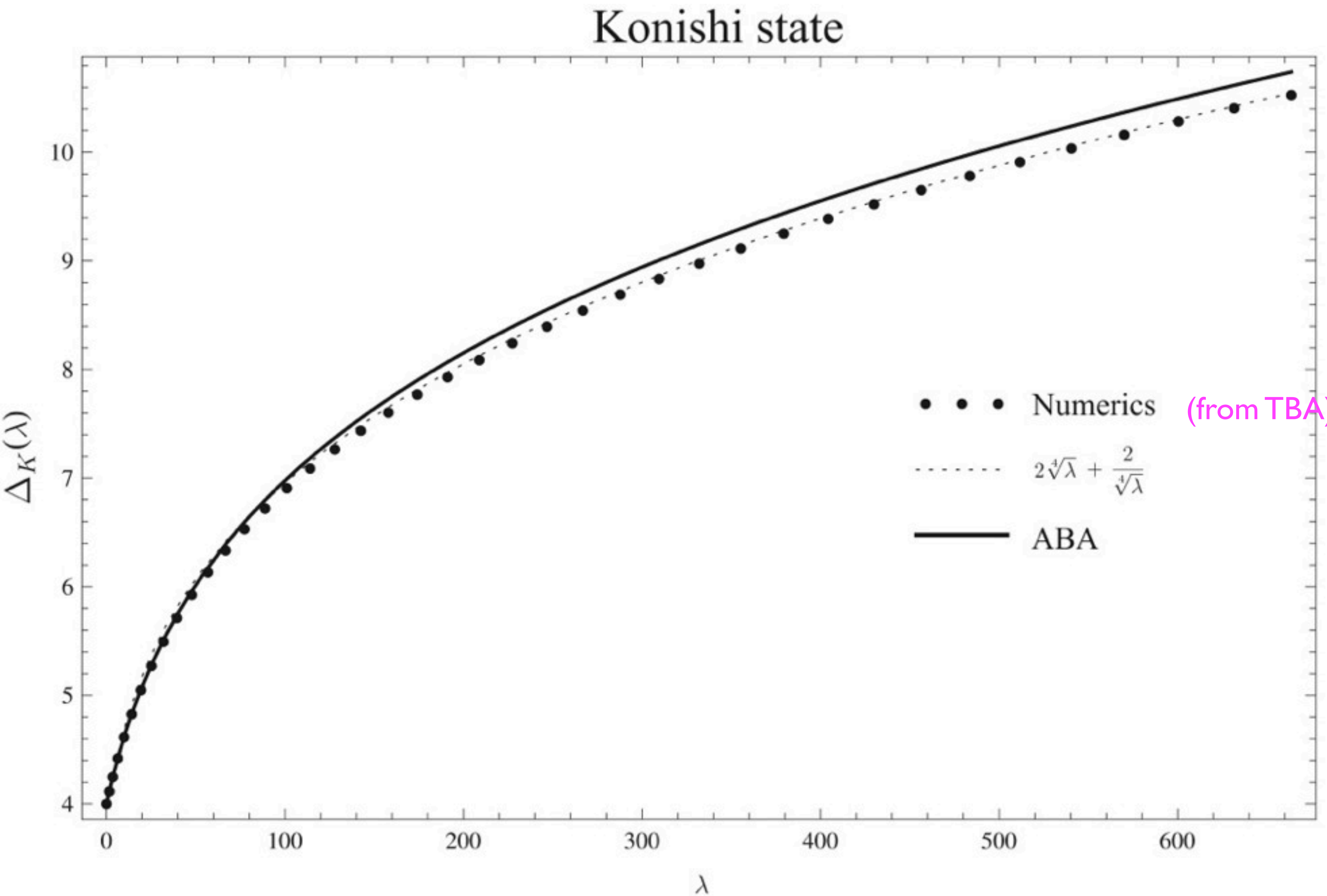
One can find b_k using semiclassical "short string" expansion
[Roiban, AT 09, 11; Gromov, Serban, Shenderovich, Volin 11]

$$b_1 = 2 \,, \qquad b_2 = a - 3\zeta_3$$

rational a was found [Gromov, Valatka 11] using "2-loop"
coefficient in exact slope function $E^2 = h(\lambda)S$ [Basso 11]

$$b_2 = \frac{1}{2} - 3\zeta_3 \;\approx -3.106...$$

Remarkable agreement with TBA - check of quantum integrability


Konishi state
10
9
8
7
6
5
4
0
100
200
300
400
500
600
λ
$\Delta_K(\lambda)$
Numerics
(from TBA)
$2\sqrt[4]{\lambda} + \frac{2}{\sqrt[4]{\lambda}}$
ABA

$$\gamma(\lambda \gg 1) \quad = \quad 2\sqrt[4]{\lambda} + \frac{b_1}{\sqrt[4]{\lambda}} + \frac{b_2}{(\sqrt[4]{\lambda})^3} + \frac{b_3}{(\sqrt[4]{\lambda})^{5/2}} + ...$$

Recent work on string side: [BGMRT 12; BT12]

• highest transcendentality terms in b_k
are $\sim \zeta_{2k-1}$ and have 1-loop origin, e.g.,

$$b_3 = a_1 + a_2\zeta_3 + a_3\zeta_5$$

rational a_1 receives contribution from 3 loops; a_2 from
2-loops, etc.; $b_4 \sim \zeta_7 + ...$, etc.

• supermultiplet structure: universality of coefficients in E
for string states with spins in different $AdS_5 \times S^5$ directions:
dual operators from Konishi multiplet have same energy
(up to constant shift depending on position in the multiplet)

• states on leading Regge trajectory:
general structure of dependence of energy on
string tension $\sqrt{\lambda}$, string level (spin) and S^5 orbital momentum J

Some open questions:

• Analytic form of strong-coupling expansion from TBA?

• only ζ_k coefficients in $\Delta(\lambda)$ in both
weak and strong coupling expansions
or other transcendental constants appear?
(cf. cusp anomalous dimension)
[2-loop string computation may shed light on this ...]

• Asymptotic form of strong coupling expansion:
$e^{-k\sqrt{\lambda}}$ corrections to cusp dimension
absent for short strings / operators like Konishi?
[no such corrections in slope function; no massless S^5 modes]

• Energies of other quantum states: general structure of spectrum?

Some details:

Konishi multiplet:

long multiplet related to singlet $[0,0,0]_{(0,0)}$ by susy

$[J_2 - J_3, J_1 - J_2, J_2 + J_3]_{(s_L, s_R)}$
$s_{L,R} = \frac{1}{2}(S_1 \pm S_2)$

$SO(6)$ (J_1, J_2, J_3) and $SO(4)$ (S_1, S_2) labels
of $SO(2,4) \times SO(6)$ global symmetry

$\Delta = \Delta_0 + \gamma(\lambda), \qquad \Delta_0 = 2, \frac{5}{2}, 3, ..., 10$
same anomalous dimension γ for all members

singlet eigen-state of anom. dim. matrix with lowest eigenvalue

Examples of gauge-theory operators in Konishi multiplet:

$[0,0,0]_{(0,0)}$:
$\mathrm{Tr}(\bar{\Phi}_i \Phi_i), \quad i = 1,2,3, \qquad \Delta_0 = 2$

$[2,0,2]_{(0,0)}$:
$\mathrm{Tr}([\Phi_1, \Phi_2]^2)$ in $su(2)$ sector, $\Delta_0 = 4$

$[0,2,0]_{(1,1)}$:
$\mathrm{Tr}(\Phi_1 D^2 \Phi_1)$ in $sl(2)$ sector, $\Delta_0 = 4$

Δ_0	
2	$[0,0,0]_{(0,0)}$
$\frac{5}{2}$	$[0,0,1]_{(0,\frac{1}{2})} + [1,0,0]_{(\frac{1}{2},0)}$
3	$[0,0,0]_{(\frac{1}{2},\frac{1}{2})} + [0,0,2]_{(0,0)} + [0,1,0]_{(0,1)+(1,0)} + [1,0,1]_{(\frac{1}{2},\frac{1}{2})} + [2,0,0]_{(0,0)}$
$\frac{7}{2}$	$[0,0,1]_{(\frac{1}{2},0)+(\frac{1}{2},1)+(\frac{3}{2},0)} + [0,1,1]_{(0,\frac{1}{2})+(1,\frac{1}{2})} + [1,0,0]_{(0,\frac{1}{2})+(0,\frac{3}{2})+(1,\frac{1}{2})} + [1,0,2]_{(\frac{1}{2},0)}$ $+[1,1,0]_{(\frac{1}{2},0)+(\frac{1}{2},1)} + [2,0,1]_{(0,\frac{1}{2})}$
4	$[0,0,0]_{(0,0)+(0,2)+(1,1)+(2,0)} + [0,0,2]_{(\frac{1}{2},\frac{1}{2})+(\frac{3}{2},\frac{1}{2})} + [0,1,0]_{2(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})} + [2,0,2]$ $+[0,1,2]_{(1,0)} + [0,2,0]_{2(0,0)+(1,1)} + [1,0,1]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [1,1,1]_{2(\frac{1}{2},\frac{1}{2})} + [2,0,$
6	$[0,0,0]_{3(0,0)+3(1,1)+(2,2)} + [0,0,2]_{3(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})} + [0,1,0]_{4(\frac{1}{2},\frac{1}{2})+2(\frac{1}{2},\frac{3}{2})+2(\frac{3}{2},}$ $+[0,1,2]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [0,2,0]_{3(0,0)+(0,1)+(0,2)+(1,0)+3(1,1)+(2,0)} + [0,2,2]_{(\frac{1}{2},\frac{1}{2}}$ $+[0,3,0]_{2(\frac{1}{2},\frac{1}{2})} + [0,4,0]_{(0,0)} + [1,0,1]_{(0,0)+3(0,1)+3(1,0)+4(1,1)+(1,2)+(2,1)} + [1,0,3]_{(\frac{1}{2},\frac{1}{2})}$ $+[1,1,1]_{4(\frac{1}{2},\frac{1}{2})+2(\frac{1}{2},\frac{3}{2})+2(\frac{3}{2},\frac{1}{2})} + [1,2,1]_{(0,0)+(0,1)+(1,0)} + [2,0,0]_{3(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},}$ $+[2,0,2]_{(0,0)+(1,1)} + [2,1,0]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [2,2,0]_{(\frac{1}{2},\frac{1}{2})} + [3,0,1]_{(\frac{1}{2},\frac{1}{2})} + [4,0,$
$\frac{17}{2}$	$[0,0,1]_{(0,\frac{1}{2})+(0,\frac{3}{2})+(1,\frac{1}{2})} + [0,1,1]_{(\frac{1}{2},0)+(\frac{1}{2},1)} + [1,0,0]_{(\frac{1}{2},0)+(\frac{1}{2},1)+(\frac{3}{2},0)} + [1,0,2]_{(0,\frac{1}{2})}$ $+[1,1,0]_{(0,\frac{1}{2})+(1,\frac{1}{2})} + [2,0,1]_{(\frac{1}{2},0)}$
9	$[0,0,0]_{(\frac{1}{2},\frac{1}{2})} + [0,0,2]_{(0,0)} + [0,1,0]_{(0,1)+(1,0)} + [1,0,1]_{(\frac{1}{2},\frac{1}{2})} + [2,0,0]_{(0,0)}$
$\frac{19}{2}$	$[0,0,1]_{(\frac{1}{2},0)} + [1,0,0]_{(0,\frac{1}{2})}$
10	$[0,0,0]_{(0,0)}$

Table 1: Long Konishi multiplet (part of it)

Comparison between gauge and string theory states:

• $\lambda \ll 1$:   gauge-theory operators built out of free fields, canonical dim. Δ_0 determines operators that can mix

• $\lambda \gg 1$: in near-flat-space expansion string states built out of free oscillators, level N determines states that can mix

(i) relate states with same global charges

(ii) assume direct interpolation (no "level crossing") for states with same quantum numbers as λ changes from small to large values

• Konishi operator dual to
"lightest" among massive $AdS_5 \times S^5$ string states

• large $\sqrt{\lambda} = \frac{\mathrm{R}^2}{\alpha'}$:
"short" strings probe near-flat limit of $AdS_5 \times S^5$

• members of supermultiplet:
strings with spins/oscillators in different $AdS_5 \times S^5$ directions

String spectrum in $AdS_5 \times S^5$:

long multiplets of $PSU(2,2|4)$

highest weight states:

$[J_2 - J_3, J_1 - J_2, J_2 + J_3]_{(s_1,s_2)}$

$s_{1,2} = \frac{1}{2}(S_1 \pm S_2)$

Flat-space string spectrum can be re-organized
in multiplets of $SO(2,4) \times SO(6) \subset PSU(2,2|4)$
[Bianchi, Morales, Samtleben 03; Beisert et al 03]
$SO(4) \times SO(5) \subset SO(9)$ rep.
lifted to $SO(4) \times SO(6)$ rep. of $SO(2,4) \times SO(6)$

Konishi multiplet:

$\mathcal{K} = (1 + Q + Q \wedge Q + ...)[0,0,0]_{(0,0)}$

determines the “floor” of 1-st excited string level

$\sum_{J=0}^{\infty} [0, J, 0]_{(0,0)} \times \mathcal{K}$

Examples:

- folded string with spin S_1 and momentum J:

$$S_1 = J = 2 \quad \rightarrow \quad [0,2,0]_{(1,1)}, \quad \Delta_0 = 4$$

- folded string with spin J_1 and momentum J:

$$J_1 = J = 2 \quad \rightarrow \quad [2,0,2]_{(0,0)}, \quad \Delta_0 = 4$$

- circular string with spins $J_1 = J_2$ and momentum J:

$$J_1 = J_2 = 1, J = 2 \quad \rightarrow \quad [0,1,2]_{(0,0)}, \quad \Delta_0 = 6$$

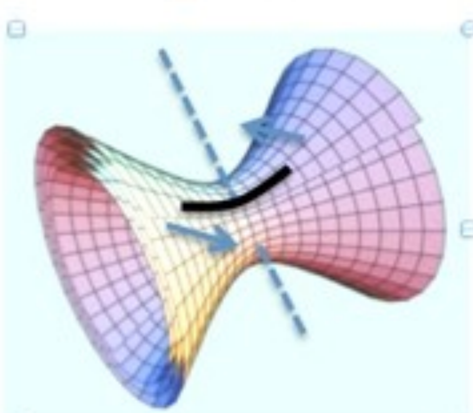
- circular string with spins $S_1 = S_2$ and momentum J:

$$S_1 = S_2 = 1, J = 2 \quad \rightarrow \quad [0,2,0]_{(0,1)}, \quad \Delta_0 = 6$$

- circular string with spins $S_1 = J_1$ and momentum J:

$$S_1 = J_1 = 1, J = 2 \quad \rightarrow \quad [1,1,1]_{(\frac{1}{2},\frac{1}{2})}, \quad \Delta_0 = 6$$

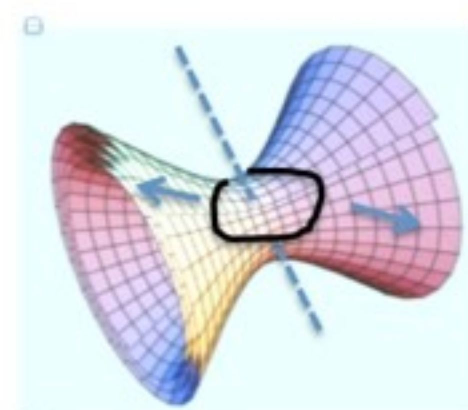
Footprints of Konishi multiplet in semiclassical string ?



folded in AdS_3

folded in $R \times S^2$

pulsating $R \times S^2$



pulsating AdS_3

Vertex operator approach

calculate 2d anomalous dimensions from "first principles"–
superstring theory in $AdS_5 \times S^5$:

$$I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \Big[\partial Y_p \bar{\partial} Y^p + \partial X_k \bar{\partial} X_k + \text{fermions} \Big]$$

$$-Y_0^2 - Y_5^2 + Y_1^2 + ... + Y_4^2 = -1 \,, \qquad X_1^2 + ... + X_6^2 = 1$$

construct marginal (1,1) operators in terms of Y_p and X_k
e.g. vertex operator for dilaton (in NSR framework)

$$\mathrm{V}_J = \ (Y_+)^{-\Delta} \ (X_x)^J \ \Big[\partial Y_p \bar{\partial} Y^p + \partial X_k \bar{\partial} X_k + \text{fermions} \Big]$$

$$Y_+ \equiv Y_0 + iY_5 = z + z^{-1} x_m x_m \sim e^{it}$$
$$X_x \equiv X_1 + iX_2 \sim e^{i\varphi}$$

$$2 = 2 + \frac{1}{2\sqrt{\lambda}} [\Delta(\Delta - 4) - J(J+4)] + O(\tfrac{1}{(\sqrt{\lambda})^2})$$

i.e. $\Delta = 4 + J$ (BPS)

Vertex operators = eigenstates of 2d anomalous dimension matrix

particular linear combinations like

$$V = f_{k_1...k_\ell m_1...m_{2s}} X_{k_1}...X_{k_\ell} \partial X_{m_1} \bar{\partial} X_{m_2} ... \partial X_{m_{2s-1}} \bar{\partial} X_{m_{2s}}$$

their renormalization studied in $O(n)$ sigma model [Wegner 90]
simplest case: $f_{k_1...k_\ell} X_{k_1}...X_{k_\ell}$ with traceless $f_{k_1...k_\ell}$
h.-w. rep. $V_J = (X_x)^J, \qquad \widehat{\gamma} = 2 - \frac{1}{2\sqrt{\lambda}} J(J+4) + ...$

$AdS_5 \times S^5$: candidates for operators on leading Regge trajectory:

$$\mathrm{V}_J = (Y_+)^{-\Delta} \left(\partial X_x \bar{\partial} X_x\right)^{J/2}, \qquad X_x \equiv X_1 + iX_2$$

$$\mathrm{V}_S = (Y_+)^{-\Delta} \left(\partial Y_u \bar{\partial} Y_u\right)^{S/2}, \qquad Y_u \equiv Y_1 + iY_2$$

+ fermionic terms
+ $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ terms from diagonalization of anom. dim. op.
– mixing with ops with same charges and dimension

General structure of dimension/energy $\Delta = E$

marginality condition – condition on quantum numbers Q_i

$Q = (E(\lambda), S_1, S_2; J_1, J_2, J_3; ...)$; $N = \sum_i a_i Q_i$ = level

$$0 = 2N + \frac{1}{\sqrt{\lambda}}\Big(\sum_{i,j} c_{ij} Q_i Q_j + \sum_i c_i Q_i\Big)$$
$$+ \frac{1}{(\sqrt{\lambda})^2}\Big(\sum_{i,j,k} c_{ijk} Q_i Q_j Q_k + \sum_{i,j} c'_{ij} Q_i Q_j + \sum_i c'_i Q_i\Big) + ...$$

States on "leading Regge trajectory": (max spin for given E)

marginality condition: $Q = (E, J; N)$, N= spin

$$0 = 2N + \frac{1}{\sqrt{\lambda}}\Big(-E^2 + J^2 + n_{02} N^2 + n_{11} N\Big)$$
$$+ \frac{1}{(\sqrt{\lambda})^2}\Big(n_{01} J^2 N + n_{03} N^3 + n_{12} N^2 + n_{21} N\Big) + ...$$

solution for E^2 takes form [Roiban, AT 09, 11; BGMRT 12]

$$E^2 = 2\sqrt{\lambda}N + J^2 + n_{02}N^2 + n_{11}N$$
$$+ \frac{1}{\sqrt{\lambda}}\big(n_{01}J^2N + n_{03}N^3 + n_{12}N^2 + n_{21}N\big)$$
$$+ \frac{1}{(\sqrt{\lambda})^2}\big(\widetilde{n}_{11}J^2N + \widetilde{n}_{02}J^2N^2 + n_{04}N^4 + n_{13}N^3 + n_{22}N^2 + n_{31}N\big) + ...$$

Expanding in large $\sqrt{\lambda}$ for fixed N, J

$$E = \sqrt{2\sqrt{\lambda}N}\Big[1 + \frac{A_1}{\sqrt{\lambda}} + \frac{A_2}{(\sqrt{\lambda})^2} + O(\frac{1}{(\sqrt{\lambda})^3})\Big] \quad = 2 + \gamma(\lambda)$$
$$A_1 = \frac{1}{4N}J^2 + \frac{1}{4}(n_{02}N + n_{11})$$
$$A_2 = -\frac{1}{2}A_1^2 + \frac{1}{4}(n_{01}J^2 + n_{03}N^2 + n_{12}N + n_{21})$$

Gives strong-coupling dimension of dual SYM operator

States on 1-st excited superstring level: $N = 2$

Konishi multiplet states: $N = 2, J = 2$

$$E = \sqrt[4]{\lambda}\Big[2 + \frac{b_1}{\sqrt{\lambda}} + \frac{b_2}{(\sqrt{\lambda})^2} + O(\frac{1}{(\sqrt{\lambda})^3})\Big]$$
$$b_1 = 1 + n_{02} + \tfrac{1}{2}n_{11}$$
$$b_2 = -4b_1^2 + 2n_{01} + 2n_{03} + n_{12} + \tfrac{1}{2}n_{21}$$

coefficients n_{km} =? – use semiclassical "short string" expansion:

• start with solitonic string carrying same charges

as vertex operator representing particular quantum string state

• perform semiclassical expansion: $\sqrt{\lambda} \gg 1$

for fixed classical parameters $\mathcal{N} = \frac{1}{\sqrt{\lambda}}N, \quad \mathcal{J} = \frac{1}{\sqrt{\lambda}}J$

• expand E in small values of $\mathcal{N}, \mathcal{J}$

• re-interpret the resulting E in terms of N, J: get n_{km}

Key point: limit $\mathcal{N} = \frac{N}{\sqrt{\lambda}} \to 0, \quad \mathcal{J} = \frac{J}{\sqrt{\lambda}} \to 0$

corresponds to $\sqrt{\lambda} \gg 1$ for fixed values of quantum charges N, J

Results: for several states on leading Regge trajectory

$$E^2 = 2\sqrt{\lambda}N + J^2 + n_{02}N^2 + n_{11}N$$
$$+ \frac{1}{\sqrt{\lambda}}(n_{01}J^2N + n_{03}N^3 + n_{12}N^2 + n_{21}N)$$
$$+ \frac{1}{(\sqrt{\lambda})^2}(\tilde{n}_{11}J^2N + \tilde{n}_{02}J^2N^2 + n_{04}N^4 + n_{13}N^3 + n_{22}N^2 + n_{31}N)$$
$$+ \frac{1}{(\sqrt{\lambda})^3}(\tilde{n}_{01}J^4N + \tilde{n}_{21}J^2N + \tilde{n}_{12}J^2N^2 + n_{05}N^5 + \ldots) + \ldots$$

- $n_{01} = 1,\ \tilde{n}_{01} = -\frac{1}{4}, \ldots$ from near-BMN expansion ($J \ll \sqrt{\lambda}$)

$E^2 = J^2 + 2N\sqrt{\lambda + J^2} + \ldots = J^2 + N(2\sqrt{\lambda} + \frac{J^2}{\sqrt{\lambda}} + \ldots)$

- "tree-level" coeffs $n_{02}, n_{03}, n_{04}, \ldots$ are all rational
- leading 1-loop n_{11} is rational [Roiban, AT 09; Gromov et al 11]
- $\tilde{n}_{11} = -n_{11}$, i.e. in general [BGMRT 12]

$$h_1 = 2\sqrt{\lambda}\sqrt{1+\mathcal{J}^2} + \frac{n_{11}}{1+\mathcal{J}^2} + \frac{1}{\sqrt{\lambda}}(n_{21} + \tilde{n}_{21}\mathcal{J}^2 + \ldots) + \ldots$$

$$h_2 = \frac{n_{02}+\mathcal{J}^2}{1+\mathcal{J}^2} + \frac{1}{\sqrt{\lambda}}(n_{12} + \tilde{n}_{12}\mathcal{J}^2 + \ldots) + \ldots$$

- $n_{12} = n'_{12} - 3\zeta_3, \qquad n'_{12} = -\frac{3}{8} - 2n_{03}$ is rational

[Tirziu, AT 08; Roiban, AT 09; Gromov-Valatka 11]

Conclusions

- progress in understanding of $AdS_5 \times S^5$ string spectrum or spectrum of conformal $\mathcal{N} = 4$ SYM operators
- agreement with numerical results from TBA: non-trivial check of quantum string integrability !
- prediction of transcendental structure of leading coefficients: reproduce them by an analytic solution of TBA at strong coupling?
- evidence of universality of some coefficients in strong coupling expansion of dimensions of states on leading Regge trajectory
- exact results for leading "slope" functions

- need systematic study of quantum string theory in $AdS_5 \times S^5$ in near-flat-space expansion

- still need first-principles solution for

spectrum of $AdS_5 \times S^5$ superstring = spectrum of $\mathcal{N} = 4$ SYM

based on integrability

... it now seems within reach...

Progress in other directions:

- Pohlmeyer reduction: towards 1st-principles solution of string theory -- from GS superstring action to gauged WZW + integrable potential
  (i) resolution of non-ultralocality problem and possible lattice version? [Delduc, Magro, Vicedo, ...]
  (ii) exact S-matrix as q-deformation of magnon S-matrix; 2-parameter generalization of TBA

  [Hoare, AT; Beisert, Koroteev; Hollowood, Miramontes; Arutyunov et al, ...]

- Similar TBA solution for other integrable supercoset models $AdS_4 \times CP^3$, $AdS_3 \times S^3 \times T^4$, $AdS_2 \times S^2 \times T^6$
- 3-point functions for "long" operators using integrability [Gromov, Vieiera, Foda, Kostov, ...]

Minkowski	$\mathrm{AdS}_5 \times \mathrm{S}^5$	$\mathrm{AdS}_4 \times \mathbb{CP}^3$	$\mathrm{AdS}_3 \times S^3$	$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3$
$\frac{\text{super-Poincaré}}{\text{Lorentz}}$	$\frac{\mathrm{PSU}(2,2\vert 4)}{\mathrm{SO}(1,4)\times \mathrm{SO}(5)}$	$\frac{\mathrm{OSP}(6\vert 2,2)}{\mathrm{U}(3)\times \mathrm{SO}(1,3)}$	$\mathrm{PSU}(1,1\vert 2)$	$\mathrm{D}(2,1;\alpha)$

Table 1: Supercosets and their applications in string theory. The supercosets and supergroup in the lower row describe a supersymmetrized version of the geometries in the first line.